Esercizio Calcolare le primitive di
$$f(x) = \cos^2 x$$
dim

1° modo $\int \cos^2 x \, dx = \int \underset{f'}{\cos x} \cdot \underset{g}{\cos x} \, dx$

$= \underset{f}{\operatorname{sen} x} \, \underset{g}{\cos x} - \int \underset{f}{\operatorname{sen} x} \, \underset{g'}{(-\operatorname{sen} x)} \, dx$

$= \operatorname{sen} x \cos x + \int (1 - \cos^2 x) \, dx$

$= \operatorname{sen} x \cos x + x - \int \cos^2 x \, dx$

$\Downarrow$

$2 \int \cos^2 x \, dx = x + \operatorname{sen} x \cos x + c \qquad c \in \mathbb{R}$

$\int \cos^2 x \, dx = \frac{1}{2} (x + \operatorname{sen} x \cos x) + c \qquad c \in \mathbb{R}$

2° modo (ricordo che $\int \operatorname{sen}^2 x \, dx = \frac{1}{2} (x - \operatorname{sen} x \cos x) + c$)

$\int \cos^2 x \, dx = \int (1 - \operatorname{sen}^2 x) dx = \int 1 dx - \int \operatorname{sen}^2 x dx$

$= x - \left(\frac{1}{2} x - \frac{1}{2} \operatorname{sen} x \cos x \right) + c$

$= \underbrace{\frac{1}{2} x + \frac{1}{2} \operatorname{sen} x \cos x + c}_{} \qquad c \in \mathbb{R}$

DA NON FARE (utilizzo $t = \operatorname{tg} \frac{x}{2}$)

$\operatorname{sen} x = \dfrac{2 \operatorname{tg} \frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}} \qquad \cos x = \dfrac{1 - \operatorname{tg}^2 \frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}} \qquad \dfrac{d \operatorname{tg}(\frac{x}{2})}{dx} = \left(1 + \operatorname{tg}^2 \left(\frac{x}{2}\right)\right) \frac{1}{2}$

$x = 2 \operatorname{arctg} t$

$\dfrac{dx}{dt} = \dfrac{2}{1+t^2}$

$$\int \cos^2 x\,dx \underset{t=\operatorname{tg}\frac{x}{2}}{=} \left(\int \left(\frac{1-t^2}{1+t^2}\right)^2 \cdot \frac{2}{1+t^2}\,dt \right)_{t=\operatorname{tg}\left(\frac{x}{2}\right)}$$

$$= \left(\int \frac{(1-t^2)^2}{(1+t^2)^3} \cdot 2\,dt \right)_{t=\operatorname{tg}\frac{x}{2}}$$

Come costruire un esercizio

$$\int \frac{1}{1+t^2}\,dt - \int \frac{1}{1+t}\,dt = \operatorname{arctg} t - \log(1+t)\,dt$$

1° passaggio

$$\int \frac{1+t-1-t^2}{(1+t^2)(1+t)}\,dt = \int \frac{t-t^2}{(1+t)(1+t^2)}\,dt$$

2° passaggio

$$\underset{\substack{t=e^x \\ dt=e^x dx}}{=} \left(\int \frac{e^x - e^{2x}}{(1+e^x)(1+e^{2x})} \cdot e^x\,dx \right)_{t=e^x}$$

Calcolare tutte le primitive di

$$f(x) = \frac{e^{2x} - e^{3x}}{1+e^{2x}+e^x+e^{3x}}$$

Esercizio

Provare che: "Data $f : \mathbb{R} \to \mathbb{R}$ derivabile con derivata continua t.c. $f'(q) = 2 \quad \forall q \in \mathbb{Q}$. Allora $\lim_{x \to +\infty} f = +\infty$"

dim

Se $f'(x) = 2 \ \forall x$ allora $\lim_{x\to+\infty} (f(x) - f(0)) = \lim_{x\to+\infty} \frac{f(x)-f(0)}{x} \cdot x$

$$= \lim_{x\to+\infty} f'(z_x) \cdot x = \lim_{x\to+\infty} 2x = +\infty$$

Ma $\overbrace{\left(f'(q) = 2 \quad \forall q \in \mathbb{Q}\right)}^{\text{per ipotesi}}$ e $\underbrace{\left(\forall x \in \mathbb{R} \ \exists \{q_n\} \subseteq \mathbb{Q} \ q_n \to x\right)}_{\mathbb{Q} \text{ è denso in } \mathbb{R}}$

