Oss: l'integrale di Riemann è definito per funzioni

$f : [a,b] \to \mathbb{R}$ limitate

ovvero

$$Graf(f) = \{(x, f(x)) : x \in [a,b]\} \subseteq$$

$$\subseteq [a,b] \times \left[\inf f([a,b]), \sup f([a,b])\right]$$

Oss: parleremo di integrale improprio quando il grafico di f NON È contenuto in un rettangolo limitato

ovvero quando

f non è limitata o l'intervallo di definizione di f non è limitato

Oss: Procedo, per studiare la convergenza di un integrale improprio, confrontando l'integrando con opportuni integrandi di cui conosco l'andamento

Esempio fondamentale 1

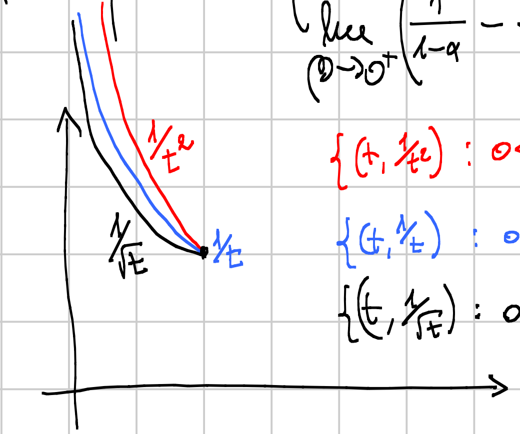
Per quali valori di $\alpha \in \mathbb{R}$ esiste finito (infinito) il seguente limite

$$\lim_{\beta \to 0} \int_\beta^1 \frac{1}{t^\alpha}\, dt$$

Soluz

$$\int_\beta^1 \frac{1}{t^\alpha}\, dt = \begin{cases} \left[\dfrac{t^{1-\alpha}}{1-\alpha}\right]_{t=\beta}^{t=1} & \alpha \neq 1 \\ \left[\log t\right]_{t=\beta}^{t=1} & \alpha = 1 \end{cases} = \begin{cases} \dfrac{1}{1-\alpha} - \dfrac{\beta^{(1-\alpha)}}{1-\alpha} & \alpha \neq 1 \\ -\log \beta & \alpha = 1 \end{cases}$$

$$\lim_{\beta \to 0^+} \int_\beta^1 \frac{1}{t^\alpha}\, dt = \begin{cases} \lim\limits_{\beta \to 0^+} \left(\dfrac{1}{1-\alpha} + \dfrac{1}{\beta^{\alpha-1} \cdot (\alpha-1)}\right) & \alpha > 1 \\ \lim\limits_{\beta \to 0^+} -\log \beta & \alpha = 1 \\ \lim\limits_{\beta \to 0^+} \left(\dfrac{1}{1-\alpha} - \dfrac{\beta^{(1-\alpha)}}{1-\alpha}\right) & \alpha < 1 \end{cases} = \begin{cases} +\infty & \alpha > 1 \\ +\infty & \alpha = 1 \\ \dfrac{1}{1-\alpha} & \alpha < 1 \end{cases}$$



$\{(t, 1/t^2) : 0 < t \leq 1\}$ ha "area" infinita

$\{(t, 1/t) : 0 < t \leq 1\}$ ha "area" infinita

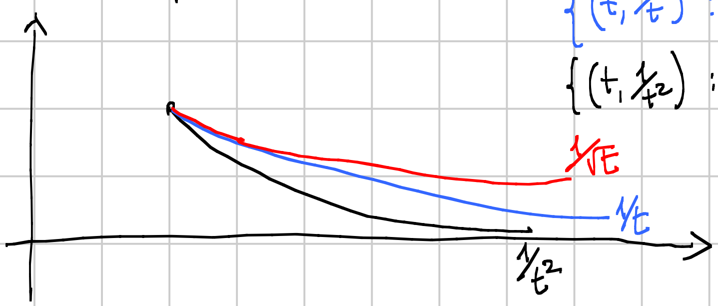
$\{(t, 1/\sqrt{t}) : 0 < t \leq 1\}$ " " finita

