2015-05-19 - Lezione 12 martedì 8,30 ÷ 10,30 1

ESERCITAZIONI

17) Data la funzione $F(x) = \int_x^{x+1} \frac{1}{t^2+t+1}\,dt \quad ,\forall\, x \in R$

a) disegnare un grafico approssimativo di F

b) determinare al variare di $k \in R$ il numero di soluzioni dell'equazione $F(x) = k$.

Oss: $f(t)=\frac{1}{t^2+t+1}$ è una funzione razionale, e dunque posso rappresentare una sua $\forall$ primitive in termini di f.ni elementari

DOMINIO DI F(x)

$$F(x) = \int_x^{x+1} \frac{dt}{t^2+t+1} = \int_x^{x+1} \frac{1}{(t^2+t+\frac{1}{4})-\frac{1}{4}+1}\,dt = \int_x^{x+1} \frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}}$$

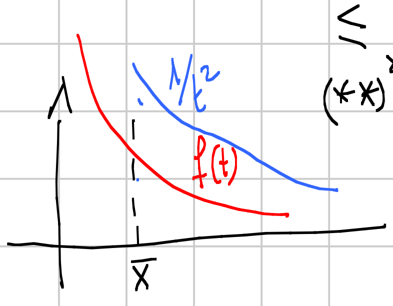
$f(t) = \frac{1}{(t+\frac{1}{2})^2+\frac{3}{4}}$ è definita, continua e limitata $\forall t \in \mathbb{R}$

$$\Rightarrow \int_x^{x+1} f(t)\,dt \in \mathbb{R} \quad \forall x \in \mathbb{R} \quad \Rightarrow \quad \text{dominio di } F(x) \equiv \mathbb{R}$$

LIMITI estremi del dominio

$$\lim_{x\to+\infty} \int_x^{x+1} \frac{dt}{t^2+t+1} = \lim_{x\to+\infty} \int_x^{x+1} \frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}} \underset{(*)}{\overset{x>0}{\leq}} \lim_{x\to+\infty} \int_x^{x+1} \frac{dt}{t^2}$$

$$\underset{(**)}{\leq} \lim_{x\to+\infty} \frac{1}{x^2}\cdot(\cancel{x}+1-\cancel{x}) = 0$$



oppure

$$\lim_{x\to+\infty} \int_x^{x+1} f(t)\,dt = \lim_{x\to+\infty} \left(\int_0^{x+1} f(t)\,dt - \int_0^{x} f(t)\,dt \right) \overset{(***)}{=} \ \boxed{\int_0^{+\infty} f(t)\,dt \in \mathbb{R}}$$

$$= \lim_{x\to+\infty} \int_0^{x+1} f(t)\,dt - \lim_{x\to+\infty}\int_0^{x} f(t)\,dt = \int_0^{+\infty} f(t)\,dt - \int_0^{+\infty} f(t)\,dt = 0$$

(***) $f(t) = \dfrac{1}{t^2+t+1} \sim \dfrac{1}{t^2} \quad t\to+\infty$ ma $\displaystyle\int_1^{+\infty} \frac{1}{t^2}\,dt \in \mathbb{R}$ $\Bigg\}$ th. confronto asintotico $\Rightarrow \displaystyle\int_1^{+\infty} f(t)\,dt \in \mathbb{R}$

ma $\displaystyle\int_0^{1} f(t)\,dt \in \mathbb{R}$

$$\Rightarrow \int_0^{+\infty} f(t)\,dt \in \mathbb{R}$$

$\displaystyle\lim_{x\to-\infty} \int_x^{x+1} f(t)\,dt = 0$ si procede come prima

REGIONI DI MONOTONIA

$$F(x) = \int_x^{x+1} f(t)\,dt = \int_0^{x+1} f(t)\,dt - \int_0^{x} f(t)\,dt$$

$$F'(x) = \left(\int_0^{x+1} f(t)\,dt\right)' - \left(\int_0^{x} f(t)\,dt\right)' = f(x+1)\cdot\frac{d(x+1)}{dx} - f(x)$$

$$= f(x+1) - f(x) = \frac{1}{\left(x+\frac{3}{2}\right)^2+\frac{3}{4}} - \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{x+\frac{1}{4}-3x-\frac{9}{4}}{\left(\left(x+\frac{3}{2}\right)^2+\frac{3}{4}\right)\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)}$$

$$= \frac{-2x-2}{(\quad)(\quad)} = 0 \quad \underline{\text{se}} \quad x=-1$$

$$F'(x) \begin{cases} >0 & x<-1 \\ <0 & x>-1 \end{cases} \Rightarrow F(x) \begin{cases} \nearrow & x<-1 \\ \searrow & x>-1 \end{cases}$$

