Teorema (del confronto)

$f,g : [a,b[\to \mathbb{R}$ t.c.

1) f,g continue $\forall x \in [a,b[$

2) $\int_a^b g(x)dx \in \mathbb{R}$ $\Rightarrow$ $\int_a^b f(x)dx \in \mathbb{R}$

3) $|f(x)| \leq g(x)$ $\forall x \in [a,b[$

Teorema

$f,g : [a,b[\to \mathbb{R}$ t.c.

1) f,g continue $\forall x \in [a,b[$ 3) $f,g \geq 0$

2) $\lim_{x \to b^-} \frac{f(x)}{g(x)} = 0$

i) Se $\int_a^b f(x)dx$ diverge a $+\infty$ allora $\int_a^b g(x)dx$ diverge $+\infty$

ii) Se $\int_a^b g(x)dx$ converge allora $\int_a^b f(x)dx$ converge

dim

2) $\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists \delta > 0$: $\forall x \in]b-\delta, b[\Rightarrow -\varepsilon < \frac{f(x)}{g(x)} < \varepsilon$

$\Downarrow f,g \geq 0$

$\forall \varepsilon > 0$ $\exists \delta > 0$: $\forall x \in]b-\delta, b[\Rightarrow 0 \leq \frac{f(x)}{g(x)} < \varepsilon$

$\Downarrow \varepsilon = 1$

$\varepsilon = 1$ $\exists \delta > 0$: $\boxed{\forall x \in]b-\delta, b[\Rightarrow 0 \leq f(x) < g(x)}$

$\Downarrow$

$$0 \leq \int_{b-\delta}^b f(x)dx \leq \int_{b-\delta}^b g(x)dx$$

Se $\int_a^b f(x)dx = +\infty$ allora $\int^b f(x)dx = +\infty$

$\left(\text{poiché } \int_a^{b-\delta} f(x)dx \in \mathbb{R}\right)$

allora $\int_{b-\delta}^b g(x)dx = +\infty$

allora $\int_a^b g(x)dx = +\infty$

Se $\int_a^b g(x)dx \in \mathbb{R}$ allora $\int_{b-\delta}^b g(x)dx \in \mathbb{R}$

allora $\int_{b-\delta}^b f(x)dx \in \mathbb{R}$

allora $\int_a^b f(x)dx \in \mathbb{R}$ $\square$

Oss: Proviamo che $\displaystyle\int_3^{+\infty} \frac{dx}{\sqrt{e^x-e^2}\,(x-2)^\alpha} \in \mathbb{R} \ \forall \alpha$

$$f(x) = \frac{1}{\sqrt{e^x-e^2}\cdot(x-2)^\alpha} \sim \frac{1}{e^{x/2}\cdot x^\alpha} = g(x) \quad x \to +\infty$$

Adesso studio $g(x)$ $\left(\text{sappiamo che } \displaystyle\int_3^{+\infty} \frac{dx}{x^2} \in \mathbb{R}\right)$

$$\lim_{x\to+\infty} \frac{g(x)}{1/x^2} = \lim_{x\to+\infty} \frac{1}{e^{x/2}\cdot x^\alpha}\cdot x^2$$

$$= \lim_{x\to+\infty} \frac{x^{2-\alpha}}{e^{x/2}} = 0 \qquad \frac{x^k}{e^x}$$

Corollario
Teorema confronto

Ma $\displaystyle\int_3^{+\infty} \frac{1}{x^2}dx \in \mathbb{R}$ allora $\displaystyle\int_3^{+\infty} g(x)dx \in \mathbb{R} \ \forall \alpha$

allora (Teorema Confronto Asintotico) $\int_{3}^{+\infty} f(x)dx \in \mathbb{R} \quad \forall \alpha$

Teorema (Confronto Asintotico)

$f, g : [a,b[\to \mathbb{R}$ t.c.

1) $f, g \geq 0 \ \forall x \in [a,b[$, $g(x) > 0$

2) f, g continue su $[a,b[$

3) $\exists \lim\limits_{x \to +\infty} \frac{f(x)}{g(x)} = \ell \in]0, +\infty[$

allora sono equivalenti

i) $\int_a^b f dx$ converge (diverge a $+\infty$)

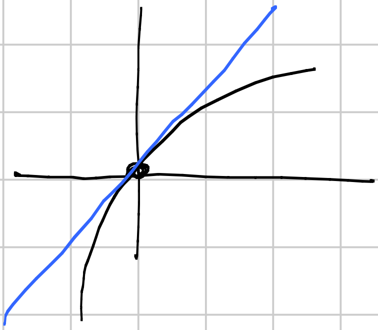
ii) $\int_a^b g dx$ " (" " ")

