$$\int_1^{+\infty} \frac{1}{x^\alpha (\log x)^\beta}\,dx$$ condizioni su α, β per la convergenza

dim

$$\int_1^2 \frac{dx}{x^\alpha(\log x)^\beta} \quad e \quad \int_2^{+\infty} \frac{dx}{x^\alpha(\log x)^\beta}$$

1° caso $\displaystyle\int_1^2 \frac{dx}{x^\alpha(\log x)^\beta}$ $\quad x \to 1$

$$f(x) = \frac{1}{x^\alpha(\log x)^\beta} \sim \frac{1}{(\log x)^\beta} \qquad x \to 1$$

$$\int_1^2 \frac{dx}{(\log x)^\beta} \underset{\substack{y=\log x\\ x=e^y}}{=} \int_0^{\log 2} \frac{e^y\,dy}{y^\beta}$$

$$\frac{e^y}{y^\beta} \sim \frac{1}{y^\beta} \qquad y \to 0$$

$$\int_0^{\log 2} \frac{dy}{y^\beta} \in \mathbb{R} \quad \text{se} \quad \beta < 1$$

Dunque $\displaystyle\int_1^2 \frac{dx}{x^\alpha(\log x)^\beta} \in \mathbb{R}$ se $\beta < 1$ $\forall\alpha$

2° caso $\displaystyle\int_2^{+\infty} \frac{dx}{x^\alpha(\log x)^\beta} \underset{\substack{y=\log x\\ x=e^y\\ dx=e^y dy}}{=} \int_{\log 2}^{+\infty} \frac{e^y\,dy}{e^{\alpha y}\cdot y^\beta}$

$$= \int_{\log 2}^{+\infty} \frac{dy}{(e^y)^{\alpha-1}\cdot y^\beta}$$

$$\begin{cases} \alpha > 1 & \text{l'integrale converge } \forall\beta \\ \alpha = 1 & \text{" " } \beta > 1 \\ \alpha < 1 & \text{" diverge} \end{cases}$$

$\displaystyle\int_2^{+\infty} \frac{dy}{y^\alpha(\log y)^\beta}$ converge se $\{\alpha > 1, \forall\beta\}$ o $\{\alpha = 1, \beta > 1\}$

$$\int_1^{+\infty} \frac{dy}{y^\alpha (\log y)^\beta} \text{ converge } \underline{\text{sse}} \quad \alpha,\beta \in \{\beta<1, \forall\alpha\} \cap (\{\alpha>1, \forall\beta\} \cup \{\alpha=1, \beta>1\})$$

$\underline{\text{sse}}$ $\quad \alpha > 1$ e $\beta < 1$

Esercizio

Studiare la convergenza di $\displaystyle\int_1^{+\infty} \frac{x-1}{\sqrt{x}\,(x+1)\log x}\,dx$

dim

Devo studiare separatamente $\displaystyle\int_1^2 f(x)$ e $\displaystyle\int_2^{+\infty} f(x)$

1° caso $\displaystyle\int_1^2 f(x)$

$$f(x) = \frac{x-1}{\sqrt{x}\,(x+1)\log x} \sim \frac{x-1}{1\cdot 2\cdot \log(1+(x-1))} \sim \frac{1}{2} \quad x \to 1$$

Dunque $f(x)$ è continua su $(1,2]$ e limitata su $[1,2]$ (f si può estendere a $\tilde f(x)$

$$\tilde f(x) = \begin{cases} f(x) & 1 < x \le 2 \\ \frac{1}{2} & x = 1 \end{cases}$$

che è continua)

Dunque $\displaystyle\int_1^2 f(x) \in \mathbb{R}$

$\underline{\text{2° caso}}$ $\displaystyle\int_2^{+\infty} f(x)\,dx$

$$f(x) = \frac{x-1}{\sqrt{x}\,(x+1)\log x} \sim \frac{x}{\sqrt{x}\cdot x\cdot \log x} = \frac{1}{x^{\frac{1}{2}}\cdot \log x}$$

Ma $\displaystyle\int_2^{+\infty} \frac{1}{x^{\frac{1}{2}}\log x}\,dx = +\infty$

quindi $\int_1^{+\infty} f(x) = +\infty$

OSS: $\int_2^{+\infty} \frac{dx}{\sqrt{x}\log x} \underset{\substack{y=\log x \\ x=e^y \\ dx=e^y dy}}{=} \int_{\log 2}^{+\infty} \frac{e^y dy}{e^{y/2}\cdot y} = \int_{\log 2}^{+\infty} \frac{e^{y/2}}{y} dy = +\infty$

in quanto $e^{y/2} = \left(e^{y/6}\right)^3 \geq \left(1+\frac{y}{6}\right)^3 \geq \frac{y^3}{6^3}$

