2015-04-14 - Lezione 10 A1-Mat

martedì 8.30-10.30

$$\{(x,y) : a \le x \le b \quad 0 \le y \le f(x)\} = A$$

$$s(f,\mathcal{A}) \le \text{area } A \le S(f,\mathcal{A})$$

$\forall \mathcal{A}$ suddivisione di $[a,b]$

$f : [a,b] \to \mathbb{R}$ limitata

$$\int_a^b f(x)dx \overset{\text{se esiste}}{=} s(f) = \sup\{s(f,\mathcal{A}) : \mathcal{A}\}$$

$$= S(f) = \inf\{S(f,\mathcal{A}) : \mathcal{A}\}$$

Teorema (CNS di integrabilità)

$f : [a,b] \to \mathbb{R}$ limitata. Sono equivalenti

i) f è R-integrabile

ii) $\forall \varepsilon > 0 \ \exists \mathcal{A}_\varepsilon, \mathcal{B}_\varepsilon$ suddivisioni di $[a,b]$: $S(f,\mathcal{A}_\varepsilon) - s(f,\mathcal{B}_\varepsilon) < \varepsilon$

iii) $\forall \varepsilon > 0 \ \exists \mathcal{C}_\varepsilon$ suddivisione di $[a,b]$: $S(f,\mathcal{C}_\varepsilon) - s(f,\mathcal{C}_\varepsilon) < \varepsilon$

unica

i) $\Rightarrow$ ii)

Per ipotesi $\quad s(f) = S(f) = \int_a^b f(x)dx$

$$s(f) = \sup_{\mathcal{B}} s(f,\mathcal{B}) = \int_a^b f(x)dx \iff \begin{cases} s(f,\mathcal{B}) \le \int_a^b f & \forall \mathcal{B} \\ \forall \varepsilon > 0 \ \exists \mathcal{B}_\varepsilon & \int_a^b f dx - \varepsilon < s(f,\mathcal{B}_\varepsilon) \quad * \end{cases}$$

$$S(f) = \inf_{\mathcal{A}} S(f,\mathcal{A}) = \int_a^b f dx \iff \begin{cases} \int_a^b f dx \le S(f,\mathcal{A}) & \forall \mathcal{A} \\ \forall \varepsilon > 0 \ \exists \mathcal{A}_\varepsilon & S(f,\mathcal{A}_\varepsilon) < \int_a^b f dx + \varepsilon \quad ** \end{cases}$$

$* + **$ $\quad \forall \varepsilon > 0 \ \exists \mathcal{A}_\varepsilon, \mathcal{B}_\varepsilon$

$$-s(f,\mathcal{B}_\varepsilon) < \varepsilon - \int_a^b f dx$$

$$S(f,\mathcal{A}_\varepsilon) < \int_a^b f dx + \varepsilon$$

$\Rightarrow \forall \varepsilon>0 \ \exists \mathcal{A}_\varepsilon, \mathcal{B}_\varepsilon \quad S(f,\mathcal{A}_\varepsilon) - s(f,\mathcal{B}_\varepsilon) < 2\varepsilon$

ii) $\Rightarrow$ i)

$$\left.\begin{aligned} s(f) &= \sup_{\mathcal{A}} s(f,\mathcal{A}) \\ S(f) &= \inf_{\mathcal{B}} S(f,\mathcal{B}) \end{aligned}\right\} \Longleftrightarrow \begin{cases} s(f,\mathcal{B}) \le s(f) \ \forall \mathcal{B} \\ \forall \varepsilon>0 \ \exists \mathcal{B}_\varepsilon \quad s(f)-\varepsilon < s(f,\mathcal{B}_\varepsilon) \end{cases} \quad \begin{cases} S(f) \le S(f,\mathcal{A}) \ \forall \mathcal{A} \\ \forall \varepsilon>0 \ \exists \mathcal{A}_\varepsilon \quad S(f,\mathcal{A}_\varepsilon) < S(f)+\varepsilon \end{cases}$$