Oss: $\sum_n 1/n$ diverge a $+\infty$

$\log(1+x) \leq x \quad \forall x > -1$

$\Downarrow x = 1/n$

$\log(1 + 1/n) \leq \frac{1}{n} \quad \forall n \geq 1$



$\log\left(\frac{n+1}{n}\right) = \log(n+1) - \log n \leq 1/n \quad \forall n \geq 1$

$\Rightarrow \log 2 - \log 1 \leq 1$

$\log 3 - \log 2 \leq 1/2$

$- - - -$

$\log(n+1) - \log(n) \leq 1/n$

sommo le disuguaglianze

$$\sum_{k=1}^{n}\left(\log(k+1)-\log(k)\right) \leq \sum_{k=1}^{n} 1/k \qquad \forall\, n \geq 1$$

$$\| \quad \log(n+1)-\log(1)$$

$$\| \quad \log(n+1)$$

$$\Downarrow$$

$$\underbrace{\lim_{n\to+\infty} \log(n+1)}_{= +\infty} \leq \lim_{n\to+\infty} \sum_{k=1}^{n} 1/k$$

$\Longrightarrow$ $\sum_n 1/n$ diverge a $+\infty$

$$\int_1^6 \frac{dx}{x} \leq \sum_{k=1}^{5} \frac{1}{k}\cdot\left(k-(k-1)\right)$$

$$\| \quad \log(1+5)$$

Si può leggere così $\mathcal{A}=\{1<2<3<4<5<6\}$ suddi. di $[1,6]$

$$\int_1^6 \frac{1}{x}\,dx \leq S\left(\frac{1}{x};\mathcal{A}\right) = \sum_{k=1}^{5}(k+1-k)\cdot \sup_{[k,k+1]} \frac{1}{x}$$

$$= \sum_{k=1}^{5} \frac{1}{k}$$

Teorema (Confronto Serie Integrale Improprio)

$f: [a,+\infty[\longrightarrow [0,+\infty[$

f debolmente decrescente

allora sono tra loro equivalenti

i) $\int_a^{+\infty} f\,dx$ converge (diverge a $+\infty$)

ii) $\sum_n f(n)$ converge (diverge a $+\infty$)

dim

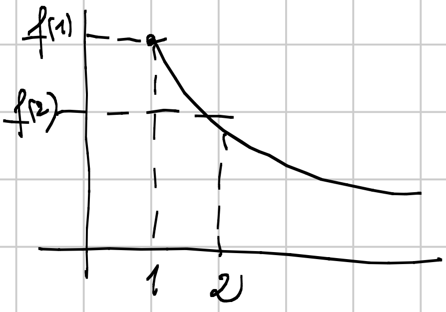
f monotona $\Rightarrow \exists \int_a^{\beta} f(x)dx \quad \forall \beta > a$

$f \geq 0 \quad \Rightarrow \int_a^{b} f(x)dx$ è finito o è diverge a $+\infty$

$\sum_n f(n)$ è convergente o è diverge a $+\infty$

Non è restrittivo prendere $a=1$

Considero $[1,n]$



$f(2) \leq f(x) \leq f(1) \qquad x \in [1,2]$

$f(3) \leq f(x) \leq f(2) \qquad x \in [2,3]$

$- - - -$

$f(n) \leq f(x) \leq f(n-1) \qquad x \in [n-1,n]$

$\Downarrow$

$f(2) = \int_1^2 f(2)dx \leq \int_1^2 f(x) \leq \int_1^2 f(1) = f(1) \quad x \in [1,2]$

$f(3) = \int_2^3 f(3) \leq \int_2^3 f(x) \leq \int_2^3 f(2) = f(2) \quad x \in [2,3]$

$- - - -$

$f(n) = \int_{n-1}^{n} f(n) \leq \int_{n-1}^{n} f(x) \leq \int_{n-1}^{n} f(n-1) = f(n-1) \quad x \in [n-1,n]$ —

$\Downarrow$ sommo

$$f(2)+f(3)+f(4)+\dots+f(n) \le \int_1^2 f(x)\,dx+\dots+\int_{n-1}^{n} f(x)\,dx \le f(1)+f(2)+\dots+f(n-1)$$

$$\underbrace{\boxed{\sum_{k=1}^{n} f(k)}}_{S_n} - f(1) \underset{(1)}{\le} \int_1^n f(x)\,dx \underset{(2)}{\le} \underbrace{\boxed{\sum_{k=1}^{n} f(k)}}_{S_n} - f(n)$$

