Lunedì 10.30-12.30

Def Data $f: I \to \mathbb{R}$ I intervallo ed f continua, una f.ne $F: I \to \mathbb{R}$ si dice "primitiva di f" (su I) se $\frac{dF}{dx}(x) = f(x)$

- Data una primitiva F, tutte le altre primitive di f sono date da

$$\int f(x)dx = F(x) + c \qquad c \in \mathbb{R}$$

- Date $f(x) = g(x)$ per ogni $x \in I \Rightarrow F(x) = G(x) + c$ con $c \in \mathbb{R}$

dove $F' = f$

$G' = g(x)$

Integrazione per parti

Ricordo che

$$\frac{d}{dx}(f \cdot g) = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}$$

$$(fg)' = f' \cdot g + f g'$$

$$f \cdot g' = (f \cdot g)' - f' g \qquad \forall x \in I$$

$$\Downarrow$$

$$\int (f \cdot g')(x) dx = \int (f \cdot g)'(x) - \int (f' \cdot g)(x) dx$$

Integraz. x Parti

$$\boxed{\int f(x) g'(x) dx = f(x) \cdot g(x) - \int f'(x) g(x) dx}$$

Esempi

Calcolare $\int x\cdot e^x\,dx$

dim

$$\int \underset{\underset{u'}{\uparrow}}{x}\cdot \underset{\underset{v}{\uparrow}}{e^x}\,dx = \underset{\underset{u}{\uparrow}}{\frac{x^2}{2}}\cdot \underset{\underset{v}{\uparrow}}{e^x} - \int \underset{\underset{u}{\uparrow}}{\frac{x^2}{2}}\cdot \underset{\underset{v'}{\uparrow}}{e^x}\,dx$$

in questo caso ho "complicato" il problema

$$\int \underset{\underset{u}{\uparrow}}{x}\cdot \underset{\underset{v'}{\uparrow}}{e^x}\,dx = \underset{\underset{u}{\uparrow}}{x}\cdot \underset{\underset{v}{\uparrow}}{e^x} - \int \underset{\underset{u'}{\uparrow}}{1}\cdot \underset{\underset{v}{\uparrow}}{e^x}\,dx$$

$$= x\cdot e^x - e^x + c \qquad c\in\mathbb{R}$$

Verifica

$$\left(xe^x - e^x + c\right)' = \cancel{e^x} + xe^x - \cancel{e^x} \qquad \checkmark$$

∎

Esempio Calcolare $\int x^2 \operatorname{sen} x\,dx$

dim

$$\int \underset{\underset{u}{\uparrow}}{x^2}\, \underset{\underset{v'}{\uparrow}}{\operatorname{sen} x}\,dx = \underset{\underset{u}{\uparrow}}{x^2}\cdot \underset{\underset{v}{\uparrow}}{(-\cos x)} + \int \underset{\underset{u'}{\uparrow}}{2x}\cdot \underset{\underset{v}{\uparrow}}{(+\cos x)}\,dx$$

$$= -x^2\cos x + 2\left\{\int \underset{\underset{u}{\uparrow}}{x}\, \underset{\underset{v'}{\uparrow}}{\cos x}\,dx\right\}$$

$$= -x^2\cos x + 2\left\{\underset{\underset{u}{\uparrow}}{x}\cdot \underset{\underset{v}{\uparrow}}{\operatorname{sen} x} - \int \underset{\underset{u'}{\uparrow}}{1}\, \underset{\underset{v}{\uparrow}}{\operatorname{sen} x}\,dx\right\}$$

$$= -x^2\cos x + 2x\operatorname{sen} x + 2\cos x + c \qquad c\in\mathbb{R}$$

Verificare!!

