2015-05-28-tutoraggio giovedì 10,30÷12,30

Esercizio

Calcolare, se esiste, $\lim_{x\to+\infty} \frac{1}{x}\int_2^x \frac{dt}{\log(t)}$

dim

$\frac{1}{x} \xrightarrow[x\to+\infty]{} 0^+$ assodato

$\int_2^x \frac{dt}{\log t} \xrightarrow[x\to+\infty]{} ?$ Il limite esiste? SI poiché $\frac{1}{\log t} > 0$ $\forall t > 2$, e quindi $f(x) = \int_2^x \frac{dt}{\log t}$ è strettamente crescente

E' finito o $+\infty$?

$$\log x = \log(1+(x-1)) = x-1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

Non trovo $f(x)$ polinomiale t.c. $f(x) \sim \log(x)$ $x\to+\infty$

Però

$$0 \leq \log(x) = \log(1+(x-1)) \leq x-1 \quad \forall x > 2$$

$$\Downarrow$$

$$\frac{1}{\log x} \geq \frac{1}{x-1} \quad \forall x > 2$$

$$\frac{1}{\log t} \geq \frac{1}{t-1} \quad \forall t > 2$$

$\Downarrow$ Teorema Confronto

$$\int_2^x \frac{dt}{\log t} \geq \int_2^x \frac{dt}{t-1} \quad \forall x > 2$$

$$\overset{\shortparallel}{\int_1^{x-1} \frac{dy}{y}} = \log(x-1)$$

$$\Rightarrow \lim_{x\to+\infty} \int_2^x \frac{dt}{\log t} \geq \lim_{x\to+\infty} \log(x-1) = +\infty$$

Ora possiamo applicare il Teorema dell'Hôpital

$$\lim_{x\to+\infty} \frac{\int_2^x \frac{dt}{\log t}}{x} = \lim_{x\to+\infty} \frac{\frac{1}{\log x}}{1} = 0^+$$

Oss: Osservato che $\frac{1}{\log t} > 0 \quad \forall t > 2$,

si ha che $F(x) = \int_2^x \frac{dt}{\log t}$ è strettamente

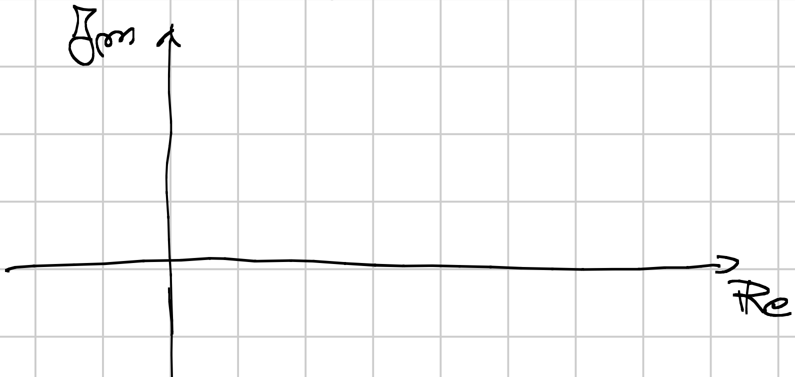
crescente e dunque $\exists \lim_{x \to +\infty} \int_2^x \frac{dt}{\log t} = \ell \in]0,+\infty]$

1 caso) $\ell \in]0,+\infty[\Rightarrow \lim_{x \to +\infty} \frac{\int_2^x \frac{dt}{\log t}}{x} = \frac{\ell}{+\infty} = 0^+$

2 caso) $\ell = +\infty \overset{\text{Hopital}}{\Rightarrow} \lim_{x \to +\infty} \frac{---}{---} \overset{H}{=} \lim_{x \to +\infty} \frac{\frac{1}{\log x}}{1} = 0^+$

dunque $\lim_{x \to +\infty} \frac{1}{x} \int_2^x \frac{dt}{\log t} = 0^+$ ////

$$e^x \qquad \begin{aligned} &e^0 = 1 \\ &e^{x+y} = e^x \cdot e^y \\ &e^x \nearrow \\ &e^x \geq 1+x \end{aligned} \qquad \boxed{x \in \mathbb{R}}$$



$$e^{x+iy} = e^x \cdot e^{iy} := e^x \cdot (\cos y + i \sin y)$$

Definito $e^z = e^{\operatorname{Re} z} \cdot (\cos(\operatorname{Im} z) + i \sin(\operatorname{Im} z))$