- Se $\lim_{n\to+\infty} S_n = +\infty$ allora per la (1) $\lim_{n\to+\infty} \int_1^n f(x)\,dx = +\infty$

Se $\quad \text{"} \quad \in \mathbb{R}$ allora per la (2) $\quad \text{"} \quad \in \mathbb{R}$

Se $\lim_{n\to+\infty} \int_1^n f(x)\,dx = +\infty$ allora per la (2) $\lim_{n\to+\infty} S_n = +\infty$

Se $\quad \text{"} \quad \in \mathbb{R}$ allora per la (1) $\quad \text{"} \quad \in \mathbb{R}$

$\blacksquare$

Oss: In modo analogo si poteva osservare che $f:[1,n]\to\mathbb{R}$ essendo decrescente è integrabile su $[1,n]$ e si ha

posto $A=\{1,2,3,\dots,n\}$

$$s(f,A) \le \int_1^n f(x)\,dx \le S(f,A)$$

ma
$$s(f,A) = \sum_{k=1}^{n-1} (k+1-k)\cdot \inf f([k,k+1]) = \sum_{k=1}^{n-1} f(k+1)$$

$$S(f,A) = \sum_{k=1}^{n-1} (k+1-k) \sup f([k,k+1]) = \sum_{k=1}^{n-1} f(k)$$

ovvero
$$\sum_{k=1}^{n-1} f(k+1) = \sum_{h=2}^{n} f(h) = \sum_{h=1}^{n} f(h) - f(1) = S_n - f(1)$$

$$\sum_{k=1}^{n-1} f(k) = \sum_{k=1}^{n} f(k) - f(n) = S_n - f(n)$$

$$\Rightarrow \quad S_n - f(1) \le \int_1^n f(x)\,dx \le S_n - f(n) \quad \forall n$$

da cui segue che

$$\lim_{m\to+\infty}\int_1^m f\,dx \in \mathbb{R} \iff \lim_m S_m \in \mathbb{R}$$

$$\lim_{m\to+\infty}\int_1^m f\,dx = +\infty \iff \lim_{m\to+\infty} S_m = +\infty$$

Esercizio

$$\sum_n \frac{1}{n^\alpha}$$ converge se $\alpha > 1$

diverge a $+\infty$ se $\alpha \le 1$

dim

$f(x) = \frac{1}{x^\alpha}$ è una funzione decrescente per $\alpha > 0$

$$\int_1^{+\infty} \frac{dx}{x^\alpha} \begin{cases} \in \mathbb{R} & \text{se } \alpha > 1 \\ +\infty & \text{se } \alpha \le 1 \end{cases}$$

per il Teorema confronto serie - integrale

$$\sum_n f(n) = \sum_n \frac{1}{n^\alpha}$$ converge $\alpha > 1$

diverge se $0 < \alpha < 1$

Il caso $\alpha = 0$ è banale $\int_1^{+\infty} dx = +\infty$

Il caso $\alpha < 0$ " " $+\infty = \int_1^{+\infty} \frac{1}{x^\alpha}\,dx \geq \int_1^{+\infty} dx = +\infty$

$\boxed{/\!/\!/}$

Esercizio

Sia $a_n = \displaystyle\int_{5-\frac{16}{n^\alpha}}^{5} f(t)\,dt \qquad f(t) = \frac{1}{\sqrt{5-t}} + \sqrt{5-t}$

Studiare la convergenza di $\sum_n a_n$ al variare di α

dim

$$\int \left[(5-t)^{-\frac12} + (5-t)^{\frac12}\right] dt \underset{5-t=x}{=} \left(\int \left(x^{-\frac12} + x^{\frac12} \right)(-1)\,dx \right)_{x=5-t}$$

$$= -\left(\frac{x^{1-\frac12}}{1-\frac12} + \frac{x^{1+\frac12}}{1+\frac12} + c \right)_{x=5-t} = -2\sqrt{5-t} - \frac{2}{3}(5-t)^{\frac32} + c$$

$c \in \mathbb{R}$

$$a_n = \int_{5-\frac{16}{n^\alpha}}^{5} f(t)\,dt = \left[-2\sqrt{5-t} - \frac{2}{3}(5-t)^{\frac32} \right]_{5-\frac{16}{n^\alpha}}^{5} = 2\sqrt{\frac{16}{n^\alpha}} + \frac{2}{3}\left(\frac{16}{n^\alpha}\right)^{\frac32}$$

$$= \underbrace{\frac{8}{n^{\alpha/2}}}_{b_n} + \underbrace{\frac{128}{3} \cdot \frac{1}{n^{\frac32\alpha}}}_{c_n}$$