Esercizio

Determinare il raggio di convergenza e la somma delle seguenti serie di potenze

i) $\sum_{n=1}^{\infty} \frac{x^{2n-1}}{(n-1)!}$ iii) $\sum_{n=1}^{\infty} n x^n$

ii) $\sum_{n=1}^{\infty} \frac{x^{n+1}}{n^2+n}$

dim

i) $\sum_{n=1}^{\infty} \frac{x^{2n-1}}{(n-1)!} \underset{n-1=h}{=} \sum_{h=0}^{\infty} \frac{x^{2(h+1)-1}}{h!} = \sum_{h=0}^{\infty} \frac{x^{2h+1}}{h!}$

$= x \cdot \sum_{h=0}^{\infty} \frac{(x^2)^h}{h!} = x \cdot \sum_{h=0}^{\infty} a_h \cdot (x^2)^h \qquad a_h = \frac{1}{h!}$

$R = \frac{1}{\lim_{h\to+\infty} \sqrt[h]{\frac{1}{h!}}} = \frac{1}{0^+} = +\infty$

$\frac{\frac{1}{(h+1)!}}{\frac{1}{h!}} = \frac{h!}{(h+1)!} = \frac{1}{h+1} \xrightarrow[h\to+\infty]{} 0 \Rightarrow \sqrt[h]{\frac{1}{h!}} \xrightarrow[h\to+\infty]{} 0$

$x \sum_{h=0}^{\infty} \frac{(x^2)^h}{h!} = x \cdot e^{x^2} = f(x) \qquad \sum_{h=0}^{\infty} \frac{y^h}{h!} = e^y$

ii) $\sum_{n=1}^{\infty} \frac{x^{n+1}}{n^2+n} = x \sum_{n=1}^{\infty} \frac{x^n}{n^2+n} = x \cdot \sum_{n=1} a_n \cdot x^n$

$a_n = \frac{1}{n^2+n}$

$\lim_{n\to+\infty} \sqrt[n]{a_n} = \lim_{n\to+\infty} \sqrt[n]{\frac{1}{n^2+n}} = \lim_{n\to+\infty} \left(\sqrt[n]{\frac{1}{n}}\right)^2 \cdot \lim_{n\to+\infty} \sqrt[n]{\frac{1}{1+\frac{1}{n}}}$

$= 1$

$$\Rightarrow R = \frac{1}{\lim_{n\to+\infty} \sqrt[n]{a_n}} = 1 \Rightarrow$$ la serie converge per $x \in (-1,1)$ e diverge per $|x| > 1$

Devo studiare cosa succede sul bordo, ovvero in $x=1$ e $x=-1$

$\sum_{n=1}^{\infty} \frac{x^{n+1}}{n^2+n}$ quando $x = \pm 1$ converge assolutamente

$$\sum_{n=1}^{\infty} \frac{|\pm 1|^{n+1}}{n^2+n} = \sum_{n=1}^{\infty} \frac{1}{n^2+n} \in \mathbb{R} \left(\text{poiché } \frac{1}{n^2+n} \sim \frac{1}{n^2} \text{ e } \sum \tfrac{1}{n^2} \in \mathbb{R} \right)$$

la serie converge puntualmente e unif. su $[-1,1]$

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{n^2+n} = f(x) = \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot(2+1)} + \frac{x^4}{3\cdot(3+1)} + \dots$$

$$f(0) = 0$$

$$f'(x) = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$f'(0) = 0$$

$$f''(x) = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\begin{cases} f'' = \frac{1}{1-x} \\ f(0) = f'(0) = 0 \end{cases} \Rightarrow \begin{cases} f'(x) = -\log|1-x| + c \\ f(0) = f'(0) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} f'(x) = -\log(1-x) \\ f(0) = 0 \end{cases}$$

OSS: $x \in (-1,1) \Rightarrow 1-x > 0$!

$$-\int \log(1-x)\,dx = -\left\{ x \log(1-x) - \int x \cdot \frac{1}{1-x} \cdot (-1)\,dx \right\}$$

$$= -x\log(1-x) - \int \frac{x-1+1}{1-x}\,dx$$

$$= -x\log(1-x) - \int \frac{x-1}{1-x}\,dx - \int \frac{dx}{1-x}$$

$$= -x\log(1-x) + x + \log(1-x) + c \quad = f(x)$$

ma $f(0) = 0 \quad \Rightarrow c = 0$

$$\Rightarrow \quad f(x) = (1-x)\log(1-x) + x$$

iii) $\displaystyle\sum_{n=1}^{\infty} n x^n = \sum_{n=1}^{\infty} a_n x^n \qquad a_n = n$

$$\sqrt[n]{a_n} = \sqrt[n]{n} \xrightarrow[n\to+\infty]{} 1 \Rightarrow R = \frac{1}{\lim_n \sqrt[n]{a_n}} = 1$$