∎

Esempio Calcolare $\int \operatorname{sen}^2 x \, dx$

dim

$$\int \underset{u'}{\operatorname{sen} x} \cdot \underset{v}{\operatorname{sen} x} \, dx = \underset{u}{(-\cos x)} \cdot \underset{v}{\operatorname{sen} x} - \int \underset{u}{(-\cos x)} \underset{v'}{\cos x} \, dx$$

$$= -\operatorname{sen} x \cos x + \int \cos^2 x \, dx$$

$$= -\operatorname{sen} x \cos x + \int (1 - \operatorname{sen}^2 x) \, dx$$

$$\int \operatorname{sen}^2 x \, dx = -\operatorname{sen} x \cos x + x - \int \operatorname{sen}^2 x \, dx$$

$$2 \int \operatorname{sen}^2 x \, dx = x - \operatorname{sen} x \cos x + c \qquad c \in \mathbb{R}$$

$$\int \operatorname{sen}^2 x \, dx = \frac{1}{2}(x - \operatorname{sen} x \cos x) + c \qquad \in \mathbb{R}$$

Verificate !!

$\blacksquare$

Esercizio Calcolare $\int \cos^2 x \, dx$; $\int \operatorname{sen}^3 x \, dx$

Esempio Calcolare $\int \log x \, dx$

dim

$$\int \log x \, dx = \int \underset{u'}{1} \cdot \underset{v}{\log x} \, dx = \underset{u}{x} \cdot \underset{v}{\log x} - \int \underset{u}{x} \cdot \underset{v'}{\frac{1}{x}} \, dx$$

$$= x \log x - x + c \qquad c \in \mathbb{R}$$

verificate !!

$\blacksquare$

Esempio Calcolare $\int \sqrt{1-x^2}\,dx$

dim

$$\int \underset{u'}{1}\cdot \underset{v}{\sqrt{1-x^2}}\,dx = \underset{u}{x}\cdot \underset{v}{\sqrt{1-x^2}} - \int \underset{u}{x}\cdot \underset{v'}{\frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}}\,dx$$

$$= x\cdot\sqrt{1-x^2} + \int \frac{x^2-1+1}{\sqrt{1-x^2}}\,dx$$

$$= x\sqrt{1-x^2} + \int \frac{x^2-1}{\sqrt{1-x^2}} + \int \frac{dx}{\sqrt{1-x^2}}$$

$$= x\sqrt{1-x^2} + \arcsin x - \int \frac{\left(\sqrt{1-x^2}\right)^2}{\sqrt{1-x^2}}\,dx$$

$$\Downarrow$$

$$\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \arcsin x - \int \sqrt{1-x^2}\,dx$$

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin x\right) + c \qquad c\in\mathbb{R}$$

verifica $\left(\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x\right)' =$

$$= \frac{1}{2}\sqrt{1-x^2} + \frac{1}{2}x\cdot\frac{-x}{\sqrt{1-x^2}} + \frac{1}{2}\frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{2}\;\frac{1-x^2-x^2+1}{\sqrt{1-x^2}} = \sqrt{1-x^2}$$

Integrazione per sostituzione

$$(f\circ\varphi)'(x) = f'(\varphi(x))\cdot\varphi'(x) \qquad x\in I$$

$$\Downarrow$$

$$\int (f\circ\varphi)'(x)\,dx = \int f'(\varphi(x))\cdot\varphi'(x)\,dx$$

$$\Downarrow$$

Integraz. per sostituzione

$$\boxed{\int f'(\varphi(x))\cdot\varphi'(x)\,dx = (f\circ\varphi)(x) + c \quad c\in\mathbb{R}}$$

Date $f: I\to\mathbb{R}$, $F: I\to\mathbb{R}$ con $F'(x)=f(x)$,

$\varphi: J\to I$ derivabile

$$\int (f\circ\varphi)(x)\cdot\varphi'(x)\,dx = (F\circ\varphi)(x) + c \quad c\in\mathbb{R}$$

