$\mathbb{R}^2$ "+" e "·"

$\Rightarrow$ $(1,0) \longrightarrow 1$

$(0,1) \longrightarrow i = \sqrt{-1}$ in quanto $(0,1)\cdot(0,1) = (-1,0)$

Rappresentazione algebrica

$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$

$\bar{z} = \mathrm{Re}\, z - i\, \mathrm{Im}\, z$

$|z| = \sqrt{\mathrm{Re}^2 z + \mathrm{Im}^2 z}$

Interpretazione della somma tra complessi

$\equiv$ Regola del Parallelogramma

Pb. Interpretazione del prodotto in $\mathbb{C}$

$$(a,b)\cdot(c,d) = (ac - bd,\ ad + bc)$$

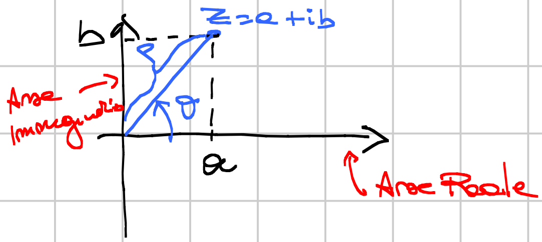
geometricamente che significa?

Pb. Quanto fa $(1+i)^{100}$?

$$(1+i)^{100} = \sum_{k=0}^{100} \binom{100}{k} 1^k \, i^{n-k}$$

Pb. Radici dei numeri complessi?

RAPPRESENTAZIONE POLARE



$\rho = |z| =$ modulo del numero complesso z

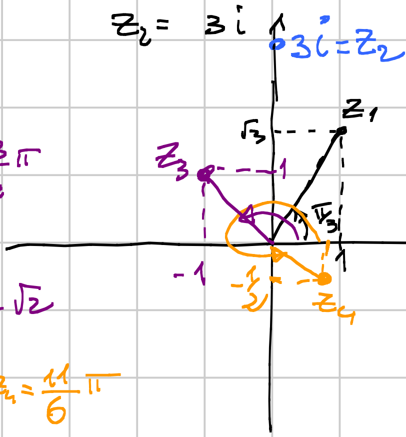
$\theta = \arg_{\min} z =$ il minimo angolo, orientato in senso antiorario, formato dal vettore $z = (a,b) \equiv a + ib$ con il semiasse positivo reale

Esempio

det. ρ e θ per i seguenti n.ri complessi

$z_1 = (1+i\sqrt{3}) \qquad z_3 = -1+i$

$z_2 = 3i \qquad z_4 = \frac{\sqrt{3}}{2} - \frac{i}{2}$



$\arg\min z_3 = \frac{3}{4}\pi = \pi + \operatorname{arctg}\frac{1}{-1}$

$|z_3| = \sqrt{1+1} = \sqrt{2}$

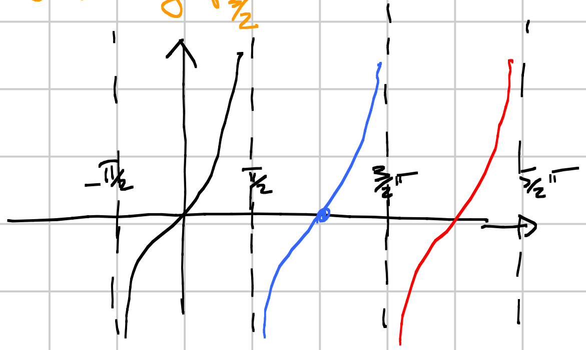
$\arg\min z_1 = \operatorname{arctg}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$

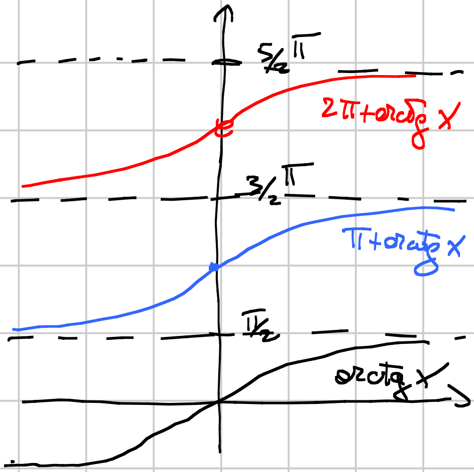
$|z_1| = \sqrt{1+3} = 2$

$\arg\min z_2 = \pi/2$

$|z_2| = \sqrt{3^2} = 3$

$\theta_4 = \arg\min z_4 = \frac{11}{6}\pi = 2\pi + \operatorname{arctg}\frac{-1/2}{\sqrt{3}/2}$