e $\underbrace{\left(f' \text{ continua}\right)}_{\text{per ipotesi}}$

$$\Rightarrow f'(x) = 2 = \lim_n f'(q_n)$$

$\boxed{///}$

Esercizio Calcolare ordine e p.p. per $x \to 0$ di $f(x) = (\operatorname{sen} 2x^2) \cdot (\sqrt{1+3x} - 1)$ 3

dim

1° metodo $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k$

mi calcolo $f^{(k)}(x)$ fino a trovare $\bar{k}$ t.c.

$f^{(\bar{k})}(0) \neq 0$

allora $f^{(\bar{k})}(0) \cdot \frac{x^{\bar{k}}}{\bar{k}!} = p.p.$

$\bar{k} = \text{ordine}$

2° metodo $f(x) = (\operatorname{sen} 2x^2) \cdot (\sqrt{1+3x} - 1)$

uso sviluppi noti

$\operatorname{sen} y = y + o(y^2) \Rightarrow \boxed{\operatorname{sen}(2x^2) = 2x^2 + o(x^5)}$

$\sqrt{1+y} = (1+y)^{1/2} = 1 + \frac{1}{2}y + o(y) \Rightarrow \boxed{\sqrt{1+3x} = 1 + \frac{1}{2}3x + o(x)}$

$f(x) = (2x^2 + o(x^5))(\cancel{1} + \frac{3}{2}x + o(x) - \cancel{1}) =$

$= 3x^3 + o(x^3)$ ordine$(f) = 3$

$pp(f) = 3x^3$

Problema (DIFFICILE!!)

$\sum_n \frac{1}{n}$ diverge ; $\sum_n \frac{1}{5n}$ diverge ;

$\sum_n \frac{1}{3^{10} \cdot n}$ diverge ; $\sum_n \frac{1}{n \log n}$ diverge

però

$\sum_n 1/n^2$ converge

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

1 4 9 16

Problema $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \dots = \sum_{p \text{ primo}} \frac{1}{p}$

converge o diverge ??

Esercizio Studiare la convergenza di:

$$\sum_n (n - \operatorname{sen} n) \cdot \left(\frac{1}{n} - \operatorname{sen}\frac{1}{n}\right)$$

dim

$$a_n = \underbrace{(n - \operatorname{sen} n)}\underbrace{\left(\frac{1}{n} - \operatorname{sen}\frac{1}{n}\right)} \sim n \cdot \left(\frac{1}{n} - \operatorname{sen}\frac{1}{n}\right)$$

$(n \to +\infty)$

quando $n \to +\infty$, $\frac{1}{n} \to 0$: per $y \to 0$

$$\boxed{\begin{aligned} y - \operatorname{sen} y &= y - \left(y - \frac{y^3}{6} + o(y^4)\right) \\ &= \frac{y^3}{6} + o(y^4) \qquad y \to 0 \end{aligned}}$$

$\Downarrow$

$$\frac{1}{n} - \operatorname{sen}\frac{1}{n} = \frac{1}{6} \cdot \frac{1}{n^3} + o\left(\frac{1}{n^4}\right) \qquad n \to +\infty$$

$\Downarrow$

$$a_n = (n - \operatorname{sen} n)\left(\frac{1}{n} - \operatorname{sen}\frac{1}{n}\right) \sim n \cdot \left(\frac{1}{6}\frac{1}{n^3} + o\left(\frac{1}{n^4}\right)\right)$$

$$a_n \sim \left(\frac{1}{6}\frac{1}{n^2} + o\left(\frac{1}{n^3}\right)\right) \qquad n \to +\infty$$

$$a_n \sim \underbrace{\frac{1}{6} \cdot \frac{1}{n^2}}_{b_n} \qquad n \to +\infty$$

Osservo che $\sum_n b_n = \sum_n \frac{1}{6}\frac{1}{n^2}$ converge
e quindi, per il Criterio Confronto Asintotico