Nell'esempio che segue

$\{(t, 1/\sqrt{t}) : 1 \leq t\}$ ha "area" infinita

$\{(t, 1/t) : 1 \leq t\}$ ha "area" infinita

$\{(t, 1/t^2) : 1 \leq t\}$ " " finita



Esempio fondamentale 2

Per quali valori di $\alpha \in \mathbb{R}$ esiste finito (infinito) il seguente limite

$$\lim_{\beta \to +\infty} \int_1^\beta \frac{dt}{t^\alpha}\, dt$$

dim

$$\int_1^\beta \frac{dt}{t^a} = \begin{cases} \frac{\beta^{(1-\alpha)}}{1-a} - \frac{1}{1-\alpha} & \alpha \neq 1 \\ \log \beta & \alpha = 1 \end{cases}$$

$$\lim_{\beta \to +\infty} \int_1^\beta \frac{dt}{t^\alpha} = \begin{cases} \lim_{\beta \to +\infty} \left(\frac{1}{\beta^{(\alpha-1)}(1-\alpha)} - \frac{1}{1-a} \right) & \alpha > 1 \\ \lim_{\beta \to +\infty} \log \beta & \alpha = 1 \\ \lim_{\beta \to +\infty} \left(\frac{\beta^{1-\alpha}}{1-\alpha} - \frac{1}{1-a} \right) & \alpha < 1 \end{cases}$$

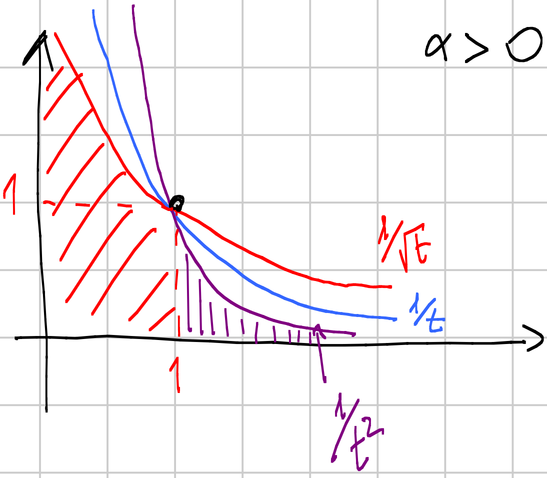
$$= \begin{cases} \frac{1}{\alpha-1} & \alpha > 1 \\ +\infty & \alpha = 1 \\ +\infty & \alpha < 1 \end{cases}$$

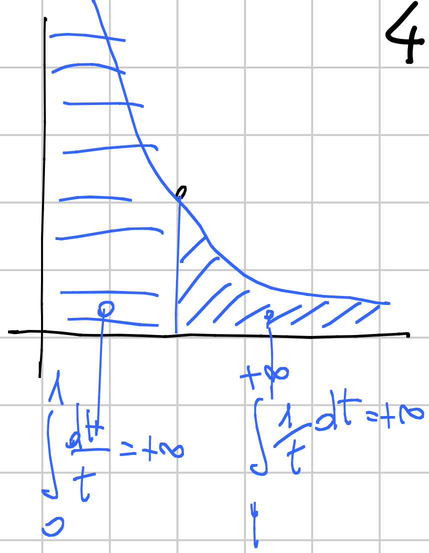
Riassumendo

$$\boxed{\int_0^1 \frac{dt}{t^\alpha} = \begin{cases} \frac{1}{1-a} & \alpha < 1 \\ +\infty & \alpha \geq 1 \end{cases}}$$

$$\boxed{\int_1^{+\infty} \frac{dt}{t^\alpha} = \begin{cases} +\infty & \alpha \leq 1 \\ \frac{1}{\alpha-1} & \alpha > 1 \end{cases}}$$

$\alpha > 0$





Def (Integrale Improprio)

$f : [a,b[\longrightarrow \mathbb{R}$, con f R-integrabile su $[a,\beta]$ $\forall \beta \in [a,b[$

f è integrabile in senso improprio su $[a,b[$ se $\exists \lim_{\beta \to b^-} \int_a^\beta f(x)\,dx$

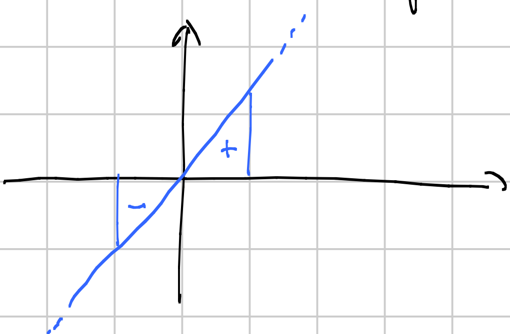
f non è " " " " " " se $\nexists \lim_{\beta \to b^-} \int_a^\beta f(x)\,dx$

Se $\lim_{\beta \to b^-} \int_a^\beta f(x)\,dx \in \mathbb{R}$ allora scriviamo $\lim_{\beta \to b^-} \int_a^\beta f\,dx = \int_a^b f(x)\,dx$ $\Uparrow \mathbb{R}$

Se " $= +\infty \; (-\infty)$ " " $\int_a^b f(x)\,dx = +\infty \; (-\infty)$

Oss: posso considerare $f :]a,b] \longrightarrow \mathbb{R}$?

