Esercizio Determinate ordine e p.p. per $x \to \frac{\pi}{2}$ dell'infinitesimo $\operatorname{sen}(\cos x) = f(x)$

dim

$$f\left(\tfrac{\pi}{2}\right) = \operatorname{sen}\left(\cos\tfrac{\pi}{2}\right) = \operatorname{sen}(0) = 0 \qquad f = o\left(1, \tfrac{\pi}{2}\right)$$

1ª strada $\quad P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}\left(\frac{\pi}{2}\right)}{k!} \cdot \left(x - \frac{\pi}{2}\right)$

Calcolate (a caso) $f'(x)$, $f''(x)$ ...
e cercate la prima derivata che non si annulla in $\frac{\pi}{2}$: sia $f^{(\bar{k})}(x)$

ordine $\equiv \bar{k}$ $\qquad$ p.p. $= \frac{f^{(\bar{k})}\left(\frac{\pi}{2}\right)}{\bar{k}!} \cdot \left(x - \frac{\pi}{2}\right)^{\bar{k}}$

2ª via $\quad f(x) = \operatorname{sen}(\cos(x)) = \operatorname{sen}\left(\cos\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right)\right)$

$$f(x) = \operatorname{sen}\left(\cos\left(x-\tfrac{\pi}{2}\right)\cdot \underbrace{\cos\tfrac{\pi}{2}}_{=0} - \operatorname{sen}\left(x - \tfrac{\pi}{2}\right) \underbrace{\operatorname{sen}\tfrac{\pi}{2}}_{=1}\right)$$

$$= \operatorname{sen}\left(-\operatorname{sen}\left(x - \tfrac{\pi}{2}\right)\right)$$

$$= -\operatorname{sen}\left(\operatorname{sen}\left(x - \tfrac{\pi}{2}\right)\right) =: g\left(x - \tfrac{\pi}{2}\right) \quad \text{con } g(y) = -\operatorname{sen}(\operatorname{sen}(y))$$

Sviluppo $g(y) = -\operatorname{sen}(\operatorname{sen}(y))$ per $y \to 0$

$$g(y) = -\operatorname{sen}\left(y + o(y^2)\right) =$$

$$= -\left\{\left(y + o(y^2)\right) + o\left(\left(y + o(y^2)\right)^2\right)\right\}$$

$$= -y + o(y^2) \qquad y \to 0$$

$$\Rightarrow f(x) = -\left(x - \tfrac{\pi}{2}\right) + o\left(\left(x - \tfrac{\pi}{2}\right)^2\right) \qquad x \to \tfrac{\pi}{2}$$

ordine $(f) = 1$ $\qquad$ p.p. $(f) = -\left(x - \frac{\pi}{2}\right)$

Oss:

$f(x) = \operatorname{sen}(\cos(x)) \qquad f' = \cos(\cos(x)) \cdot (-\operatorname{sen} x)$

$$f'\left(\frac{\pi}{2}\right) = \cos\left(\cos\frac{\pi}{2}\right)\cdot\left(-\operatorname{sen}\frac{\pi}{2}\right) = -1$$

$$P_1(x) = -\left(x-\frac{\pi}{2}\right) \qquad \text{fine}$$

Esercizio

Calcolare $\lim\limits_{x\to+\infty} \overbrace{x^2\left(e^{1/x}-e^{\frac{1}{1+x}}\right)}^{f(x)} =$

dim

Se volessi applicare l'Hospital

$$\lim_{x\to+\infty} f(x) = \lim \frac{e^{1/x}-e^{\frac{1}{1+x}}}{1/x^2} \overset{H}{=} \lim \frac{-\frac{1}{x^2}e^{\frac{1}{x}}+\frac{1}{(x+1)^2}e^{\frac{1}{x+1}}}{-2/x^3}$$

non sembra interessante come approccio!

$$e^y = 1+y+\frac{y^2}{2}+o(y^2) \qquad y\to 0$$

$$\Downarrow$$

$$e^{1/x} = 1+\frac{1}{x}+\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right) \qquad x\to+\infty$$

$$e^{\frac{1}{1+x}} = 1+\frac{1}{x+1}+\frac{1}{2(x+1)^2}+o\left(\frac{1}{(x+1)^2}\right)$$

$$e^{1/x}-e^{\frac{1}{x+1}} = \frac{1}{x}-\frac{1}{x+1}+\frac{1}{2}\left(\frac{1}{x^2}-\frac{1}{(x+1)^2}\right)+o\left(\frac{1}{x^2}\right)+o\left(\frac{1}{(x+1)^2}\right)$$

$$= \frac{1}{x(x+1)}+\frac{1}{2}\left(\frac{\cancel{x^2}+2x+1-\cancel{x^2}}{x^2(x+1)^2}\right)+o\left(\frac{1}{x^2}\right)+o\left(\frac{1}{(x+1)^2}\right)$$

$$= \frac{1}{x(x+1)}+o\left(\frac{1}{x^2}\right)$$

infatti $o\left(\frac{1}{(x+1)^2}\right) = o\left(\frac{1}{(x+1)^2}-\frac{1}{x^2}+\frac{1}{x^2}\right)$

$$= o\left(\frac{1}{x^2}-\frac{2x+1}{(x+1)^2\cdot x^2}\right) \qquad (x\to+\infty)$$

$$= o\left(\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)\right) = o\left(\frac{1}{x^2}\right)$$