$x=-1$ p.to di max assoluto

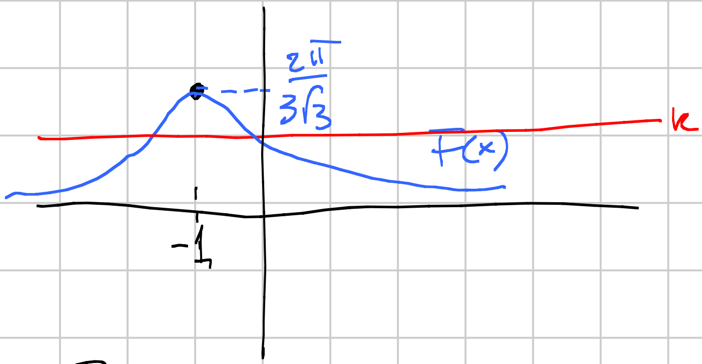
$$F(-1) = \int_{-1}^{0} \frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{4}{3}\int_{-1}^{0} \frac{dt}{1+\left[\frac{2}{\sqrt{3}}\cdot\left(t+\frac{1}{2}\right)\right]^2}$$

$$y=\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)$$

$$dy=\frac{2}{\sqrt{3}}dt$$

$$=\frac{4}{3}\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\frac{\frac{\sqrt{3}}{2}dy}{1+y^2}=\frac{2}{\sqrt{3}}\left[\arctan y\right]_{y=-1/\sqrt{3}}^{y=1/\sqrt{3}}$$

$$=\frac{2}{\sqrt{3}}\left(\frac{\pi}{6}+\frac{\pi}{6}\right)=\frac{2\pi}{3\sqrt{3}}=F(-1)$$



$F(x)=k$ NON HA SOLUZIONI REALI se $k\le 0$

$F(x)=k$ HA 2 soluzioni REALI se $0<k<\frac{2\pi}{3\sqrt{3}}$

$F(x)=k$ HA 1 sol. REALI se $\frac{2\pi}{3\sqrt{3}}\le k$

$F(x)=k$ NON HA SOL. REALI se $\frac{2\pi}{3\sqrt{3}}<k$

$$F((-\infty,-1))=\left(0,\frac{2\pi}{3\sqrt{3}}\right)\qquad F((-1,+\infty))=\left(0,\frac{2\pi}{3\sqrt{3}}\right)$$

$k\in\left(0,\frac{2\pi}{3\sqrt{3}}\right)\Rightarrow\{F^{-1}(k)\}$ è formato da 2 el.ti,

$\{F^{-1}\left(\frac{2\pi}{3\sqrt{3}}\right)\}$ " " " 1 "

$k\notin\left(0,\frac{2\pi}{3\sqrt{3}}\right]\Rightarrow\{F^{-1}(k)\}=\emptyset$

Esercizio studiare, al variare di $\alpha \in \mathbb{R}$, la convergenza del seguente integrale improprio

g) $$\int_0^{+\infty} \frac{1}{x \cdot (x^2 + x^{-2})^{\alpha/2}} dx$$

$$f(t) = \frac{1}{t\left(t^2 + \frac{1}{t^2}\right)^{\alpha/2}} \qquad \begin{cases} f > 0 \quad \forall t \in (0,+\infty) \\ f \text{ continua su } (0,+\infty) \\ \alpha < -2 \quad \lim_{t \to +\infty} f(t) = +\infty \end{cases}$$

$f > 0$ su $(0,+\infty)$ $\forall \alpha \Rightarrow \int_0^{+\infty} f(t)dt$ esiste finito o infinito

$$\int_{\boxed{0}}^{\boxed{+\infty}} f(t)dt = \int_0^1 f(t)dt + \int_1^{+\infty} f(t)dt$$

Il mio integrale converge se entrambi gli integrali convergono

1 caso $\int_0^1 f(t)dt$

$$f(t) = \frac{1}{t\left(t^2 + \frac{1}{t^2}\right)^{\alpha/2}} = \frac{1}{t\left(\frac{t^4}{t^2} + \frac{1}{t^2}\right)^{\alpha/2}} = \frac{1}{t^{1-\alpha}(t^4+1)}$$

