2015-03-03 - Lezione Analisi Matematica 1 ②

martedì 8,30-10,30

①

$$f(x) = x + 2x^3 + o(x^4) \quad (x \to 0)$$

$$g(x) = 1 - x^2 + o(x^4) \quad (x \to 0)$$

osservo che $\underbrace{f(x) = x + o(x)}$

$$(g \circ f)(x) = 1 - (f(x))^2 + o(f^4(x))$$

$$= 1 - (x + 2x^3 + o(x^4))^2 + o\left((x + o(x))^4\right) \quad x \to 0$$

$$= 1 - (x^2 + 4x^4) + o(x^4 + o(x^4))$$

$$= 1 - x^2 - 4x^4 + o(x^4)$$

Teorema (Formula Taylor Resto di Peano)

$f:]a,b[\to \mathbb{R}$, $x_0 \in]a,b[$

f derivabile $(n-1)$ volte in $]a,b[$

" " n " " x_0

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Allora $f(x) - P_n(x) = o\left((x - x_0)^n; x_0\right)$

dim

$$\exists \lim_{x \to x_0} \frac{f(x) - P_n(x)}{(x - x_0)^n} = 0 \qquad \%$$

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \qquad P_n(x_0) = f(x_0)$$

$$P_n'(x) = f'(x_0) + f''(x_0)(x - x_0) + \dots + \frac{f^{(n)}(x_0)}{(n-1)!}\cdot(x - x_0)^{n-1} \qquad P_n'(x_0) = f'(x_0)$$

$$P_n''(x) = f''(x_0) + f'''(x_0)(x - x_0) + \dots + \frac{f^{(n)}(x_0)}{(n-2)!}(x - x_0)^{n-2} \qquad P_n''(x_0) = f''(x_0)$$

$$P_n^{(n-1)}(x) = f^{(n-1)}(x_0) + f^{(n)}(x_0)(x - x_0) \qquad P_n^{(n-1)}(x_0) = f^{(n-1)}(x_0)$$

$$\lim_{x\to x_0}\frac{f(x)-P_n(x)}{(x-x_0)^n} \underset{\substack{\uparrow\\ H}}{=} \lim_{x\to x_0}\frac{f'(x)-P_n'(x)}{n(x-x_0)^{n-1}} = \lim_{x\to x_0}\frac{f''(x)-P_n''(x)}{n(n-1)(x-x_0)^{n-2}}$$

$\frac{0}{0}$ $\quad$ $\frac{0}{0}$

$$\underset{\substack{\uparrow\\ H}}{=} \dots \underset{\substack{\uparrow\\ H}}{=} \lim_{x\to x_0}\frac{f^{(n-1)}(x)-P_n^{(n-1)}(x)}{n!\,(x-x_0)} = \lim_{x\to x_0}\frac{f^{(n-1)}(x)-f^{(n-1)}(x_0)-f^{(n)}(x_0)(x-x_0)}{n!\,(x-x_0)}$$

$$= \frac{1}{n!}\lim_{x\to x_0}\frac{f^{(n-1)}(x)-f^{(n-1)}(x_0)}{(x-x_0)} - \frac{f^{(n)}(x_0)}{n!} \underset{\substack{\uparrow\\ \text{f deriv. n volte in } x_0}}{=} \frac{f^{(n)}(x_0)}{n!} - \frac{f^{(n)}(x_0)}{n!} = 0$$

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Oss : f derivabile in $x_0 \Rightarrow f$ è differenziabile in x_0 e dunque

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + o(x-x_0) \qquad x\to x_0$$
$$\Updownarrow$$
$$f(x) - f(x_0) - f'(x_0)(x-x_0) = o(x-x_0) \qquad x\to x_0$$
$$\Updownarrow$$
$$\lim_{x\to x_0}\frac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{|x-x_0|} = 0$$

$P_1(x) \equiv r(x) \equiv$ retta tangente al grafico di f nel punto $(x_0, f(x_0))$

Esempio

Sviluppo di Taylor in $x_0 = 1$ di $f(x) = e^x$

" " " " $x_0 = 0$ " "

dire

$$f(x) = e^x = f'(x) = f''(x) = f'''(x) = \dots = f^{(n)}(x)$$

$$f(1) = f'(1) = \dots = f^{(n)}(1) = e^1$$

$$P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(1)}{k!}(x-1)^k = e\cdot\sum_{k=0}^{n}\frac{(x-1)^k}{k!}$$

$$= e\left(1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \dots + \frac{(x-1)^n}{n!}\right)$$

Se voglio lo sviluppo di e^x centrato in $x_0=0$

$$P_n(x) = e^0\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}\right)$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + o(x^n) \quad x \to 0}$$