Osservo che, quando $z = x \in \mathbb{R}$ (ovvero $y=0$) ritrovo $e^z = e^x$

Osservo che $e^0 = 1$

Osservo che $e^z \in \mathbb{C}$, ovvero non è una funzione a valori reali ovvero $e^z : \underset{\substack{\uparrow \\ 2\text{ var.}}}{\mathbb{C}} \longrightarrow \underset{2\text{ var.}}{\mathbb{C}}$

Osservo che $e^{z+w} = e^z \cdot e^w$

infatti $z = a+ib \qquad w = c+id$

$$e^{a+ib+c+id} = e^{a+ib} \cdot e^{c+id}$$

$$\cancel{e}^{a+c} \cdot \left[\cos(b+d) + i \,\text{sen}(b+d)\right] =$$

$$= e^a e^c \cdot \left[\cos b + i \,\text{sen}\, b\right]\left[\cos d + i \,\text{sen}\, d\right]$$

$$\cos(b+d) + i\,\text{sen}\,(b+d) \overset{?}{=} (\cos b + i\,\text{sen}\, b)(\cos d + i\,\text{sen}\, d)$$

$$\overset{?}{=} \cos b \cos d - \text{sen}\, b\, \text{sen}\, d + i(\text{sen}\, b \cos d + \text{sen}\, d \cos b)$$

$$\Rightarrow e^{z+w} = e^z \cdot e^w$$

Osservo $e^{z+2\pi i \cdot k} = e^z \quad \forall k \in \mathbb{Z}$

(verifico banale)

$$\text{sen}\, x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - - - = \sum_{k=0}^{+\infty} (-1)^k \cdot \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + - - - = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$\text{sen}(ix) = ix - i^3 \frac{x^3}{3!} + i^5 \frac{x^5}{5!} - i^7 \frac{x^7}{7!} + i^9 \frac{x^9}{9!} + - -$$

$$= ix + i\frac{x^3}{3!} + i\frac{x^5}{5!} + i\frac{x^7}{7!} + i\frac{x^9}{9!} - - -$$

$$= i \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos(ix) = 1 - i^2\frac{x^2}{2!} + i^4\frac{x^4}{4!} - i^6\frac{x^6}{6!} + i^8\frac{x^8}{8!} + \cdots$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + - -$$

$$= \sum_{k=0}^{+\infty} \frac{x^{2k}}{(2k)!}$$

$$\frac{d}{dx}\cos(ix) = -\operatorname{sen}(ix)\cdot i = -i\operatorname{sen}(ix)$$

$$\operatorname{sen}(ix) = i\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!} \qquad -i\cdot\operatorname{sen}(ix) = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}$$

$$\frac{d}{dx}\cos(ix) = \frac{d}{dx}\sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!} =$$

$$= \frac{d}{dx}\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\right)$$

$$= x+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}$$

$$= -i\operatorname{sen}(ix)$$

$$\cos ix - i\operatorname{sen}(ix) = e^x$$

$$\sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}+\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

$$e^x = e^{i(-ix)} \overset{\text{Def}}{=} \cos(-ix)+i\operatorname{sen}(-ix)$$

$$= \cos(ix) - i\operatorname{sen}(ix)$$

OSSERVAZIONE

Dalla definizione $e^{x+iy} = e^x(\cos y + i\operatorname{sen} y)$ segue che

$$e^{iy} = \cos y + i\operatorname{sen} y$$

ovvero

$$\left.\begin{aligned} e^{iy} &= \cos y + i\operatorname{sen} y \\ e^{-iy} &= \cos y - i\operatorname{sen} y \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \begin{cases} \cos y = \dfrac{e^{iy}+e^{-iy}}{2} \\ \operatorname{sen} y = \dfrac{e^{iy}-e^{-iy}}{2i} \end{cases}$$