$\forall \varepsilon>0 \ \exists \mathcal{A}_\varepsilon \, \mathcal{B}_\varepsilon \quad S(f,\mathcal{A}_\varepsilon) - s(f,\mathcal{B}_\varepsilon)$

$$\Rightarrow \forall \varepsilon>0 \ \exists \mathcal{A}_\varepsilon, \mathcal{B}_\varepsilon \quad \begin{cases} s(f)-\varepsilon < s(f,\mathcal{B}_\varepsilon) \le s(f) \\ S(f) \le S(f,\mathcal{A}_\varepsilon) < S(f)+\varepsilon \\ S(f,\mathcal{A}_\varepsilon) - s(f,\mathcal{B}_\varepsilon) < \varepsilon \end{cases}$$

$$\Rightarrow \forall \varepsilon>0 \ \exists \mathcal{A}_\varepsilon \, \mathcal{B}_\varepsilon \quad \begin{cases} -s(f) \le -s(f,\mathcal{B}_\varepsilon) \\ S(f) \le S(f,\mathcal{A}_\varepsilon) \\ S(f,\mathcal{A}_\varepsilon) - s(f,\mathcal{B}_\varepsilon) < \varepsilon \end{cases}$$

$\Rightarrow \forall \varepsilon>0 \ \exists \mathcal{A}_\varepsilon \, \mathcal{B}_\varepsilon \quad S(f) - s(f) \le S(f,\mathcal{A}_\varepsilon) - s(f,\mathcal{B}_\varepsilon) < \varepsilon$

$\Rightarrow S(f) = s(f)$

ii) $\Rightarrow$ iii) $\quad \mathcal{C}_\varepsilon = \mathcal{A}_\varepsilon \cup \mathcal{B}_\varepsilon \qquad \mathcal{A}_\varepsilon \subseteq \mathcal{C}_\varepsilon \Rightarrow S(f,\mathcal{C}_\varepsilon) \le S(f,\mathcal{A}_\varepsilon)$

$\mathcal{B}_\varepsilon \subseteq \mathcal{C}_\varepsilon \Rightarrow s(f,\mathcal{B}_\varepsilon) \le s(f,\mathcal{C}_\varepsilon)$

$\Rightarrow \forall \varepsilon>0 \ \exists \mathcal{C}_\varepsilon = \mathcal{A}_\varepsilon \cup \mathcal{B}_\varepsilon : S(f,\mathcal{C}_\varepsilon) - s(f,\mathcal{C}_\varepsilon) \le S(f,\mathcal{A}_\varepsilon) - s(f,\mathcal{B}_\varepsilon) \overset{\text{per la ii)}}{<} \varepsilon$

iii) $\Rightarrow$ ii) basta prendere $\mathcal{A}_\varepsilon = \mathcal{B}_\varepsilon = \mathcal{C}_\varepsilon$ e la ii) segue ∎

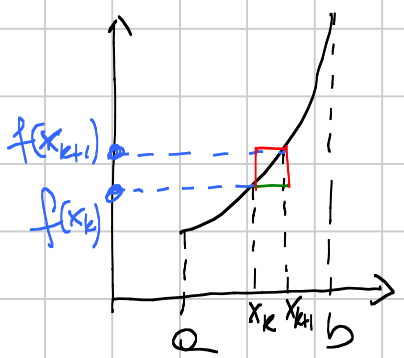
Teorema (funzioni monotone sono integrabili)

$f:[a,b] \to \mathbb{R}$, f limitata, f monotona

$\Rightarrow f$ è integrabile

dim

Sia $f:[a,b] \to \mathbb{R}$ debolmente crescente



Voglio provare che

$\forall \varepsilon>0 \ \exists \mathcal{C}_\varepsilon : S(f,\mathcal{C}_\varepsilon) - s(f,\mathcal{C}_\varepsilon) < \varepsilon$