$$(\rho,\theta) \longrightarrow z = \rho\cos\theta + i\rho\operatorname{sen}\theta = \rho(\cos\theta + i\operatorname{sen}\theta)$$

$$\begin{array}{l} z \neq 0 \\ z = a+ib \end{array} \longrightarrow \begin{cases} \rho = |z| = \sqrt{a^2+b^2} \\ \theta = \arg\min z \end{cases}$$

$$\theta = \begin{cases} \operatorname{arctg}\frac{b}{a} & a>0,\ b\geq 0 \\ \pi/2 & a=0 \quad b>0 \\ \pi + \operatorname{arctg}\frac{b}{a} & a<0 \\ \frac{3}{2}\pi & a=0,\ b<0 \\ 2\pi + \operatorname{arctg}\frac{b}{a} & a>0,\ b<0 \end{cases}$$

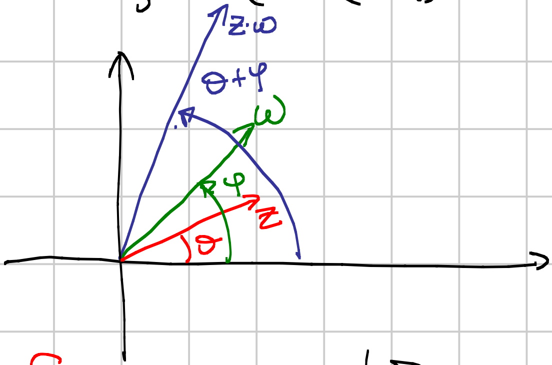
Interpretazione prodotto

$z = a+ib = \rho(\cos\theta + i\operatorname{sen}\theta)$

$w = c+id = r(\cos\varphi + i\operatorname{sen}\varphi)$

$$z\cdot w = (ac-bd) + i(ad+bc) = \rho\cdot r\cdot\left(\overbrace{\cos\theta\cos\varphi - \operatorname{sen}\theta\operatorname{sen}\varphi} + i(\operatorname{sen}\varphi\cos\theta + \operatorname{sen}\theta\cos\varphi)\right)$$

$$= \rho \cdot r \left(\cos(\vartheta+\varphi) + i \operatorname{sen}(\vartheta+\varphi)\right)$$



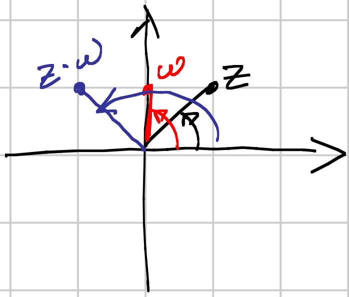
Esempio dati $z = (1+i)$ e $\omega = i$

$z \cdot \omega = i - 1$

$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i \operatorname{sen}\frac{\pi}{4}\right)$

$\omega = 1 \cdot \left(\cos\frac{\pi}{2} + i \operatorname{sen}\frac{\pi}{2}\right)$

$z \cdot \omega = \sqrt{2}\left(\cos\frac{3}{4}\pi + i \operatorname{sen}\frac{3}{4}\pi\right)$



Teorema (Formula di De Moivre)

Dato $z = \rho(\cos\vartheta + i \operatorname{sen}\vartheta)$, dato $n \in \mathbb{N}$ si ha

$$z^n = \rho^n\left(\cos(n\vartheta) + i \operatorname{sen}(n\vartheta)\right)$$

Esercizio Calcolare $(1+i)^{100}$

dim

$1+i = \rho(\cos\vartheta + i \operatorname{sen}\vartheta)$ $\qquad \rho = |z| = \sqrt{1^2+1^2} = \sqrt{2}$