Esempio Sviluppo di $f = \operatorname{sen} x$ centrato in $x_0 = 0$

dim

$f = \operatorname{sen} x$ $\qquad f(0) = 0$

$f' = \cos x$ $\qquad f'(0) = 1$

$f'' = -\operatorname{sen} x$ $\qquad f''(0) = 0$

$f''' = -\cos x$ $\qquad f'''(0) = -1$

$f^{(IV)} = \operatorname{sen} x$ $\qquad f^{(IV)}(0) = 0$

$$P_1(x) = \sum_{k=0}^{1} \frac{f^{(k)}(0)}{k!} \cdot x^k = 0 + 1 \cdot x = \boxed{x} \quad P_1$$

$$P_2(x) = \sum_{k=0}^{2} \quad '' \quad = 0 + 1 \cdot x + 0 \cdot \frac{x^2}{2} = x$$

$$P_3(x) = \sum_{k=0}^{3} \quad '' \quad = P_2(x) + (-1) \cdot \frac{x^3}{6} = \boxed{x - \frac{x^3}{6}} \quad P_3$$

$$P_4(x) = P_3(x)$$

$$P_5(x) = P_4(x) + 1 \cdot \frac{x^5}{5!} = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$P_{2n+1}(x) = \sum_{k=0}^{n} (-1)^k \cdot \frac{x^{2k+1}}{(2k+1)!}$$

$$\boxed{\operatorname{sen} x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} + o\left(x^{2n+2}\right)}$$

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + o\left(x^{2n+1}\right)}$$

$f = \log(1+x) \qquad f(0) = 0$

$f' = \dfrac{1}{1+x} \qquad f'(0) = 1$

$f'' = -\dfrac{1}{(1+x)^2} \qquad f''(0) = -1$

$f''' = +\dfrac{2}{(1+x)^3} \qquad f'''(0) = 2$

$f^{(IV)} = -\dfrac{6}{(1+x)^4} \qquad f^{(IV)}(0) = -3!$

$f^{(V)} = \dfrac{4!}{(1+x)^5}$

$f^{(n)} = (-1)^{n+1} \dfrac{(n-1)!}{(1+x)^n} \qquad f^{(n)}(0) = (-1)^{n+1} \cdot (n-1)!$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k = 1 \cdot x - 1 \cdot \frac{x^2}{2!} + 2 \cdot \frac{x^3}{3!} - 3!\,\frac{x^4}{4!}$$

$$+ 4!\,\frac{x^5}{5!} + \dots + (-1)^{n+1} \cdot \frac{(n-1)!}{n!} \cdot x^n$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots + (-1)^{n+1} \cdot \frac{x^n}{n}$$

$$\boxed{\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^n \frac{x^{n+1}}{n+1} + o\left(x^{n+1}\right)}$$

Parentesi $\frac{1}{1-x} = 1+x+x^2+ \dots +x^n+ \dots = \sum_{k=0}^{\infty} x^k \qquad |x|<1$

$= 1+x+\dots+x^n+o(x^n) \qquad x\to 0$

dunque, posto $y=-x$, trovo

$$\frac{1}{1+y} = 1-y+y^2-y^3+\dots+(-1)^n\cdot y^n+o(y^n) \qquad y\to 0$$

e dunque, supponendo di sapere cosa significa

$$\int_0^x \frac{dy}{1+y} = \int_0^x \left[1-y+\dots+y^n+o(y^n)\right]dy$$

si ottiene $\log(1+y) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots+(-1)^n\frac{x^{n+1}}{n+1}+$
$+o(x^{n+1})$

Allo stesso modo, osservando che

$$\frac{1}{1+y} = 1-y+\dots+(-1)^n\cdot y^n+o(y^n) \quad y\to 0$$

posto $z^2=y$ si ha

$$\frac{1}{1+z^2} = 1-z^2+z^4+\dots+(-1)^n\cdot z^{2n}+o(z^{2n+1})$$

da cui segue, "integrando" tra 0 e x

$$\operatorname{arctg}(x) = x-\frac{x^3}{3}+\frac{x^5}{5}+\dots+(-1)^n\frac{x^{2n+1}}{2n+1}+o(x^{2n+2})$$
$$x\to 0$$

Chiusa Parentesi

$$f(x) = \operatorname{arcTg} x \qquad f(0)=0$$

$$f' = \frac{1}{1+x^2} \qquad f'(0) = 1$$

$$f'' = \frac{-2x}{(1+x^2)^2} \qquad f''(0) = 0$$

$$f''' = -\frac{2}{(1+x^2)^2} + 2x \cdot \frac{4x}{(1+x^2)^3} \qquad f'''(0) = -2$$

$$= -\frac{2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3}$$

$$f^{(IV)} = +\frac{8x}{(1+x^2)^3} + \frac{16x}{(1+x^2)^3} - \frac{48x^3}{(1+x^2)^4} \qquad f^{(IV)}(0) = 0$$

$$= \frac{24x}{(1+x^2)^3} - \frac{48x^3}{(1+x^2)^4}$$

$$f^{(V)} = \frac{24}{(1+x^2)^3} - \frac{144x^2}{(1+x^2)^4} - \frac{144x^2}{(1+x^2)^4} + \frac{384x^4}{(1+x^2)^5} \qquad f^{(V)}(0) = 24 = 4!$$