$f(t) \sim \frac{1}{t^{1-\alpha}}$ $t \to 0^+$

$\int_0^1 \frac{dt}{t^{1-\alpha}} \in \mathbb{R}$ se $1-\alpha < 1$ se $\alpha > 0$

Th. Confronto $\Rightarrow$ integrali impropri

$\int_0^1 f(t)dt$ converge se $\alpha > 0$

2° caso $\int_1^{+\infty} f(t)\,dt$

$$f(t)=\frac{1}{t^{1-\alpha}(t^4+1)^{\alpha/2}} \sim \frac{1}{t^{1-\alpha}\cdot t^{2\alpha}} = \frac{1}{t^{1+\alpha}} \quad t\to+\infty$$

$\int_1^{+\infty}\frac{dt}{t^{1+\alpha}}\in\mathbb{R}$ se $1+\alpha>1$ se $\alpha>0$ $\Rightarrow$ Teorema Confronto

$\Rightarrow \int_1^{+\infty} f(t)\,dt\in\mathbb{R}$ se $\alpha>0$

Dunque $\int_0^{+\infty} f(t)\,dt\in\mathbb{R}$ se $\alpha>0$

Esercizio Studiare, al variare di $\alpha\in\mathbb{R}$, la convergenza dell'integrale improprio

d) $\int_2^{+\infty}\frac{dx}{\sqrt{e^x-e^2}\cdot(x-2)^\alpha}$;

dim

$f(t)=\frac{1}{\sqrt{e^t-e^2}\cdot(t-2)^\alpha}$ è ben definita, continua, positiva $\forall t>2$

$\forall\alpha\in\mathbb{R}$ esistono, finiti o infiniti, $\int_2^3 f(t)\,dt$ e $\int_3^{+\infty} f(t)\,dt$

1) Studio $\int_2^3 f(t)\,dt$ $\qquad e^y=1+y+o(y)\quad y\to0$

$$f(t)=\frac{1}{\sqrt{e^t-e^2}\cdot(t-2)^\alpha}=\frac{1}{e\sqrt{e^{t-2}-1}\cdot(t-2)^\alpha}$$

$$= \frac{1}{e\sqrt{\cancel{1}+(t-2)+o(t-2)-\cancel{1}}} \cdot \frac{1}{(t-2)^\alpha} \qquad t \to 2^+$$

$$= \frac{1}{e \cdot \sqrt{(t-2)+o(t-2)} \cdot (t-2)^\alpha} \sim \frac{1}{e\,(t-2)^{1/2}\,(t-2)^\alpha} \qquad t\to 2^+$$

$$= \frac{1}{e\,(t-2)^{1/2+\alpha}}$$

$$\int_2^3 \frac{1}{e\,(t-2)^{1/2+\alpha}}\,dt \underset{t-2=y}{=} \int_0^1 \frac{1}{e\,y^{1/2+\alpha}}\,dy \in \mathbb{R} \text{ se } \frac{1}{2}+\alpha<1$$

se $\alpha < \frac{1}{2}$

$$f(t) \sim \frac{1}{e\,(t-2)^{\alpha+1/2}} \quad t\to 2^+ \longrightarrow \text{(teo. confronto asintotico)}$$

$$\int_2^3 f(t)\,dt \in \mathbb{R} \text{ se } \alpha<\tfrac{1}{2}$$

2) Caso $\int_3^{+\infty} f(t)\,dt$

$$f(t) = \frac{1}{\sqrt{e^t-e^2}\cdot(t-2)^\alpha} \sim \frac{1}{\sqrt{e^t}\cdot t^\alpha} = \frac{1}{e^{t/2}\cdot t^\alpha} \qquad t\to+\infty$$

e dunque se converge $\int_3^{+\infty} \frac{dt}{e^{t/2}\cdot t^\alpha}$ allora converge $\int_3^{+\infty} f(t)\,dt$.