$= \sqrt{2}\left(\cos\frac{\pi}{4} + i \operatorname{sen}\frac{\pi}{4}\right)$ $\qquad \vartheta = \operatorname{arctg}\frac{1}{1} = \frac{\pi}{4}$

$$(1+i)^{100} = (\sqrt{2})^{100}\left(\cos\frac{100}{4}\pi + i \operatorname{sen}\frac{100}{4}\pi\right)$$
$$= 2^{50}\left(\cos(24\pi+\pi) + i \operatorname{sen}(24\pi+\pi)\right)$$
$$= 2^{50}\left(\cos\pi \; i \operatorname{sen}\pi\right) = -2^{50}$$

dim (di de Moivre)

$n=0 \qquad z^0 = 1 \overset{?}{=} \rho^0(\cos 0\cdot\vartheta + i \operatorname{sen} 0\cdot\vartheta) = 1$ ✓

Suppongo valga per n $\qquad z^n = \rho^n(\cos n\vartheta + i \operatorname{sen} n\vartheta)$

devo provare che $z^{n+1} = \rho^{n+1}\left(\cos(n+1)\vartheta + i \operatorname{sen}(n+1)\vartheta\right)$

$$z^{n+1} = z^n \cdot z \overset{\text{Hip. Ind.}}{=} \rho^n(\cos n\vartheta + i \operatorname{sen} n\vartheta) \cdot \rho(\cos\vartheta + i \operatorname{sen}\vartheta)$$

$$= \rho^{n+1}\left(\cos n\vartheta \cos\vartheta - \operatorname{sen} n\vartheta \operatorname{sen}\vartheta + i\left(\cos n\vartheta \operatorname{sen}\vartheta + \operatorname{sen} n\vartheta \cos\vartheta\right)\right)$$

$$= \rho^{n+1}\left(\cos(n+1)\vartheta + i \operatorname{sen}(n+1)\vartheta\right) \quad \boxed{\text{cvd}}$$

de Moivre + Binomio di Newton

$z = \cos\vartheta + i \operatorname{sen}\vartheta$, ovvero $|z| = 1$

$z^2 = \cos 2\vartheta + i \operatorname{sen} 2\vartheta$ (de Moivre)

$$\overset{!}{=} \sum_{k=0}^{2} \binom{2}{k} (\cos\vartheta)^k \cdot (i \operatorname{sen}\vartheta)^{2-k} = \cos^2\vartheta + i\, 2\operatorname{sen}\vartheta\cos\vartheta - \operatorname{sen}^2\vartheta$$

(Binomio Newton)

$$\Downarrow$$

$$\begin{cases} \cos 2\vartheta = \cos^2\vartheta - \operatorname{sen}^2\vartheta \\ \operatorname{sen} 2\vartheta = 2\operatorname{sen}\vartheta\cos\vartheta \end{cases}$$

ma anche

$$(\cos\vartheta + i\operatorname{sen}\vartheta)^3 = \cos 3\vartheta + i \operatorname{sen} 3\vartheta$$

$$\|$$

$$\cos^3\vartheta + 3\cos^2\vartheta \cdot i \operatorname{sen}\vartheta + 3\cos\vartheta \cdot i^2 \cdot \operatorname{sen}^2\vartheta + i^3 \operatorname{sen}^3\vartheta$$

$$\| $$

$$\cos^3\vartheta - 3\cos\vartheta \operatorname{sen}^2\vartheta + i\left(3\cos^2\vartheta \operatorname{sen}\vartheta - \operatorname{sen}^3\vartheta\right)$$

$$\begin{cases} \cos 3\vartheta = \cos^3\vartheta - 3\cos\vartheta \operatorname{sen}^2\vartheta \\ \operatorname{sen} 3\vartheta = 3\cos^2\vartheta \operatorname{sen}\vartheta - \operatorname{sen}^3\vartheta \end{cases}$$

Radici in $\mathbb{C}$

in $\mathbb{R}$

$\longrightarrow x^2 = 2$ ha 2 soluzioni

$x^3 = 2$ ha 1 soluzione

$x^2 = -1$ non ha sol.

$x^3 = -1$ ha 1 sol.

$\longrightarrow x^{300} = 2$ ha 2 sol

Def Dato $\omega \in \mathbb{C}$, un numero $z \in \mathbb{C}$ è detto "radice n-esima di ω" se $z^n = \omega$

Pb: quante sono le radici n-esime?