Chiave della dim: prendo

$$\mathcal{C}_n = \left\{a;\ a+\frac{b-a}{n};\ a+2\cdot\frac{b-a}{n};\ \dots;\ b\right\} = \left\{a+k\,\frac{b-a}{n} : k=0,1,2,\dots,n\right\}$$

ovvero $\forall k \quad x_{k+1} - x_k = \frac{b-a}{n}$

$$S(f, \mathcal{P}_n) - s(f, \mathcal{P}_n) = \sum_{k=0}^{n-1} (x_{k+1} - x_k) \cdot \left(\sup f([x_k, x_{k+1}]) - \inf f([x_k, x_{k+1}]) \right)$$

$$= \sum_{k=0}^{n-1} \left(\frac{b-a}{n}\right) \cdot \left(f(x_{k+1}) - f(x_k) \right)$$

$$= \frac{b-a}{n} \cdot \sum_{k=0}^{n-1} \left(f(x_{k+1}) - f(x_k) \right)$$

$$= \frac{b-a}{n} \left(\left(f(\cancel{x_1}) - f(x_0) \right) + \left(f(\cancel{x_2}) - f(\cancel{x_1}) \right) + \cancel{\ldots} + \left(f(x_n) - f(\cancel{x_{n-1}}) \right) \right)$$

$$= \frac{b-a}{n} \cdot \left(f(x_n) - f(x_0) \right) = (b-a) \cdot \left(f(b) - f(a) \right) \cdot \frac{1}{n}$$

$\forall \varepsilon > 0 \ \exists n = n(\varepsilon) \ : \ S(f, \mathcal{P}_{n(\varepsilon)}) - s(f, \mathcal{P}_{n(\varepsilon)}) = (b-a)(f(b) - f(a)) \frac{1}{n(\varepsilon)} < \varepsilon$

$\forall \varepsilon > 0 \ \exists \mathcal{P}_\varepsilon \ : \ S(f, \mathcal{P}_\varepsilon) - s(f, \mathcal{P}_\varepsilon) < \varepsilon$ $\boxed{M}$

Teorema (f continua ⇒ R-integrabile)

$f : [a,b] \to \mathbb{R}$ continua $\forall x \in [a,b]$

Allora f è R-integrabile

dim

Siamo nelle ipotesi di Weierstrass (Thm) e Heine Cantor e dunque

f è limitata su $[a,b]$ (x Weierstrass)

f è unif. continua su $[a,b]$ (x H. C.)

$\to \forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) > 0 \ : \ \forall x, y \in [a,b] \quad |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Per scegliere la suddivisione mi baso su δ

Preso $\mathcal{C}_\delta = \{a \le x_0 < x_1 < \dots < x_n = b\}$

tale che $x_{k+1} - x_k = d < \delta$

Con questa scelta si ha

$\sup f([x_k, x_{k+1}]) - \inf f([x_k, x_{k+1}]) =$

Weiers.

$= \max f([x_k, x_{k+1}]) - \min f([x_k, x_{k+1}]) = f(M_k) - f(m_k) < \varepsilon$

$M_k, m_k \in [x_k, x_{k+1}]$

f è unif. c. H.C.

$$\Delta S(f, \mathcal{C}_{\delta(\varepsilon)}) = \sum_{k=0}^{n-1} (x_{k+1} - x_k) \cdot \left(\sup f([x_k, x_{k+1}]) - \inf f([x_k, x_{k+1}])\right)$$

$$< \sum_{k=0}^{n-1} (x_{k+1} - x_k) \cdot \varepsilon =$$

$$= \varepsilon \sum_{k=0}^{n-1} (x_{k+1} - x_k) = \varepsilon \cdot (b-a)$$

$\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) \ \exists \mathcal{C}_{\delta(\varepsilon)} : S(f, \mathcal{C}_\delta) - s(f, \mathcal{C}_\delta) < \varepsilon \cdot (b-a)$

$\forall \varepsilon > 0 \ \exists \mathcal{C}_\varepsilon : S(f, \mathcal{C}_\varepsilon) - s(f, \mathcal{C}_\varepsilon) < \varepsilon \cdot (b-a)$ $\blacksquare$