Ma $\forall \alpha \in \mathbb{R}$

$$\lim_{t\to+\infty} \frac{1/(e^{t/2}t^\alpha)}{1/t^2} = \lim_{t\to+\infty} \frac{t^{2-\alpha}}{e^{t/2}} = 0$$

da cui segue che $\exists K>0$ t.c. $\frac{1}{e^{t/2}t^\alpha} \le K\cdot\frac{1}{t^2} \quad \forall t\ge 3$

e dunque, essendo $\int_3^{+\infty} \frac{1}{t^2}\,dt \in \mathbb{R}$, per il Teorema del confronto $\displaystyle\int_3^{+\infty} \frac{dt}{e^{\frac{t}{2}} t^{\alpha}} \in \mathbb{R} \quad \forall \alpha$

Dunque $\displaystyle\int_2^3 f(t)\,dt$ converge $\forall \alpha < \frac{1}{2}$

$\displaystyle\int_3^{+\infty} f(t)\,dt$ " $\forall \alpha$

$$\Longrightarrow \int_2^{+\infty} f(t)\,dt \in \mathbb{R} \quad \forall \alpha < \frac{1}{2} \qquad \square$$

$\square$

Esercizio

Provare che $\displaystyle\int_1^{+\infty} \frac{\cos x}{x^2}\,dx$ converge ①

" " $\displaystyle\int_0^{+\infty} \frac{\operatorname{sen} x}{x}\,dx$ converge ②

'' '' $\int_0^{+\infty} \frac{|\operatorname{sen} x|}{x}\,dx$ diverge ③

dim

① $\int_1^{+\infty} \frac{\cos x}{x^2}\,dx$

$|f(x)| = \left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2}$ $\quad x \geq 1$

ma $\int_1^{+\infty} \frac{dx}{x^2} \in \mathbb{R}$

th. confronto $\Rightarrow \int_1^{+\infty} \frac{\cos x}{x^2}\,dx \in \mathbb{R}$

② $\int_0^{+\infty} \frac{\operatorname{sen} x}{x}\,dx = \underbrace{\int_0^{1} \frac{\operatorname{sen} x}{x}\,dx}_{\in \mathbb{R}} + \int_1^{+\infty} \frac{\operatorname{sen} x}{x}\,dx$

$$\int_1^{y} \frac{\operatorname{sen} x}{x}\,dx = \left[\frac{-\cos x}{x}\right]_{x=1}^{x=y} - \int_1^{y} (-\cos x)\left(-\frac{1}{x^2}\right)dx$$

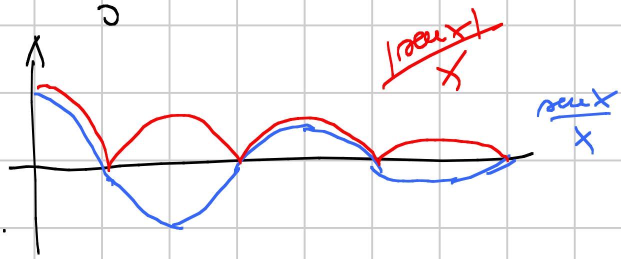
$$= -\frac{\cos y}{y} + \frac{\cos 1}{1} - \int_1^{y} \frac{\cos x}{x^2}\,dx$$

$$\int_1^{+\infty} \frac{\operatorname{sen} x}{x}\,dx = \lim_{y\to+\infty}\left(-\frac{\cos y}{y} + \cos 1 - \int_1^{y} \frac{\cos x}{x^2}\,dx\right)$$

$$= 0 + \cos 1 - \underbrace{\int_1^{+\infty} \frac{\cos x}{x^2}\,dx}_{\in \mathbb{R}}$$

per il punto 1 dell'eserc.

③ $\int_0^{+\infty} \frac{|\operatorname{sen} x|}{x}\,dx = +\infty$



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$$\int_0^{+\infty} \frac{|\operatorname{sen} x|}{x} dx = \int_0^{\pi} \frac{\operatorname{sen} x}{x} dx - \int_{\pi}^{2\pi} \frac{\operatorname{sen} x}{x} + \int_{2\pi}^{3\pi} \frac{\operatorname{sen} x}{x} - \int_{3\pi}^{4\pi} \frac{\operatorname{sen} x}{x} + \dots$$

$$= \sum_{k=0}^{+\infty} (-1)^k \int_{k\pi}^{(k+1)\pi} \frac{\operatorname{sen} x}{x} dx$$

$$\int_0^{\pi} \frac{\operatorname{sen} x}{x} dx \geq \inf_{x \in [0,\pi]} \left(\frac{1}{x}\right) \cdot \int_0^{\pi} \operatorname{sen} x \, dx = \frac{2}{\pi}$$