OSS

$$\cos iy = \frac{e^{i(iy)} + e^{-i(iy)}}{2} = \boxed{\frac{e^{-y}+e^{y}}{2}}$$

$$\operatorname{sen} iy = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y}-e^{y}}{2i} \overset{!}{=} - \boxed{\frac{e^{y}-e^{-y}}{2i}}$$

Definizione (f.ni iperboliche)

$\operatorname{senh} x$ = seno iperbolico di x $= \dfrac{e^{x}-e^{-x}}{2}$

$\cosh x$ = coseno iperbolico di x $= \dfrac{e^{x}+e^{-x}}{2}$

$\operatorname{Tgh} x$ = tangente iperbolico di x $= \dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = \dfrac{\operatorname{senh} x}{\cosh x}$

OSSERVAZIONE

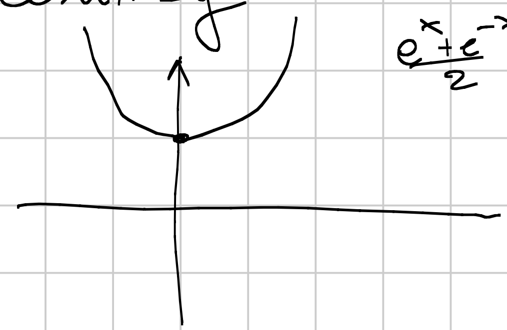
1) senh, cosh, Tgh sono definiti $\forall x \in \mathbb{R}$

2) sono illimitate

3) $-\operatorname{senh}^2 x + \cosh^2 x = 1$

4) $\operatorname{senh} x$ è invertibile su tutto $\mathbb{R}$

5) $\cosh x = y$



$\dfrac{e^{x}+e^{-x}}{2} = \cosh x$ è pari

e $-\cosh(x) = y$ è la "catenaria"

6) $\operatorname{senh}(x)$ è dispari $\cosh x$ è pari

Vediamo la relazione che intercorre tra

$\operatorname{sen} x$ e $\operatorname{senh} x$

$\cos x$ e $\cosh x$

$$\operatorname{sen} x = \frac{e^{ix}-e^{-ix}}{2i} = \frac{1}{i}\left[\frac{e^{ix}-e^{-ix}}{2}\right] = \frac{1}{i}\operatorname{senh}(ix)$$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2} = \cosh(ix)$$

$$1 = \operatorname{sen}^2 x + \cos^2 x = \left(\frac{1}{i}\operatorname{senh}(ix)\right)^2 + \left(\cosh(ix)\right)^2$$
$$= -\operatorname{senh}^2(ix) + \cosh^2(ix)$$

adesso

$$\operatorname{sen}(ix) = \frac{e^{i(ix)}-e^{-i(ix)}}{2i} = \frac{e^{-x}-e^{x}}{2i}$$
$$= -\frac{1}{i}\,\frac{e^{x}-e^{-x}}{2} = -\frac{1}{i}\operatorname{senh}(x)$$

$$\cos(ix) = \frac{e^{i(ix)}+e^{-i(ix)}}{2} = \frac{e^{-x}+e^{x}}{2} = \cosh(x)$$

$$\Downarrow$$

$$\begin{cases} \operatorname{senh}(x) = -i\operatorname{sen}(ix) \\ \cosh(x) = \cos(ix) \end{cases}$$

ora, sapendo che $\cosh^2(x) - \operatorname{senh}^2(x) = 1$

ne deduco che $\cos^2(ix) - \left(-i\operatorname{sen}(ix)\right)^2 = 1$

ovvero $\cos^2(ix) - \left(-\operatorname{sen}^2(ix)\right) = 1$

ovvero $\cos^2(ix) + \operatorname{sen}^2(ix) = 1$

OSS: $\operatorname{senh}(ix + 2k\pi i) = \operatorname{senh}(ix)$ $\quad \forall x \in \mathbb{R} \quad \forall k \in \mathbb{Z}$

infatti $\operatorname{senh}(ix) = i\operatorname{sen} x$, ed essendo $\operatorname{sen} x$

periodica, lo è pure $\operatorname{senh}(ix)$

Analogamente $\cosh(ix)$ è periodica

Riassumendo

$$e^{x+iy} \underset{\text{Def}}{:=} e^x(\cos y + i \operatorname{sen} y)$$

$$e^z e^w = e^{z+w}$$

$$e^{z+2\pi i k} = e^z \quad \forall k \in \mathbb{Z} \quad \forall z \in \mathbb{R}$$