Teorema (radici n-esime in $\mathbb{C}$)
Dato $\omega = \rho(\cos\vartheta + i \operatorname{sen}\vartheta) \in \mathbb{C}$, dato $n \geq 2$ esistono n radici n-esime di ω date da

$$z_k = \rho^{1/n}\left(\cos\left(\frac{\vartheta}{n} + k\frac{2\pi}{n}\right) + i \operatorname{sen}\left(\frac{\vartheta}{n} + k\frac{2\pi}{n}\right)\right) \quad k=0,\dots,n-1$$

dim

Se $z = r(\cos\varphi + i \operatorname{sen}\varphi)$ una radice n-esima di ω allora dovrà valere $z^n = \omega$ cioè (x de Moivre)

$$z^n = r^n(\cos(n\varphi) + i \operatorname{sen}(n\varphi)) = \rho(\cos\vartheta + i \operatorname{sen}\vartheta) = \omega$$
$$\Updownarrow$$
$$\begin{cases} |z|^n = r^n = \rho = |\omega| \\ \cos n\varphi = \cos\vartheta \\ \operatorname{sen} n\varphi = \operatorname{sen}\vartheta \end{cases} \iff \begin{cases} r = \rho^{1/n} & \text{(radici reali !!!)} \\ n\varphi = \vartheta + 2\pi \cdot h & h \in \mathbb{Z} \end{cases}$$

$$\iff \begin{cases} r = \rho^{1/n} \\ \varphi = \dfrac{\vartheta}{n} + \dfrac{2\pi}{n}\cdot h & h \in \mathbb{Z} \end{cases}$$

$h = s \cdot n + k \qquad 0 \leq k \leq n-1 \qquad h > 0$

$-h = s n + k \qquad 0 \leq k \leq n-1$

$$\iff \begin{cases} r = \rho^{1/n} \\ \varphi = \vartheta/n + k \cdot \dfrac{2\pi}{n} + s \cdot 2\pi & 0 \leq k \leq n-1 \quad s \in \mathbb{Z} \end{cases}$$

$$\implies \begin{cases} r = \rho^{1/n} \\ \varphi_1 = \vartheta/n + s \cdot 2\pi \\ \varphi_2 = \vartheta/n + 2\pi/n + s \cdot 2\pi & s \in \mathbb{Z} \\ \dots \\ \varphi_n = \vartheta/n + \frac{2\pi}{n}\cdot(n-1) + s\, 2\pi \end{cases}$$

$$\implies z_k = \rho^{1/n}\left(\cos\left(\frac{\vartheta}{n} + k\frac{2\pi}{n}\right) + i \operatorname{sen}\left(\frac{\vartheta}{n} + k\cdot\frac{2\pi}{n}\right)\right) \quad k=0\dots n-1$$

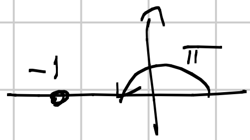
$\boxed{\text{////}}$

Esempio

Calcolare le radici cubiche di $\omega = -1$

dim

$\omega = -1 = \cos(\pi) + i \,\text{sen}(\pi)$

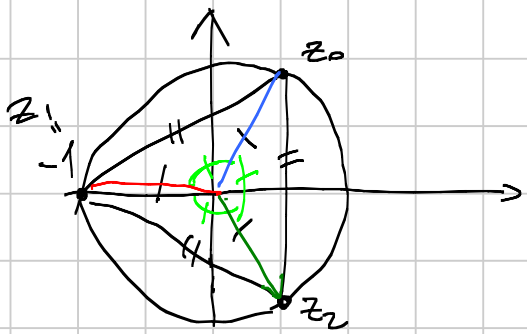


$$z_k = 1^{1/3}\left(\cos\left(\frac{\pi}{3} + k\frac{2\pi}{3}\right) + i \,\text{sen}\left(\frac{\pi}{3} + k\frac{2\pi}{3}\right)\right) \qquad k = 0, 1, 2$$

$$z_0 = \cos\frac{\pi}{3} + i \,\text{sen}\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$z_1 = \cos\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) + i \,\text{sen}\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) = -1 \leftarrow$$ che esistesse lo sapevo anche in $\mathbb{R}$

$$z_2 = \cos\left(\frac{\pi}{3} + \frac{4\pi}{3}\right) + i \,\text{sen}\left(\frac{\pi}{3} + \frac{4}{3}\pi\right) = \cos\left(2\pi - \frac{\pi}{3}\right) + i \,\text{sen}\left(2\pi - \frac{\pi}{3}\right)$$
$$= \frac{1}{2} - i\frac{\sqrt{3}}{2}$$