Oss: $\int_{-\infty}^{+\infty} x\,dx$ esiste? è finito?



Con la nostra definizione

$$\int_{-\infty}^{+\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{+\infty} x\,dx = -\infty + \infty \quad \text{forma indet.}$$

$$\int_{-\infty}^{0} x\,dx = \lim_{n\to-\infty} \int_{-n^2}^{0} x\,dx = \lim_{n\to-\infty} \int_{-n^3}^{0} x\,dx$$

$$\int_{0}^{+\infty} x\,dx = \lim_{n\to+\infty} \int_{0}^{n^4} x\,dx = \lim_{n\to+\infty} \int_{0}^{e^n} x\,dx$$

$$\int_{-\infty}^{+\infty} x\,dx = \lim_{n\to+\infty} \int_{-n^2}^{100} x\,dx + \lim_{n\to+\infty} \int_{100}^{n} x\,dx = -\infty$$

$$= \lim_{n\to+\infty} \int_{-n}^{0} x\,dx + \lim_{n\to+\infty} \int_{0}^{e^n} x\,dx = +\infty$$

Oss $f(x) = e^{-|x|}$ $\quad \exists \int_{-\infty}^{+\infty} e^{-|x|}\,dx$

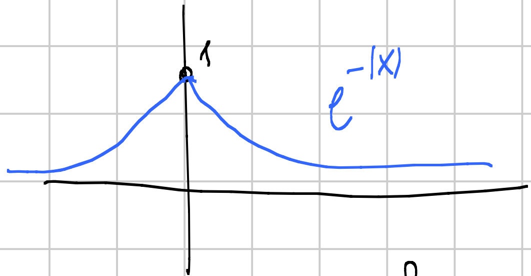
infatti,

$$\int_{-\infty}^{+\infty} e^{-|x|}\,dx = \int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{+\infty} e^{-x}\,dx$$

$$= \lim_{\beta\to-\infty} \Big[e^{x}\Big]_{x=\beta}^{x=0} - \lim_{\beta\to+\infty} \Big[e^{-x}\Big]_{x=0}^{x=\beta}$$

$$= \lim_{\beta\to-\infty} \left(1 - e^{\beta}\right) - \lim_{\beta\to+\infty} \left(e^{-\beta} - 1\right)$$

$$= 1 - (-1) = 2$$

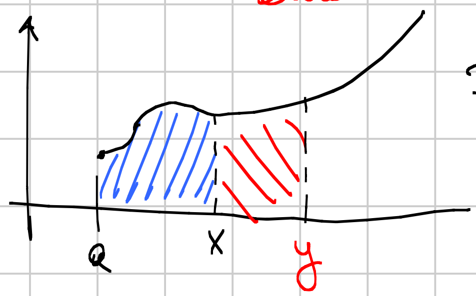


osserviamo che $\lim_{\beta\to-\infty}\int_{\beta}^{100} e^{-|x|} + \lim_{\beta\to+\infty}\int_{100}^{\beta} e^{-|x|} = 2$

Teorema (integrazione funzioni positive/negative)

$f:[a,b[\to\mathbb{R}$ t.c. $\exists \int_a^{\beta} f(x)\,dx \quad \forall \beta\in[a,b[$

Se $f\geq 0$ (≤ 0) $\forall x\in[a,b[$ allora $\int_a^b f(x)\,dx$ esiste (finito o infinito)



$$F(y)=\int_a^y f(t)\,dt = \int_a^x f(t)\,dt + \int_x^y f(t)\,dt = F(x) + \int_x^y f(t)\,dt \geq F(x)$$

quindi $y>x \Rightarrow F(y)\geq F(x)$

quindi F è monotona crescente debolmente

quindi $\exists \lim_{x\to b^-} F(x) = \int_a^b f(t)\,dt$

Devo provare che $f\geq 0 \Rightarrow \int_x^y f(t)\,dt \geq 0 \quad \forall y>x$

Teorema del confronto in quanto $f\geq 0 \Rightarrow \int_x^y f \geq \int_x^y 0$

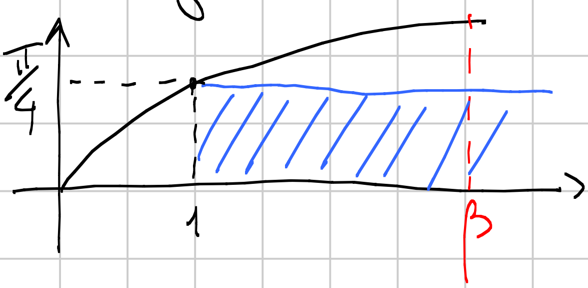
Esercizio

Studiare la convergenza del seguente integrale improprio

$$\int_0^{+\infty} \arctan t^2 \, dt$$

Sol.