Infine, dalla formula precedente

$$\boxed{\int (f\circ\varphi)(x)\cdot\varphi'(x)\,dx = (F\circ\varphi)(x)+c = \left(\int f(t)\,dt\right)_{t=\varphi(x)}}$$

i) $\displaystyle\int f'(\varphi(x))\cdot\varphi'(x)\,dx = f(\varphi(x)) + c$

ii) $\displaystyle\int f(\varphi(x))\cdot\varphi'(x)\,dx = F(\varphi(x)) + c$

iii) $\displaystyle\int f(\varphi(x))\cdot\varphi'(x)\,dx = \left(\int f(t)\,dt\right)_{t=\varphi(x)}$

Genero ∞ tabelle di primitive utilizzando le ii) e iii)

$$\underbrace{\int e^x dx = e^x + c \quad c\in\mathbb{R}}_{\text{NOTO}} \Rightarrow \underbrace{\int e^{\varphi(x)} \cdot \varphi'(x) dx = e^{\varphi(x)} + c \quad c\in\mathbb{R} \quad \forall \varphi \text{ derivabile}}_{\infty \text{ primitive delle funzioni } e^{\varphi(x)} \cdot \varphi'(x)}$$

Esempio

Calcolare $\int e^{x^3} \cdot x^2 dx$

diventa

$$\int e^{\varphi(x)} \cdot \frac{\varphi'(x)}{3} dx \qquad \varphi(x) = x^3$$

$$= \frac{1}{3} \int e^{x^3} \cdot (x^3)' dx = \frac{1}{3} e^{x^3} + c \quad c\in\mathbb{R}$$

$$\int \operatorname{sen} x \, dx = -\cos x + c \Rightarrow \int \operatorname{sen}(\varphi(x)) \cdot \varphi'(x) dx = -\cos(\varphi(x)) + c \quad c\in\mathbb{R}$$

$\forall \varphi : J \to I$ derivabile

$$\int \cos x \, dx = \operatorname{sen} x + c \Rightarrow \int \cos(\varphi(x)) \cdot \varphi'(x) dx = \operatorname{sen}(\varphi(x)) + c \quad c\in\mathbb{R}$$

$\forall \varphi$ derivabile

Esercizio Calcolare $\displaystyle\int \frac{dx}{x\log x}$

dim

$$\int \frac{dx}{x\log x} = \left(\int \frac{\cancel{e^y}\,dy}{\cancel{e^y}\cdot y}\right)_{y=\log x} = \left(\int \frac{dy}{y}\right)_{y=\log x}$$

$y=\log x$
$x=e^y$
$\frac{dx}{dy}=e^y$
$dx=e^y dy$

$$= \left(\log|y|+c\right)_{y=\log x}$$

$$= \log|\log x| + c \quad c\in\mathbb{R}$$

oppure

$$\int \frac{dx}{x\log x} = \int \frac{1}{\log x}\cdot(\log x)'\,dx = \log|\log x| + c$$

$$\int \frac{dy}{y} = \log|y| + c \Rightarrow \int \frac{\varphi'(x)\,dx}{\varphi(x)} = \log|\varphi(x)| + c$$

$\boxed{\text{III}}$

Esempio Calcolare $\displaystyle\int \sqrt{1-x^2}\,dx$

dim

$$\int \sqrt{1-x^2}\,dx = \left(\int \sqrt{1-\operatorname{sen}^2 t}\cdot \cos t\,dt\right)_{t=\arcsin x}$$

$x=\operatorname{sen} t$
$\frac{dx}{dt}=\cos t$
$dx=\cos t\,dt$

sen t invertibile

restringo $t\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$

$$= \left(\int |\cos t|\cos t\,dt\right)_{t=\arcsin x}$$

$$= \left(\int \cos^2 t \, dt \right)_{t = \arcsin x} = \left(\frac{1}{2} t + \frac{1}{2} \operatorname{sen} t \cos t + c \right)_{t = \arcs.}$$

$$= \left(\frac{1}{2} t + \frac{1}{2} \operatorname{sen} t \cdot \sqrt{1 - \operatorname{sen}^2 t} + c \right)_{t = \arcsin x} = \frac{1}{2} \arcsin x + \frac{1}{2}$$

$$= \frac{1}{2} \arcsin x + \frac{1}{2} x \sqrt{1 - x^2} + c \qquad c \in \mathbb{R}$$