$\triangle z_0 z_1 z_2$ è un triangolo equilatero

(le radici n-esime formano un n-agono regolare)

Esercizio Determinare tutte le radici dell'equazione $|z|^2 \cdot z^2 = i$

dim

Posso sempre sostituire a $z \longrightarrow a + ib$ e risolvere il sistema reale, ma non sempre conviene

$|z|^2 \cdot z^2 = i$

passando ai moduli $\left||z|^2 \cdot z^2\right| = |i|$

$$\Rightarrow |z|^4 = 1 \quad (\text{e } |z| \geq 0) \Rightarrow |z| = 1$$

Condizione necessaria (le soluzioni sono nella circonf. raggio 1 centro 0)

$\Rightarrow$ l'eq. di partenza diventa $z^2 = i = \left(\cos\frac{\pi}{2} + i \,\text{sen}\frac{\pi}{2}\right)$

$$z_0 = \cos\frac{\pi}{4} + i \,\text{sen}\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$z_1 = \cos\left(\frac{\pi}{4} + \pi\right) + i \,\text{sen}\left(\frac{\pi}{4} + \pi\right) = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

(verificare!)

Oss: sostituendo $z = x+iy$ in $|z|^2 z^2 = i$

Trovo $(x^2+y^2)(x^2-y^2+2ixy) = i$

da cui

$$\begin{cases} x^4 - y^4 = 0 \\ (x^2+y^2)\cdot 2xy = 1 \end{cases} \iff \begin{cases} x = y \\ 2x^2 \cdot 2x^2 = 1 \end{cases} \text{ o } \begin{cases} x = -y \\ 2x^2 \cdot (-2x^2) = 1 \end{cases}$$

$$\iff \begin{cases} x = y \\ x^4 = 1/4 \end{cases} \text{ o } \cancel{\begin{cases} x = -y \\ x^4 = -1/4 \end{cases}} \text{ impossibile}$$

$$\iff \begin{cases} x = y \\ x = \pm \frac{1}{\sqrt{2}} \end{cases} \iff z_1 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \quad z_2 = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

e in questo caso arrivo alla soluzione in modo ragionevolmente semplice.

Esercizio Calcolate tutte le radici di

Non è un polin. in z!! $\Longrightarrow$ $z^3 \cdot \bar{z} + 3z^2 - 4 = 0$

dim

$$z^2 \cdot z \cdot \bar{z} + 3z^2 - 4 = 0 \qquad z \in \mathbb{C}$$

$$\Longleftrightarrow z^2 \cdot |z|^2 + 3z^2 - 4 = 0$$

non posso passare ai moduli direttamente, ma... $\Longrightarrow$

$$\Longleftrightarrow z^2\left(|z|^2+3\right) = 4 \quad \Longrightarrow \quad z^2 = \frac{4}{|z|^2+3} \in \mathbb{R}^+$$

Deduco $\operatorname{Im} z^2 = 0$!

Deduco (passando ai moduli!) $|z|^2 = \left|\frac{4}{|z|^2+3}\right| = \frac{4}{|z|^2+3}$!

e quindi $|z|^4 + 3|z|^2 - 4 = 0$

$$\left(|z|^2+4\right)\left(|z|^2-1\right) = 0$$

$$\Longrightarrow |z|^2 = 1 \quad \text{ma } |z| \geq 0 \quad \Longrightarrow \quad |z| = 1$$

Condizione Necessaria dunque le soluzioni stanno su circonferenza centro O raggio 1.

L'equazione diventa

$$z^2 \cdot |z|^2 + 3z^2 - 4 = z^2(1+3) - 4 = 0$$

$$\Longrightarrow z^2 = 1 \quad \Longrightarrow \quad \boxed{z_1 = 1 \quad \text{e} \quad z_2 = -1}$$

soluzioni

Esercizio

Trovate le soluzioni (z, w), con $z, w \in \mathbb{C}$, del sistema $\begin{cases} 4|z|^2 + w^2 = \dfrac{1}{3} \\ \dfrac{z\bar{w}}{w} + 2\bar{z} = \mathrm{i}\,. \end{cases}$