$\Rightarrow$ la serie converge quando $|x|<1$

" " diverge " $|x|>1$

Inoltre la serie NON converge quando $|x|=1$ infatti

$$a_n \cdot (-1)^n = n\cdot(-1)^n \not\xrightarrow[n\to+\infty]{} \quad \Rightarrow \text{la serie non converge}$$

$$a_n \cdot 1^n = n \xrightarrow[n\to+\infty]{} +\infty \Rightarrow \text{la serie diverge}$$

$$f(x) = \sum_{n=1}^{\infty} n\cdot x^n = x + 2x^2 + 3\cdot x^3 + \ldots$$

$$\int f(x) = \frac{x^2}{2} + \cancel{\frac{2x^3}{3} + \ldots} = \sum_{n=1}^{\infty} \frac{n}{n+1} x^{n+1}$$

non funziona

$$f(x) = \sum_{n=1}^{\infty} n x^n = x \sum_{n=1}^{\infty} n x^{n-1} \qquad x \neq 0$$

$$\int \frac{f(x)}{x} = \int \sum_{n=1}^{\infty} n x^{n-1} = \int \left(1 + 2\cdot x + 3x^2 + \ldots\right)$$

$$= x + x^2 + x^3 + x^4 + \ldots$$

$$\int \frac{f(x)}{x}\,dx = \sum_{n=1}^{\infty} x^n + c$$

$$= \sum_{n=0}^{\infty} x^n - 1 + c$$

$$= \frac{1}{1-x} - 1 + c$$

$$\frac{f(x)}{x} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$$

$$f(x) = \frac{x}{(1-x)^2}$$

Altrimenti

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Longrightarrow \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1 + x + x^2 + \dots\right)$$

$$= 1 + 2x + 3x^2 + \dots$$

$$= \sum_{n=1}^{\infty} n x^{n-1}$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \Longrightarrow \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n$$

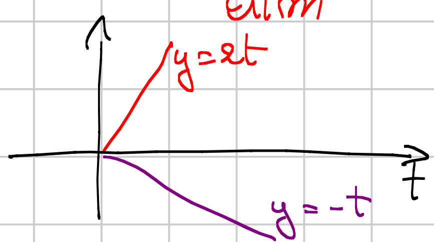
Esercizio

Determinare l'eq. differenziale di cui sono soluzioni le seguenti famiglie di curve

i) $y = c \cdot t \qquad t > 0 \qquad c \in \mathbb{R}$

ii) $y = \dfrac{1}{t^2 + c^2} \qquad c \in \mathbb{R}$

dim

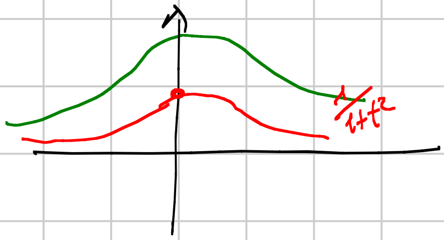
i) 

$$y' = \frac{d}{dt}(c \cdot t) = c = \frac{y}{t}$$

$$\boxed{y' = \frac{y}{t}}$$

$$y = \frac{1}{t^2+c^2}$$

ii)



$$y' = \frac{dy}{dt} = \frac{-2t}{(t^2+c^2)^2}$$

$$y' = -2t \cdot \frac{1}{(t^2+c^2)^2} = -2t \cdot y^2$$

$$\boxed{y' = -2t\, y^2}$$

Esercizio

Determinare l'integrale generale di

$$y'' + 2y' - 3y = e^x + e^{3x}$$

dim

1° passo: Int. generale omogenea associata

$$y'' + 2y' - 3y = 0$$

Eq. caratteristica $\lambda^2 + 2\lambda - 3 = 0$

$$(\lambda+3)(\lambda-1) = 0$$

$$\lambda_1 = -3 \qquad \lambda_2 = 1$$

$$\downarrow \qquad\qquad \downarrow$$

$$e^{-3x} \qquad\qquad e^x$$

$$y_0(x) = c_1 e^{-3x} + c_2 e^x \qquad c_1, c_2 \in \mathbb{R}$$

2° parte $f(x) = e^x + e^{3x}$ e conviene det.

due sol. particolari: una relativa a e^x

" " " e^{3x}

$$y'' + 2y' - 3y = e^{3x}$$

e^{3x} non è soluz. dell'omog. associata

$$\Downarrow$$

posso cercare una sol. partic.