Teorema (L'insieme delle f.ni integrabili su $[a,b]$ è uno spazio vettoriale su $\mathbb{R}$)

date $f, g : [a,b] \to \mathbb{R}$ limitate e integrabili allora

i) $\alpha f(x) + \beta g(x)$ è integrabile $\forall \alpha, \beta \in \mathbb{R}$

ii) $\displaystyle\int_a^b (\alpha f(x) + \beta g(x))\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx$

dim (omessa a lezione 2014/15)

la dim che $\int_a^b \alpha f\,dx = \alpha \int_a^b f\,dx$ è molto semplice

Proviamo che $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

Data $\mathcal{A}$ suddivisione si ha

$$s(f,\mathcal{A}) + s(g,\mathcal{A}) \leq s(f+g,\mathcal{A})$$

$$s(f,\mathcal{A}) + s(g,\mathcal{B}) \leq s(f,\mathcal{A}\cup\mathcal{B}) + s(g,\mathcal{A}\cup\mathcal{B}) \leq s(f+g,\mathcal{A}\cup\mathcal{B}) \leq s(f+g)$$

$$\Downarrow$$

$$\sup_{\mathcal{A}} \left(s(f,\mathcal{A}) + s(g,\mathcal{B}) \right) = s(f) + s(g,\mathcal{B}) \leq \sup_{\mathcal{A}} s(f+g) = s(f+g)$$

$$\sup_{\mathcal{B}} \left(s(f) + s(g,\mathcal{B}) \right) = s(f) + s(g) \leq \sup_{\mathcal{B}} \left(s(f+g) \right) = s(f+g)$$

e dunque $s(f) + s(g) \leq s(f+g)$ $\quad *$

Analogamente $S(f+g,\mathcal{A}\cup\mathcal{B}) \leq S(f,\mathcal{A}\cup\mathcal{B}) + S(g,\mathcal{A}\cup\mathcal{B}) \leq S(f,\mathcal{A}) + S(f,\mathcal{B})$

e procedendo come sopra si ottiene

$$S(f+g) \leq S(f) + S(g) \quad **$$

Da $(*) + (**)$, tenuto conto che $s(f) = S(f) = \int_a^b f$ e $S(g) = s(g) = \int_a^b g$

si ottiene $s(f+g) = S(f+g) = \int_a^b f+g = \int_a^b f + \int_a^b g$

$\boxed{\text{cvd}}$

Oss: $\phi : [a,b] \to \mathbb{R}$ integrabile $\Rightarrow \phi^2$ integrabile

(non fatto a lezione)

Suppongo $\phi > 0$ (il caso $\phi \lesseqgtr 0$ conviene spezzarlo in ϕ^+ e ϕ^-)

in tal caso
$$\sup f^2([x_k, x_{k+1}]) = [\sup f(\ )]^2$$
$$\inf f^2([x_k, x_{k+1}]) = [\inf f(\ )]^2$$

e dunque $S(\phi^2,\mathcal{A}) - s(\phi^2,\mathcal{A}) = \sum_{k=0}^{n-1} (x_{k+1} - x_k) \cdot \left\{ \left(\sup \phi([x_k, x_{k+1}]) \right)^2 - \left(\inf \phi([x_k, x_{k+1}]) \right)^2 \right\}$

$$\leq \left\{ \sum_{k=0}^{n-1} (x_{k+1} - x_k) \cdot \left\{ \sup \phi([x_k, x_{k+1}]) - \inf \phi([x_k, x_{k+1}]) \right\} \right\} \cdot 2 \sup \phi([a,b])$$

$$= \left\{ S(\phi,\mathcal{A}) - s(\phi,\mathcal{A}) \right\} \cdot \underbrace{2 \sup \phi([a,b])}_{K}$$

e quindi

$$\forall \varepsilon > 0 \ \exists \mathcal{A}_\varepsilon : \ S(\phi^2,\mathcal{A}_\varepsilon) - s(\phi^2,\mathcal{A}_\varepsilon) \leq \underbrace{\left(S(\phi,\mathcal{A}_\varepsilon) - s(\phi,\mathcal{A}_\varepsilon) \right) \cdot K}_{< K \cdot \varepsilon}$$

$\boxed{\text{cvd}}$

Teorema

$f, g : [a,b] \to \mathbb{R}$ limitate e R-integrabili

allora 1) $f \cdot g$ è R-integrabile

2) $\int_a^b f \cdot g \neq \int_a^b f \cdot \int_a^b g$ in generale

dim (cenno) (non richiesta all'esame)

$$f \cdot g = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$$

$f+g$ e $f-g$ sono integrabili per il Teorema preced.

dovete provare che se ϕ è integrabile allora ϕ^2 integ.