Non so calcolare esplicitamente una primitiva di $f(t) = \arctan(t^2)$, dunque posso solo studiare la convergenza



$f = \arctan t^2$

$f' = \frac{2t}{1+t^4} > 0 \quad \forall t > 0$

$\Downarrow$

$\boxed{f(t) > f(1) = \frac{\pi}{4} \quad \forall t > 1}$

$$\int_0^\beta \arctan t^2 = \int_0^1 \arctan t^2 \, dt + \int_1^\beta \arctan t^2 \, dt \qquad \forall \beta > 1$$

Monotonia integrale Riemann

$$\geq \int_0^1 \arctan(t^2) \, dt + \int_1^\beta \frac{\pi}{4} \, dt$$

$$= \int_0^1 \arctan(t^2) \, dt + \frac{\pi}{4}(\beta - 1)$$

$$\Rightarrow \lim_{\beta \to +\infty} \int_0^\beta \arctan t^2 \, dt \geq \int_0^1 \arctan(t^2) \, dt + \lim_{\beta \to +\infty} \frac{\pi}{4}(\beta - 1)$$

Teorema confronto tra limiti

$$= \int_0^1 \arctan t^2 \, dt + \infty = +\infty$$

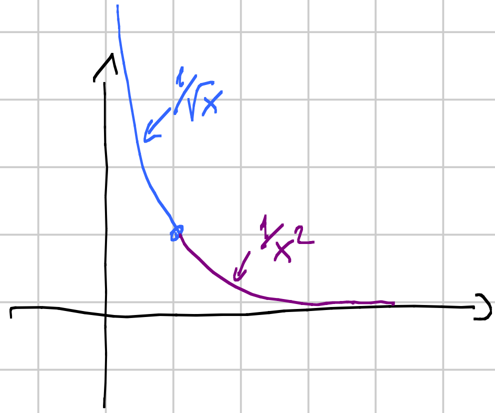
$$\Rightarrow \int_0^{+\infty} \arctan(t^2) \, dt = +\infty \qquad \square$$

Oss: $\arctan(t^2) \geq 0 \quad \forall t \geq 0 \Rightarrow \exists \int_0^{+\infty} \arctan(t^2) \, dt$

Pb: $f: I \to \mathbb{R}$, I illimitato, $\sup f(I) = +\infty$, $f \geq 0 \overset{?}{\Longrightarrow} \int_I f(t)\,dt$ diverge a $+\infty$??

NO

$$f(x) = \begin{cases} 1/\sqrt{x} & 0<x\le 1 \\ 1/x^2 & 1<x \end{cases}$$



$$\int_0^{+\infty} f(x)\,dx = \int_0^1 \frac{1}{\sqrt{x}}\,dx + \int_1^{+\infty} \frac{1}{x^2}\,dx \in \mathbb{R}$$

Pb: Data $f: [0,+\infty[\to \mathbb{R}$ t.c. $\sup f([0,+\infty[) = +\infty$

$$f \geq 0 \overset{??}{\Longrightarrow} \int_0^{+\infty} f(t)\,dt = +\infty \quad ???$$

Teorema (l'eq. delle C.N. per le serie numeriche)

$f: [a,+\infty[\to \mathbb{R}$, f continua, $f \geq 0$, $\exists \lim_{x\to+\infty} f(x) = l$

Se $\int_a^{+\infty} f \in \mathbb{R}$ allora $l = 0$

Procedo per assurdo dico $l > 0$

$$\exists \lim_{x\to+\infty} f(x) = l \in \mathbb{R} \iff \forall \varepsilon>0\ \exists N>0 : \forall x>N \quad l-\varepsilon < f(x) < l+\varepsilon$$

$$\varepsilon = l/2 \quad \exists N = N(l/2) > 0 : \boxed{\forall x>N \quad l/2 < f(x)}$$

$$\Rightarrow \int_N^\beta f(x)\,dx \geq \int_N^\beta \frac{l}{2}\,dx = \frac{l}{2}(\beta - N) \qquad \beta > N$$