$$\begin{cases} e^{iy} = \cos y + i \operatorname{sen} y \\ e^{-iy} = \cos y - i \operatorname{sen} y \end{cases} \Rightarrow \begin{cases} \operatorname{sen} y = \dfrac{e^{iy} - e^{-iy}}{2i} \\ \cos y = \dfrac{e^{iy} + e^{-iy}}{2} \end{cases} \quad \forall y \in \mathbb{R}$$

$$\begin{cases} \operatorname{sen} y = \displaystyle\sum_{k=0}^{\infty} (-1)^k \cdot \frac{y^{2k+1}}{(2k+1)!} \\ \cos y = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!} \end{cases} \Rightarrow \begin{cases} \operatorname{sen}(iy) = i \displaystyle\sum_{k=0}^{\infty} \frac{y^{2k+1}}{(2k+1)!} \\ \cos(iy) = \displaystyle\sum_{k=0}^{\infty} \frac{y^{2k}}{(2k)!} \end{cases}$$

$$\Rightarrow \quad e^y = \cos(iy) - i \operatorname{sen}(iy)$$

Definiamo

$$\begin{cases} \operatorname{senh}(x) \underset{\text{Def}}{:=} \dfrac{e^x - e^{-x}}{2} \\ \cosh(x) \underset{\text{Def}}{:=} \dfrac{e^x + e^{-x}}{2} \end{cases} \Rightarrow \begin{cases} \operatorname{senh}(x) + \cosh(x) = e^x \\ \cosh^2(x) - \operatorname{senh}^2(x) = 1 \end{cases}$$

$$\begin{cases} \operatorname{senh}(iy) = \dfrac{e^{iy} - e^{-iy}}{2} = i \operatorname{sen}(y) \\ \cosh(iy) = \dfrac{e^{iy} + e^{-iy}}{2} = \cos(y) \end{cases} \Leftrightarrow \begin{cases} \operatorname{senh}(iy) = i \operatorname{sen} y \\ \cosh(iy) = \cos y \end{cases}$$

$$\begin{cases} \operatorname{sen}(iy) = \dfrac{e^{i(iy)} - e^{-i(iy)}}{2i} = -\dfrac{1}{i} \cdot \operatorname{senh}(y) \\ \cos(iy) = \dfrac{e^{i(iy)} + e^{-i(iy)}}{2} = \cosh(y) \end{cases} \Leftrightarrow \begin{cases} \operatorname{senh}(y) = i \operatorname{sen}(iy) \\ \cosh(y) = \cos(iy) \end{cases}$$

Pb f funzione invertibile e continua e derivabile
F primitiva di f

calcolare in modo esplicito $\int f^{-1}(x)\,dx$

dim

$$\int f^{-1}(x)\,dx \underset{\text{per parti}}{=} \int 1\cdot f^{-1}(x)\,dx$$

$$= x\cdot f^{-1}(x) - \int x\cdot \left(f^{-1}(x)\right)'\,dx$$

$$= x\,f^{-1}(x) - \int x\cdot \frac{1}{f'(f^{-1}(x))}\,dx$$

$$= x\,f^{-1}(x) - \left(\int f(t)\cdot dt\right)_{t=f^{-1}(x)}$$

$f^{-1}(x)=t$

$(f^{-1}(x))'\,dx = dt$

$$\frac{dx}{f'(f^{-1}(x))} = dt$$

$$= x\,f^{-1}(x) - \left(F(t)+c\right)_{t=f^{-1}(x)}$$

$$= x\,f^{-1}(x) - F(f^{-1}(x)) + c$$

$$\int f^{-1}(x)\,dx = x\,f^{-1}(x) - F(f^{-1}(x)) + c \quad c\in\mathbb{R}$$

$$\left(x f^{-1}(x) - F(f^{-1}(x)) + c\right)' = \cancel{x\left(f^{-1}(x)\right)'} + f^{-1}(x) - \cancel{f(f^{-1}(x))\cdot\left(f^{-1}(x)\right)'}$$