$\sum_n a_n = \sum_n (b_n + c_n)$ converge sse $\sum_n b_n$ converge

$\sum_n c_n$ "

$\sum_n \frac{8}{n^{\alpha/2}}$ converge se $\frac{\alpha}{2} > 1$ se $\alpha > 2$

$\sum_n \frac{128}{3} \left(\frac{1}{n}\right)^{\frac{3}{2}\alpha}$ " se $\frac{3}{2}\alpha > 1$ se $\alpha > \frac{2}{3}$

$\Downarrow$

$\sum_n a_n$ converge se $\alpha > 2$ $\blacksquare$

Esercizio

Studiare $F(x) = \int_{\boxed{0}}^{x} \frac{e^{-t^2}}{\sqrt[3]{t}}\, dt$ al variare di x

dim

DOMINIO di F

proviamo che $\int_0^x f(t)dt \in \mathbb{R}$ $\forall x \in \mathbb{R}$

$f(t) = \frac{e^{-t^2}}{\sqrt[3]{t}}$

quando $t \to 0$ $f(t) = \frac{e^{-t^2}}{\sqrt[3]{t}} \sim \frac{1}{\sqrt[3]{t}}$

ma $\int_0^x \frac{dt}{\sqrt[3]{t}} = \int_0^x \left(\frac{1}{t}\right)^{1/3} dt \in \mathbb{R}$ poiché $\frac{1}{3} < 1$

PARITÀ di $F(x)$

voglio provare che $F(-x) = F(x)$

$$F(x) = \int_0^{-x} f(t)\,dt \underset{\substack{t=-y \\ dt=-dy}}{=} -\int_0^{x} f(-y)\cdot dy = \int_0^x [-f(-y)]\,dy = \int_0^x f(y)dy$$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \| $

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \quad F(x)$

la Tesi segue dal fatto che $f(t)$ è dispari

quindi studio solo per $x > 0$

LIMITI agli estremi del dominio

devo fare $\lim_{x\to+\infty} \int_0^x f(t)\,dt = \int_0^{+\infty} f(t)\,dt$

$f \geq 0$ dunque si tratta di scoprire se $\int_0^{+\infty} f$ diverge a $+\infty$ o converge

$$f(t) = \frac{e^{-t^2}}{\sqrt[3]{t}} \underset{t\geq 1}{\leq} \frac{e^{-t^2}}{\sqrt[3]{1}} \underset{t\geq 1}{\leq} e^{-t}$$

$\sqrt[3]{t} \geq 1 \quad \forall t \geq 1$

$t^2 \geq t \quad t \geq 1$

$\Downarrow$

$-t^2 \leq -t \quad t \geq 1$

$e^{-t^2} \leq e^{-t} \quad t \geq 1$

Ho scoperto che

$f(t) \leq e^{-t} \quad \forall t \geq 1$

$\Downarrow$

$$\int_1^{+\infty} f(t)\,dt \leq \int_1^{+\infty} e^{-t}\,dt = \left[-e^{-t}\right]_{t=1}^{t=+\infty} = e^{-1}$$

ma $\int_0^1 f(t)\,dt \in \mathbb{R}$

$\Rightarrow \int_0^{+\infty} f(t)\,dt \in \mathbb{R}$

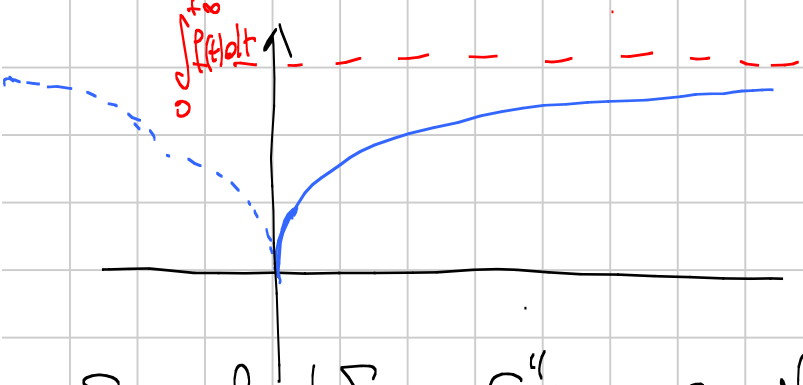
STUDIO MONOTONIA

$$F'(x) = \left(\int_0^x f(t)\,dt\right)' = f(x) = \frac{e^{-x^2}}{\sqrt[3]{x}}$$