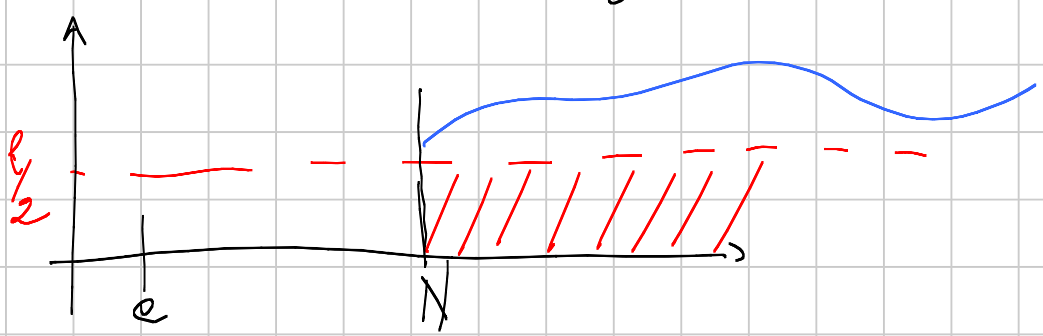
$$\Rightarrow \int_a^\beta f(x)\,dx = \int_a^N f(x)\,dx + \int_N^\beta f(x)\,dx \geq \int_a^N f(x)\,dx + \frac{\ell}{2}(\beta - N)$$

facendo il limite $+\infty$

$$\Rightarrow \int_a^{+\infty} f(x)\,dx = \lim_{\beta \to +\infty} \int_a^\beta f(x)\,dx \geq \int_a^N f(x)\,dx + \lim_{\beta \to +\infty} \frac{\ell}{2}(\beta - N)$$

$$= \int_a^N f(x)\,dx + \infty = +\infty$$

ASSURDO



Oss: Dove si usa la continuità?

si utilizza poiché devono esistere finiti

$$\int_a^\beta f(x)\,dx \quad \forall \beta > a$$

e questo è assicurato dal fatto che f continua su $[a, \beta]$ $\forall \beta > a$!!

Teorema

$f, g: [a, b[\to \mathbb{R}$

i) f, g continue su $[a, b[$

ii) $|f(x)| \leq g(x)$ $\forall x \in [a, b[$

iii) $\int_a^b g(x)\,dx \in \mathbb{R}$

$$\Rightarrow \int_a^b f(x)\,dx \in \mathbb{R}$$

$$\left(\int_a^b |f(x)|\,dx \in \mathbb{R}\right)$$

dice

$f(x) = f^+(x) - f^-(x)$ $\qquad f^+ = \max\{f(x), 0\} \geq 0$

$|f(x)| = f^+(x) + f^-(x)$ $\qquad f^- = \max\{-f(x), 0\} \geq 0$

per ipotesi $\quad 0 \leq f^+(x) + f^-(x) \leq g(x) \quad \forall x \in [a,b[$

$$\Rightarrow \boxed{\begin{cases} 0 \leq f^+(x) \leq g(x) \\ 0 \leq f^-(x) \leq g(x) \end{cases} \quad \forall x \in [a,b[}$$

equivale alla ii)

f continua su $[a,b[\Rightarrow f$ è R-integrabile su $[a,\beta]$ $\forall \beta \in [a,b[$

$$\Downarrow$$

$$\boxed{f^+ \text{ ed } f^- \text{ sono R-integr. su } [a,\beta] \quad \forall \beta \in [a,b[}$$

segue dalla i)

Dimostro che $\displaystyle\int_a^b f^+(x)dx \in \mathbb{R}$

$\exists \displaystyle\int_a^b f^+(x)dx$ poiché $f^+ \geq 0$

$0 \leq f^+(x) \leq g(x) \quad \forall x \in [a,b[\Rightarrow$

$\forall \beta \in [a,b[\quad \displaystyle\int_a^\beta f^+(x)dx \leq \int_a^\beta g(x)dx$

Teorema Confronto

$$\Rightarrow \lim_{\beta \to b^-} \int_a^\beta f^+(x)dx = \int_a^b f^+(x)dx \leq \int_a^b g(x)dx \in \mathbb{R}$$

ipotesi ii)

$$\Rightarrow \int_a^b f^+(x)dx \in \mathbb{R}$$

In modo identico

$\displaystyle\int_a^b f^-(x)dx \in \mathbb{R}$ $\qquad$ da cui la tesi! $\square$

Teorema (Confronto Asintotico)

$f, g : [a,b[\to \mathbb{R}$

f, g continue in $[a,b[$, $f, g \geq 0$

$\exists \lim_{x \to b^-} \frac{f(x)}{g(x)} = \ell \in \,]0,+\infty[$

allora sono equivalenti

i) $\int_a^b f(x)\,dx \in \mathbb{R}$ $(+\infty)$

ii) $\int_a^b g(x)\,dx \in \mathbb{R}$ $(+\infty)$