$$-\int_{\pi}^{2\pi} \frac{\operatorname{sen} x}{x} dx \geq \inf_{[\pi,2\pi]} \left(\frac{1}{x}\right) \cdot \int_{\pi}^{2\pi} \operatorname{sen} x = \frac{2}{2\pi} = \frac{1}{\pi}$$

$$\int_{2\pi}^{3\pi} \frac{\operatorname{sen} x}{x} dx \geq \frac{1}{3\pi} \cdot 2 = \frac{2}{3\pi}$$

$$-\int_{3\pi}^{4\pi} \dots \geq \frac{1}{4\pi} \cdot 2 = \frac{1}{2\pi}$$

— — —

$$\int_{k}^{(k+1)\pi} \dots \geq \frac{2}{(k+1)\pi}$$

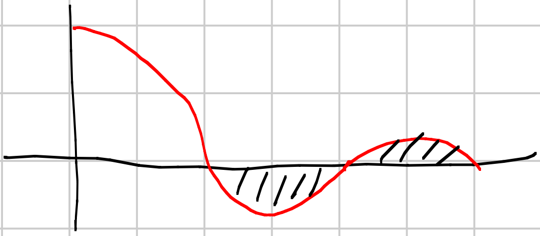
— —

$$\int_0^{+\infty} \frac{|\operatorname{sen} x|}{x} dx = \sum_{k=0}^{\infty} (-1)^k \int_{k\pi}^{(k+1)\pi} \frac{\operatorname{sen} x}{x} dx \geq \sum_{k=0}^{+\infty} \frac{2}{(k+1)\pi}$$

$$\frac{2}{\pi} \sum_{m=1}^{+\infty} \frac{1}{m} = +\infty$$

Oss

$$\int_0^{+\infty} \frac{\operatorname{sen} x}{x} dx = \underbrace{\int_0^{\pi} \frac{\operatorname{sen} x}{x}}_{\vee\ 0} + \underbrace{\int_{\pi}^{2\pi} \frac{\operatorname{sen} x}{x}}_{\wedge\ 0} + \underbrace{\int_{2\pi}^{3\pi} \frac{\operatorname{sen} x}{x}}_{\vee\ 0}$$



$$\left|\int_{k\pi}^{(k+1)\pi} \frac{\operatorname{sen} x}{x}\,dx\right| \underset{y=x-k\pi}{=} \left|\int_0^{\pi} \frac{\operatorname{sen}(y+k\pi)}{y+k\pi}\,dy\right| = \left|\int_0^{\pi} \frac{\operatorname{sen} y}{y+k\pi}\,dy\right|$$

dunque $$\left|\int_{\pi}^{2\pi} \frac{\operatorname{sen} x}{x}\,dx\right| = \left|\int_0^{\pi} \frac{\operatorname{sen} x}{x+\pi}\,dx\right| \le \int_0^{\pi} \frac{\operatorname{sen} x}{x}\,dx$$

$$\left|\int_{2\pi}^{3\pi} \frac{\operatorname{sen} x}{x}\,dx\right| = \left|\int_{\pi}^{2\pi} \frac{\operatorname{sen} x}{x+\pi}\,dx\right| \le \left|\int_{\pi}^{2\pi} \frac{\operatorname{sen} x}{x}\,dx\right|$$

Ne segue che

$$\underbrace{\int_0^{\pi} \frac{\operatorname{sen} x}{x}\,dx}_{= a_0} \ge \underbrace{\left|\int_{\pi}^{2\pi} \frac{\operatorname{sen} x}{x}\,dx\right|}_{= a_1} \ge \underbrace{\left|\int_{2\pi}^{3\pi} \frac{\operatorname{sen} x}{x}\,dx\right|}_{= a_2} \ge \dots$$

ovvero $a_n \downarrow$

Inoltre $$\lim_{n\to+\infty} \left|\int_{n\pi}^{(n+1)\pi} \frac{\operatorname{sen} x}{x}\,dx\right| \le \lim_n \frac{1}{(n+1)\pi} \left|\int_{n\pi}^{(n+1)\pi} \operatorname{sen} x\,dx\right| = 0$$

e dunque

$$\int_0^{+\infty} \frac{\operatorname{sen} x}{x}\,dx = \sum_{n=0}^{\infty} (-1)^n \cdot \left|\int_{n\pi}^{(n+1)\pi} \frac{\operatorname{sen} x}{x}\,dx\right|$$

è convergente per il Criterio di Leibniz.