Teo. fond. Calcolo Integr.

$$\lim_{x\to 0^+} F'(x) = \lim_{x\to 0^+} \frac{e^{-x^2}}{\sqrt[3]{x}} = \frac{1}{0^+} = +\infty$$

$$\lim_{x\to+\infty} F'(x) = \lim_{x\to+\infty} \frac{e^{-x^2}}{\sqrt[3]{x}} = \lim_{x\to+\infty} \frac{1}{e^{x^2}\sqrt[3]{x}} = \frac{1}{+\infty} = 0^+$$



$f'(x) > 0$
$\forall x > 0$

Se calcolate $f''(x)$ scoprite che $f''(x) < 0$ $\forall x > 0$

$$F''(x) = \left(\frac{e^{-x^2}}{\sqrt[3]{x}}\right)' = -2xe^{-x^2} \cdot \frac{1}{x^{1/3}} - \frac{1}{3}x^{-\frac{4}{3}} \cdot e^{-x^2}$$

$$= -2x^{2/3} \cdot e^{-x^2} - \frac{1}{3} x^{-4/3} e^{-x^2}$$

$$= -x^{-4/3} e^{-x^2} \left(2x^2 + \frac{1}{3}\right)$$

e si ha che $F'' < 0$ $\forall x \neq 0$ (in $x=0$ f'' non esiste)

Dunque la funzione è concava su $\mathbb{R} \setminus \{0\}$

Teorema (Confronto Asintotico Integrali Impropri)

$f, g : [a,b[\longrightarrow \mathbb{R}$

1) f, g continue in $[a,b[$ (serve per avere $\int_a^\beta f \in \mathbb{R}$ $\int_a^\beta g \in \mathbb{R}$ $\forall \beta \in [a,b[$)

2) $f \geq 0$ $g > 0$ $\forall x \in [a,b[$ — ($\int_a^b f$ finito/$+\infty$ $\int_a^b g$ finito/$+\infty$)

3) $\exists \lim_{x \to +\infty} \frac{f(x)}{g(x)} = \ell \in]0, +\infty[$ — (mi permette di legare il comportamento di f e g in $x = b$)

Allora sono equivalenti

i) $\int_a^b f(x)\,dx$ converge (diverge a $+\infty$)

ii) $\int_a^b g(x)\,dx$ converge (diverge a $+\infty$)

Dim

f, g continue $\Rightarrow \exists \int_a^\beta f(x)\,dx \quad \int_a^\beta g(x)\,dx \quad \forall \beta \in [a,b[$

$f \geq 0$ $g > 0$ $\Rightarrow \int_a^b f(x)\,dx$ è finito o diverge a $+\infty$

$\int_a^b g(x)\,dx$ " " " " "

$3) \Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 : \forall x \in [a,b[\quad b-\delta < x < b \Rightarrow \ell - \varepsilon < \frac{f(x)}{g(x)} < \ell + \varepsilon$

$\Downarrow \varepsilon = \frac{\ell}{2}$ $\quad \exists \delta > 0 : \forall x \in]b-\delta, b[\quad \frac{\ell}{2} < \frac{f(x)}{g(x)} < \frac{3}{2}\ell$

$\Downarrow g > 0$

$$\boxed{\exists \delta > 0 : \frac{\ell}{2} \cdot g(x) \underset{(1)}{<} f(x) \underset{(2)}{<} \frac{3}{2}\ell \cdot g(x) \quad \forall x \in]b-\delta, b[}$$

Se $\int_a^b f(x)dx \in \mathbb{R}$ allora $\int_{b-\delta}^b f dx \in \mathbb{R}$ allora per la (1)

• $\frac{\ell}{2}\int_{b-\delta}^b g(x)dx \in \mathbb{R}$

<u>ma</u> • $\int_a^{b-\delta} g\, dx \in \mathbb{R}$ per l'ipotesi continua

allora $\int_a^b g(x)dx \in \mathbb{R}$

Se $\int_a^b f dx = +\infty$ allora $\int_{b-\delta}^b f dx = +\infty$ allora per la (2)

$\frac{3}{2}\ell \int_{b-\delta}^b g\, dx = +\infty$

allora $\int_a^b g\, dx = +\infty$

Se $\int_a^b g\, dx \in \mathbb{R}$ allora $\int_a^b f dx \in \mathbb{R}$

Se $\int_a^b g\, dx = +\infty$ allora $\int_a^b f dx = +\infty$ $\square$