$$= \frac{24}{(1+x^2)^3} - \frac{288x^2}{(1+x^2)^4} + \frac{384x^4}{(1+x^2)^5}$$

$$f^{(VI)} = -\frac{144x}{(1+x^2)^4} - \frac{576x}{(1+x^2)^4} + \frac{8\cdot 288\cdot x^3}{(1+x^2)^5} + \dots \qquad f^{(VI)}(0) = 0.$$

$$= -\frac{720x}{(1+x^2)^4} + \dots$$

$$f^{(VII)} = -\frac{720}{(1+x^2)^4} + \dots \qquad f^{(VII)}(0) = 720 = 6!$$

dunque
$$f^{(2k)}(0) = 0$$
$$f^{(2k+1)}(0) = (2k)!$$

Ne segue che
$$\operatorname{arctg} x = \sum_{k=0}^{2m+1} \frac{f^{(k)}(0)}{k!} \cdot x^k + o\left(x^{2m+2}\right)$$

$$= \sum_{k=0}^{m} \frac{f^{(2k+1)}(0)}{(2k+1)!} x^{2k+1} + o\left(x^{2m+2}\right)$$

$$\boxed{\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^m \cdot \frac{x^{2m+1}}{2m+1} + o\left(x^{2m+2}\right)}$$

Sviluppo di $(1+x)^\alpha$ in $\boxed{x=0}$

$$(1+x)^\alpha = e^{\alpha \log(1+x)}$$

osservo che, se $\alpha=-1$, $\frac{1}{1+x} = 1-x+x^2-x^3+x^4 \dots + (-1)^n \cdot x^n + o(x^n)$

$\Big($ infatti $\frac{1}{1-x} = 1+x+x^2+\dots+x^n+o(x^n)$

e dunque $\frac{1}{1+z} = 1-z+z^2-z^3+\dots+(-1)^n \cdot z^n + o(z^n)\Big)$

Ricordo che $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^{n+1} \cdot \frac{x^n}{n} + o(x^n)$

$$\alpha \log(1+x) = \alpha x - \frac{\alpha x^2}{2} + \alpha \frac{x^3}{3} + \dots + (-1)^{n+1} \frac{\alpha x^n}{n} + o(x^n)$$

$$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \dots + \frac{y^n}{n!} + o(y^n)$$

Polinomio 3° ordine

$$e^{\alpha \log(1+x)} = 1 + \left(\alpha x - \frac{\alpha x^2}{2} + \frac{\alpha x^3}{3} + o(x^3)\right)$$
$$+ \frac{1}{2}\left(\alpha x - \frac{\alpha x^2}{2} + \frac{\alpha x^3}{3} + o(x^3)\right)^2$$
$$+ \frac{1}{6}\left(\alpha x - \frac{\alpha x^2}{2} + \frac{\alpha x^3}{3} + o(x^3)\right)^3 + o(x^3)$$

$$= 1 + \alpha x - \frac{\alpha x^2}{2} + \frac{\alpha x^3}{3} + \frac{1}{2}\left(\alpha^2 x^2 - \alpha^2 x^3\right) + \frac{1}{6}\left(\alpha^3 x^3\right) + o(x^3)$$

$$= 1 + \alpha x + x^2\left(-\frac{a}{2} + \frac{\alpha^2}{2}\right) + x^3\left(\frac{\alpha}{3} - \frac{\alpha^2}{2} + \frac{\alpha^3}{6}\right) + o(x^3)$$

$$= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} \cdot x^2 + \frac{\alpha}{3!}\left(\alpha^2 - 3\alpha + 2\right)x^3 + o(x^3)$$

$$= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} \cdot x^3 + o(x^3)$$

$$\boxed{(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \dots + \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}x^n + o(x^n)}$$

Esercizio

Calcolare il polinomio di ordine 4 centrato in $x=0$ di

$$f(x) = 3e^{x^2} - 2\log(1+x^2) - \frac{x}{1-2x} - 3\cos x + \operatorname{sen} x$$

lo sviluppo di $e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \frac{y^4}{4!} + o(y^4)$ $y \to 0$

e sostituendo $y = x^2$ trovo $e^{x^2} = \dots$

lo sviluppo di $\log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + o(y^4)$

da cui, sostituendo $y = x^2$ trovo $\log(1+x^2) = \dots$

inoltre $\frac{1}{1-y} = 1 + y + y^2 + y^3 + y^4 + o(y^4)$

da cui, posto $y = 2x$, trovo

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$$\frac{x}{1-2x} = x\left(1+2x+4x^2+8x^3+16x^4+o(x^4)\right)$$

$$= \dots$$

$$\operatorname{sen} x = x - \frac{x^3}{6} + o(x^4)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$$

Sostituendo e semplificando trovo il polinomio cercato.