della forma $\boxed{k \cdot e^{3x}}$

$$v = k e^{3x}$$

$$v' = 3k e^{3x}$$

$$v'' = 9k e^{3x}$$

$$9k e^{3x} + 2(3k e^{3x}) - 3k e^{3x} = e^{3x}$$

$12k = 1 \qquad k = 1/12$ $\qquad \boxed{y_{p_1} = \frac{e^{3x}}{12}}$

Cerco sol. di

$$y'' + 2y' - 3y = e^{x}$$

Attenzione e^x è soluz. dell'omog. associata

prendo $v(x) = x \cdot (k e^{x})$ e impongo sia sol.

$v' = ke^{x} + kxe^{x}$

$v'' = ke^{x} + ke^{x} + kxe^{x}$

$$2ke^{x} + \cancel{kxe^{x}} + 2(ke^{x} + \cancel{kxe^{x}}) - \cancel{3kxe^{x}} = e^{x}$$

$$4k\, e^{x} = e^{x} \Rightarrow k = 1/4 \qquad \boxed{y_{p_2} = \frac{1}{4} x e^{x}}$$

$$y_g(x) = y_0(x) + y_{p_1}(x) + y_{p_2}(x) =$$

$$= c_1 e^{-3x} + c_2 e^{x} + \frac{e^{3x}}{12} + \frac{1}{4} x e^{x} \qquad c_1, c_2 \in \mathbb{R}$$

Esercizio

Calcolare integrale generale di

$$y''' + 3y'' = x + e^{-3x}$$

dim

Omogenea Associata $\quad y''' + 3y'' = 0$

Equazione caratteristica è $\quad \lambda^3 + 3\lambda^2 = 0$

$$\lambda^2(\lambda + 3) = 0$$

$\lambda = 0$ con molt. 2 $\quad \boxed{1} = e^{0 \cdot x}$, $\quad x \cdot 1 = \boxed{x}$

$\lambda = -3$ " " 1 $\longrightarrow \boxed{e^{-3x}}$

$$y_o(x) = c_1 \cdot 1 + c_2 \cdot x + c_3 e^{-3x} \qquad c_1, c_2, c_3 \in \mathbb{R}$$

Sol. particolare relative a x $\qquad y''' + 3y'' = x$

Attenzione : x è soluzione dell'omogenea

lo vorrei cercare una sol. del tipo "polinomio 1° grado"

ovvero $\cancel{(ax+b)\,x = v(x)}$

$$\cancel{2ax + b = v'(x)}$$
$$\cancel{2a = v''(x)}$$
$$\cancel{0 = v'''(x)}$$

$$\cancel{0 + (2a) \cdot 3 = x} \qquad \text{IMPOSSIBILE}$$

$$v(x) = (ax+b) \cdot x^{2} \qquad \left(\text{N.B. } \lambda = 0 \text{ ha molt. } 2 \,!!!\right)$$
$$v' = 3ax^2 + 2bx$$
$$v'' = 6ax + 2b$$
$$v''' = 6a$$

sostituendo

$$6a + 3(6ax + 2b) = x$$

$$\begin{cases} 6a + b \cdot 6 = 0 \\ 18a = 1 \end{cases} \qquad \begin{cases} a = 1/18 \\ b = -1/18 \end{cases}$$

$$\boxed{y_{p_1} = \frac{1}{18}x^3 - \frac{1}{18}x^2}$$

Cerco sol. particolare di: $\quad y''' + 3y'' = e^{-3x}$

Attenzione : e^{-3x} è soluzione dell'omogenea

$$v(x) = \left(ke^{-3x}\right) \cdot x^{1} \qquad \left(\text{N.B. } \lambda = -3 \text{ ha molt. } 1 \,!!!\right)$$
$$v' = ke^{-3x} - 3ke^{-3x} \cdot x$$
$$v'' = -3ke^{-3x} - 3ke^{-3x} + 9kxe^{-3x}$$
$$v''' = 18ke^{-3x} + 9ke^{-3x} - 27kxe^{-3x}$$

$$27ke^{-3x} - 27k\cancel{xe^{-3x}} + 3\left(-6ke^{-3x} + \cancel{9kxe^{-3x}}\right) = e^{-3x}$$

$$9k = 1 \qquad k = 1/9 \qquad \boxed{y_{P_2}(x) = \frac{1}{9}xe^{-3x}}$$

$$y_g(x) = c_1 + c_2x + c_3e^{-3x} + \frac{1}{18}x^3 - \frac{1}{18}x^2 + \frac{1}{9}xe^{-3x}$$