Sostituzione $t = \operatorname{Tg} \frac{x}{2}$ $\quad \frac{x}{2} = \operatorname{arctg} t$ $\quad x = 2 \operatorname{arctg} t$

$$\operatorname{sen} 2 \frac{x}{2} = \frac{2 \operatorname{sen} \frac{x}{2} \cos \frac{x}{2}}{\operatorname{sen}^2 \frac{x}{2} + \cos^2 \frac{x}{2}} = \frac{2 \operatorname{Tg} \frac{x}{2}}{1 + \operatorname{Tg}^2 \frac{x}{2}}$$

$$\cos x = \frac{\cos^2 \frac{x}{2} - \operatorname{sen}^2 \frac{x}{2}}{\operatorname{sen}^2 \frac{x}{2} + \cos^2 \frac{x}{2}} = \frac{1 - \operatorname{Tg}^2 \frac{x}{2}}{1 + \operatorname{Tg}^2 \frac{x}{2}}$$

$$\frac{dx}{dt} = 2 \frac{1}{1 + t^2} \qquad \frac{dt}{dx} = \left(1 + \operatorname{Tg}^2 \frac{x}{2} \right) \cdot \frac{1}{2}$$

Esempio Calcolare $\displaystyle \int \frac{\operatorname{sen} x}{1 + \cos x} \, dx$

dim

$t = \operatorname{Tg} \frac{x}{2}$

$$\int \frac{\operatorname{sen} x}{1 + \cos x} \, dx \underset{\substack{t = \operatorname{Tg} \frac{x}{2} \\ dx = \frac{2}{1+t^2} dt}}{=} \left(\int \frac{2t}{1+t^2} \cdot \frac{1}{1 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} \, dt \right)_{t = \operatorname{Tg} \frac{x}{2}}$$

$$= \left(4 \int \frac{t}{(1+t^2)^{\cancel{2}}} \cdot \frac{\cancel{1+t^2}}{\cancel{1+t^2} + 1 - t^2} \, dt \right)_{t = \operatorname{Tg} \frac{x}{2}}$$

$$= \left(\int \frac{2t}{1+t^2} \, dt \right)_{t = Tg\frac{x}{2}} = \left(\log(1+t^2) + c \right)_{t = Tg\frac{x}{2}}$$

$$= \boxed{\log\left(1 + Tg^2 \frac{x}{2}\right)} + c \qquad c \in \mathbb{R}$$

$$-\int \frac{-\operatorname{sen} x}{1+\cos x} \, dx = -\int \frac{(1+\cos x)'}{(1+\cos x)} \, dx =$$

$$= \boxed{-\log(1+\cos x)} + c \in \mathbb{R}$$

Oss $\log\left(1 + Tg^2 \frac{0}{2}\right) = 0 \neq -\log(1+\cos 0) = -\log 2$

sono due primitive (VERIF!!) di $\frac{\operatorname{sen} x}{1+\cos x}$ che differiscono per una costante!!!

Esempio Calcolare $\int \frac{dx}{\sqrt{x}\,(x+1)}$

dim

$$\int \frac{dx}{\sqrt{x}\,(x+1)} \underset{t = \sqrt{x}}{=} \left(\int \frac{2\cancel{t}\,dt}{\cancel{t}\,(t^2+1)} \right)_{t = \sqrt{x}}$$

$t = \sqrt{x}$ $x > 0$

$x = t^2$

$\frac{dx}{dt} = 2t$

$\boxed{dx = 2t\,dt}$

$$= \left(2 \int \frac{dt}{t^2+1} \right)_{t = \sqrt{x}} = \left(2 \operatorname{arctg} t + c \right)_{t = \sqrt{x}}$$

$$= 2 \operatorname{arctg} \sqrt{x} + c \qquad c \in \mathbb{R}$$