Dunque

$$\lim x^2\left(e^{1/x}-e^{\frac{1}{x+1}}\right) = \lim_{x\to+\infty} x^2\left(\frac{1}{x(x+1)}+o\left(\frac{1}{x^2}\right)\right)$$

$$= \lim_{x\to+\infty}\left(\frac{x^2}{x(x+1)}+o(1)\right) = 1$$

Oss: Attenzione all'uso indiscriminato dei limiti notevoli

$$\lim_{x\to 0^+}\left(\operatorname{sen} x + f(x)\right) = \lim_{x\to 0^+}\left(\frac{\operatorname{sen} x}{x}\cdot x + f(x)\right) \underset{\text{in generale}}{\neq} \lim_{x\to 0^+}\left(x+f(x)\right)$$

infatti

$$\operatorname{sen} x = x - \frac{x^3}{6} + \frac{x^5}{5!} + o(x^5)$$

$$\lim_{x\to 0^+}\frac{1}{x^4}\left(\operatorname{sen} x - x + \frac{x^3}{6}\right) = \lim_{x\to 0^+}\frac{1}{x^4}\left(\cancel{x} - \cancel{\frac{x^3}{6}} + \frac{x^5}{120} + o(x^5) - \cancel{x} + \cancel{\frac{x^3}{6}}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{x}{120} + o(x)\right) = 0$$

mentre si può essere tentati di

$$\lim_{x\to 0^+}\frac{1}{x^4}\left(\operatorname{sen} x - x + \frac{x^3}{6}\right) =$$

$$= \lim_{x\to 0^+}\frac{1}{x^4}\left(\frac{\operatorname{sen} x}{x}\cdot x - x + \frac{x^3}{6}\right) =$$

sfrutto il fatto che $\frac{\operatorname{sen} x}{x} \xrightarrow[x\to 0]{} 1$

$$\overset{???}{=} \boxed{\lim_{x\to 0^+}\frac{1}{x^4}\left(x - x + \frac{x^3}{6}\right) = \lim_{x\to 0^+}\frac{1}{6x} = +\infty}$$

Questa conclusione è scorretta: siamo passati al limite in $\frac{\operatorname{sen} x}{x}$ senza passare al limite ovunque

L'errore ~~sta~~ nel passare al limite solo in una parte: questo non è corretto!

$$\lim_{x\to 0}\frac{\operatorname{sen} x}{x} = 1 \qquad \text{mentre} \qquad \lim_{x\to 0}\frac{\lim_{x\to 0}\operatorname{sen} x}{x} = \lim_{x\to 0}\frac{0}{x} = 0 \,!$$

Esercizio

Calcolare $\lim_{x\to+\infty} \underbrace{\left(\sqrt{\frac{x^2+3}{x^2}} - \sqrt{\frac{x^3+3x+3}{x^3}}\right)}_{f(x)} x^{\alpha}$

al variare di $\alpha > 0$

dim

$$f(x) = \left(\sqrt{1+3/x^2} - \sqrt{1+\frac{3x+3}{x^3}}\right) \cdot \frac{\sqrt{1+3/x^2} + \sqrt{1+\frac{3x+3}{x^3}}}{\sqrt{1+3/x^2} + \sqrt{1+\frac{3x+3}{x^3}}}$$

$$= \frac{\cancel{1} + \cancel{3/x^2} - \cancel{1} - \cancel{\frac{3}{x^2}} - \frac{3}{x^3}}{\sqrt{1+3/x^2} + \sqrt{1+\frac{3x+3}{x^3}}}.$$

$$= \frac{-3/x^3}{\sqrt{1+3/x^2} + \sqrt{1+\frac{3x+3}{x^3}}} = -\frac{3}{x^3} \cdot \left(\frac{1}{2+o(1)}\right)$$

$$\lim_{x\to+\infty} f(x)\cdot x^{\alpha} = \lim_{x\to+\infty} -\frac{3}{x^{3-\alpha}} \cdot \left(\frac{1}{2+o(1)}\right)$$

$$= \begin{cases} 0^- & \alpha < 3 \\ -3/2 & \alpha = 3 \\ -\infty & \alpha > 3 \end{cases}$$

Esercizio

Calcolare polinomio ordine 4 per $x \to 0$ di f

" ordine e pp. di f per $x \to 0$

$$f(x) = 3e^{x^2} - 2\log(1+x^2) - \frac{x}{1-2x} - 3\cos x + \operatorname{sen} x$$

dim

e^{x^2}: $e^y = 1 + y + \frac{y^2}{2} + o(y^2) \Rightarrow e^{x^2} = 1 + x^2 + \frac{x^4}{2} + o(x^5)$