$\boxed{///}$

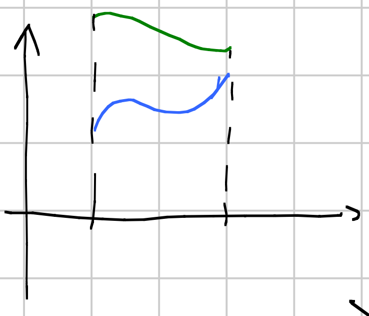
Teorema (confronto)

$f, g : [a,b] \to \mathbb{R}$ limitate e integrabili

$f(x) \leq g(x) \quad \forall x \in [a,b]$

$$\Rightarrow \quad \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

dim



$$s(f, A) \leq s(g, A) \qquad \forall A$$
$$S(f, A) \leq S(g, A)$$

$$\Rightarrow \quad s(f) \leq s(g)$$
$$S(f) \leq S(g)$$

$$\Rightarrow \quad \int_a^b f \leq \int_a^b g$$

$\boxed{///}$

Teorema

$f:[a,b]\to\mathbb{R}$ limitata integrabile

i) $f^+(x)=\max\{f,0\}$ è integrabile

ii) $f^-(x)=\max\{-f,0\}$ " "

iii) $|f|(x)=f^+(x)+f^-(x)$ è integrabile

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx$$

dim (cenno) (non richiesta all'esame)

i) vorrei provare che, essendo per ipotesi

$\forall\varepsilon>0\ \exists\mathcal{P}_\varepsilon\qquad S(f,\mathcal{P}_\varepsilon)-s(f,\mathcal{P}_\varepsilon)<\varepsilon$

si ottiene

$\forall\varepsilon>0\ \exists\mathcal{P}_\varepsilon\qquad S(f^+,\mathcal{P}_\varepsilon)-s(f^+,\mathcal{P}_\varepsilon)<\varepsilon$

$f=f^+\geq0$ il Thm. è vero

Devo ragionare sul caso $f\gtreqless0$

$\sup f^+([x_k,x_{k+1}])-\inf f^+([x_k,x_{k+1}])\le\sup f([x_k,x_{k+1}])-\inf f(\ldots)$

infatti, $-\inf f^+([x_k,x_{k+1}])<-\inf f([x_k,x_{k+1}])$

ii) equivale a i)

iii) $\displaystyle\int_a^b|f| = \int_a^b(f^++f^-)dx \overset{\text{Thm. linearità}}{=} \int_a^b f^+ + \int_a^b f^-$ e quindi

$|f|$ è integrabile

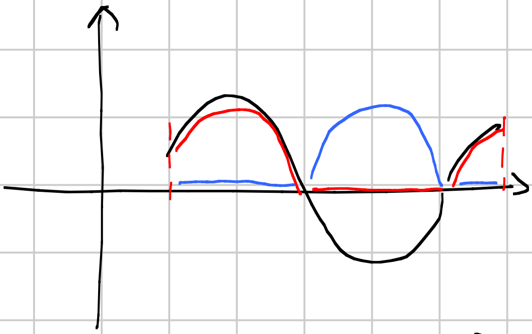
$$-(f^+(x)+f^-(x))\le f=f^+(x)-f^-(x)\le f^+(x)+f^-(x)=|f(x)|$$

$\Downarrow$ Thm. confronto

$$-\int_a^b |f|\,dx \le \int_a^b f\,dx \le \int_a^b |f|\,dx$$

$$\Downarrow$$

$$\left|\int_a^b f\,dx\right| \le \left|\int_a^b |f|\,dx\right| = \int_a^b |f|\,dx \quad \blacksquare$$



$f^+(x) = \max\{f(x), 0\}$

$f^-(x) = \max\{-f, 0\}$

$$f(x) = f^+(x) - f^-(x)$$

$$|f(x)| = f^+(x) + f^-(x)$$

$$f^+(x) = \frac{|f(x)| + f(x)}{2}$$

$$f^-(x) = \frac{|f(x)| - f(x)}{2}$$

Pb $f : [a,b] \to \mathbb{R}$ $|f|$ sia R-integrabile

$\overset{?}{\Longrightarrow}$ f è R-integrabile ??