$\log(1+x^2)$: $\log(1+y) = y - \frac{y^2}{2} + o(y^2) \Rightarrow \log(1+x^2) = x^2 - \frac{x^4}{2} + o(x^5)$

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + y^4 + o(y^4) \Rightarrow x \cdot \frac{1}{1-2x} = x(1 + 2x + 4x^2 + 8x^3 + 16x^4 + o(x^4))$$

$$= x + 2x^2 + 4x^3 + 8x^4 + o(x^4)$$

$$-3\cos x + \operatorname{sen} x = -3\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) + \left(x - \frac{x^3}{6}\right) + o(x^4) = -3 + x + \frac{3}{2}x^2 - \frac{x^3}{6} - \frac{x^4}{8} + o(x^4)$$

$$f(x) = 3\left(\cancel{1} + x^2 + \frac{x^4}{2}\right) - 2\left(x^2 - \frac{x^4}{2}\right) - \left(\cancel{x} + 2x^2 + 4x^3 + 8x^4\right) \cancel{-3} + \cancel{x} + \frac{3}{2}x^2 - \frac{x^3}{6} - \frac{x^4}{8} + o(x^4)$$

$$= x^2\left(3 - 2 - 2 + \frac{3}{2}\right) + x^3\left(-4 - \frac{1}{6}\right) + x^4\left(\frac{3}{2} + 1 - 8 - \frac{1}{8}\right) + o(x^4)$$

$$= \boxed{\frac{x^2}{2}} - \frac{25}{6}x^3 - \frac{-20+64+1}{8}x^4 + o(x^4)$$

$$= \frac{x^2}{2} - \frac{25}{6}x^3 - \frac{45}{8}x^4 + o(x^4)$$

ordine$(f) = 2$ $\quad$ $\mathrm{Pp}(f) = \frac{x^2}{2}$

Esercizio

Determinare le soluzioni di

$$\begin{cases} \left|\frac{w}{2}\right| = |z| = \sqrt{2} \\ 2z + w = 4 \end{cases}$$

dim

$z = x + iy \quad w = a + ib$

$$\begin{cases} |w| = 2|z| = 2\sqrt{2} \\ 2z + w = 4 \end{cases} \quad \begin{cases} x^2+y^2=2 \\ a^2+b^2=8 \\ 2x+2iy+a+ib=4 \end{cases} \Rightarrow \begin{cases} x^2+y^2=2 \\ a^2+b^2=8 \\ 2x+a=4 \\ 2y+b=0 \end{cases}$$

$$\Rightarrow \begin{cases} x^2+y^2=2 \\ (4-2x)^2+4y^2=8 \\ a=4-2x \\ b=-2y \end{cases} \Rightarrow \begin{cases} x^2+y^2=2 \\ (x-2)^2+y^2=2 \\ '' \\ '' \end{cases} \Rightarrow \begin{cases} x^2-(x-2)^2=0 \\ x^2+y^2=2 \\ '' \\ '' \end{cases} \Rightarrow$$

$$\begin{cases} x=1 \\ y=\pm 1 \\ a=2 \\ b=-2y \end{cases} \Rightarrow \begin{matrix} (z,w)=(1+i,\, 2-2i) \\ + \\ (z,w)=(1-i;\, 2+2i) \end{matrix}$$

oppure

$$\begin{cases} \left|\frac{w}{2}\right| = |z| = \sqrt{2} \\ 2z + w = 4 \end{cases} \quad \begin{cases} |2-z| = |z| \\ |z| = \sqrt{2} \\ w = 2(2-z) \end{cases}$$

$$\begin{cases} (2-x)^2+y^2 = x^2+y^2 \\ x^2+y^2 = 2 \\ w=2(2-z) \end{cases} \Leftrightarrow \begin{cases} x=1 \\ y=\pm 1 \\ w=2(2-z) \end{cases} \Leftrightarrow \begin{cases} z=1+i \\ w=2(2-1-i) \\ \quad = 2-2i \end{cases} \text{ or } \begin{cases} z=1-i \\ w=2+2i \end{cases}$$

Esercizio

Calcolare le soluzioni di:

$$\begin{cases} |w|^2+\bar{z}=1-i \\ z+w=1+i \end{cases}$$

dim

il caso $z=x+iy$ $w=a+ib$ per caso

$$\begin{cases} |w|^2+\overline{(1+i-w)}=1-i \\ z=1+i-w \end{cases} \qquad \begin{cases} |w|^2+\cancel{1-i}-\bar{w}=\cancel{1-i} \\ // \end{cases}$$

$$\begin{cases} \bar{w}\,(w-1)=0 \\ // \end{cases} \iff \begin{cases} \bar{w}=0=w \\ z=1+i \end{cases} \quad \text{o} \quad \begin{cases} w=1 \\ z=i \end{cases}$$