NO

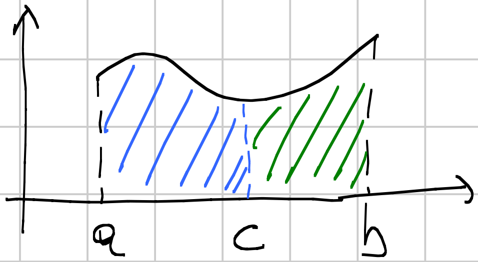
Prese $f(x) = \begin{cases} 5 & \text{se } x \in \mathbb{Q} \\ -5 & \text{se } x \notin \mathbb{Q} \end{cases}$

$|f(x)| = 5$ costante $\Rightarrow$ $|f|$ è integrabile su $[a,b]$ $\forall a,b \in \mathbb{R}$

ma

$f(x)$ NON è integrabile su $[a,b]$ $\forall a,b \in \mathbb{R}$

(vedi esempio di ieri)



$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Teorema (di spezzamento)

1) $f:[a,b] \to \mathbb{R}$ f integrabile su $[a,b]$

$\forall c \in [a,b]$ f è integrabile su $[a,c]$ e su $[c,b]$

e si ha $\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$

2) $\forall c \in [a,b]$ f è integrabile su $[a,c]$ e su $[c,b]$

allora f è integrabile su $[a,b]$ e si ha

$$\int_a^b f(x)dx = \int_a^c f\,dx + \int_c^b f\,dx$$

dim (non richiesta all'esame) (omessa!)

Teorema (della Media Integrale)

$f:[a,b] \to \mathbb{R}$ limitata

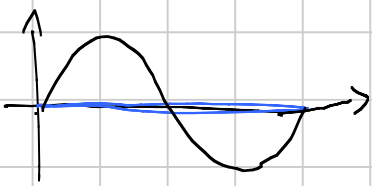
1) f integrabile $\Rightarrow \inf f([a,b]) \le \frac{1}{b-a}\int_a^b f(x)dx \le \sup f([a,b])$

2) f continua su $[a,b] \Rightarrow \exists z \in [a,b] : f(z) = \frac{1}{b-a}\int_a^b f(x)dx$

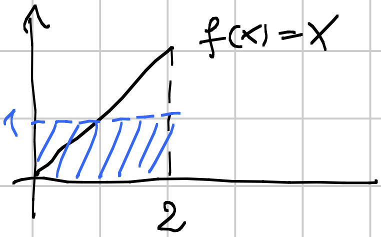
$$\frac{1}{b-a}\int_a^b f(x)dx = \text{media integrale di } f \text{ su } [a,b]$$

Oss: $a_1, \dots, a_n$ $\quad \text{media}\{a_1, \dots, a_n\} = \frac{\sum_{k=1}^{n} a_k}{n}$

Oss:



$$\frac{1}{2\pi}\int_0^{2\pi} \sin x\,dx = 0$$



$$\frac{1}{2-0}\int_0^2 x\,dx = \frac{4/2}{2} = 1$$

Il rettangolo altezza 1 base (2-0) ha = valore a $\int_0^2 x\,dx$!!

dim

1) $\inf f([a,b]) \leq f(x) \leq \sup f([a,b]) \quad \forall x \in [a,b]$

$\Downarrow$ Th. confronto

$$\int_a^b \inf f([a,b])\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b \sup f([a,b])$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$(b-a)\cdot \inf f([a,b]) \qquad\qquad (b-a)\sup f([a,b])$$

$$\inf f([a,b]) \leq \frac{1}{b-a}\int_a^b f\,dx \leq \sup f([a,b])$$

2) f continua $\Rightarrow$ f è integrabile e

$\inf f([a,b]) = \min f([a,b])$

$\sup f([a,b]) = \max f([a,b])$

dal punto 1) si ottiene

$$\min f([a,b]) \leq \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{numero reale}} \leq \max f([a,b])$$

$\Rightarrow$ $\exists z \in [a,b] : f(z) = \dfrac{1}{b-a}\displaystyle\int_a^b f\,dx$

(Th. Valori Intermedi)

$\blacksquare$