$\sum_n a_n$ converge

Esercizio Per quali valori di x si ha

$$-\frac{1}{x^2}\log(1+x)+\frac{1}{1+x}\cdot\left(\frac{1}{x}+\frac{1}{2}\right)\leq 0$$

dim

$$\frac{1}{1+x}\cdot\frac{2+x}{2x}\leq\frac{1}{x^2}\log(1+x)$$

$$\frac{2+x}{1+x}\cdot\frac{x}{2}\leq\log(1+x)$$

$$f=\log(1+x)-\frac{x}{2}\,\frac{2+x}{1+x}$$

$$\lim_{x\to-1}f(x)=\lim_{x\to-1}\frac{1}{1+x}\left((1+x)\log(1+x)-(2+x)\cdot\frac{x}{2}\right)=$$

$$=\lim_{x\to-1}\frac{1}{1+x}\cdot\frac{1}{2}=+\infty$$

$$\lim_{x\to+\infty}f(x)=-\infty$$

$$f'=\frac{1}{1+x}-\frac{1}{2}\,\frac{2+x}{1+x}-\frac{x}{2}\cdot\frac{1+\cancel{x}-2-\cancel{x}}{(1+x)^2}=$$

$$=\frac{1}{1+x}-\frac{1}{2}\,\frac{2+x}{1+x}+\frac{x}{2}\cdot\frac{1}{(1+x)^2}$$

$$=\frac{\cancel{2(1+x)}-(\cancel{2}+x)(1+x)+x}{2(1+x)^2}=-\frac{x^2}{2\,(1+x)^2}$$

$f(0)=0 \qquad f'\leq 0 \quad \forall x>-1 \implies$

$$f(x)\begin{cases}>0 & -1<x<0\\ <0 & 0<x\end{cases}$$

$\blacksquare$

Esempio

Calcolare $\int \operatorname{arctg} x \, dx$

dire

$$\int 1 \cdot \operatorname{arctg} x \, dx = x \cdot \operatorname{arctg} x - \int x \cdot \frac{1}{1+x^2} dx$$

$$= x \cdot \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$$

$$= x \cdot \operatorname{arctg} x - \frac{1}{2} \log(1+x^2) + c \quad c \in \mathbb{R}$$

In generale

$$\int f^{-1}(x) \, dx = \int 1 \cdot f^{-1}(x) \, dx$$

$$= x \cdot f^{-1}(x) - \int x \cdot \left(f^{-1}\right)'(x) \, dx$$

$$= x \cdot f^{-1}(x) - \int x \cdot \frac{1}{f'(f^{-1}(x))} dx$$

$$\underset{\substack{t = f^{-1}(x) \\ x = f(t) \\ dx = f'(t)dt}}{=} x \cdot f^{-1}(x) - \left(\int f(t) \cdot \frac{1}{\cancel{f'(t)}} \cdot \cancel{f'(t)} dt \right)_{t = f^{-1}(x)}$$

$$= x \cdot f^{-1}(x) - \left(F(t) + c \right)_{t = f^{-1}(x)}$$

$$\int f^{-1}(x) \, dx = x \cdot f^{-1}(x) - F\left(f^{-1}(x)\right) + c \quad c \in \mathbb{R}$$

dove $\boxed{F' = f}$

$f = \operatorname{Tg} x \qquad F = -\log|\cos x|$

$$\int \operatorname{arctg} x = x \cdot \operatorname{arctg} x + \log|\cos(\operatorname{arctg} x)| + c$$

$$= x \cdot \operatorname{arctg} x + \log\left|\sqrt{\frac{1}{1+x^2}}\right| + c$$

$$= x \cdot \operatorname{arctg} - \frac{1}{2} \log(1+x^2) + c \quad c \in \mathbb{R}$$

$$\int f^{-1}(x)\,dx = x\,f^{-1}(x) - \int x \cdot \left(f^{-1}(x)\right)' dx$$

$$= x\,f^{-1}(x) - \int x \frac{1}{f'(f^{-1}(x))} dx \qquad y = f^{-1}(x)$$

$$= x\,f^{-1}(x) - \left(\int f(y) \frac{1}{\cancel{f'(y)}} \cancel{f'}(y)\,dy\right)_{y=f^{-1}(x)}$$

$$= x\,f^{-1}(x) - \left(F(y) + c\right)_{y=f^{-1}(x)}$$

$$= x\,f^{-1}(x) - \left(F \circ f^{-1}\right)(x) + c$$

Verifica $\quad f^{-1} + x\left(f^{-1}(x)\right)' - f\left(f^{-1}(x)\right) \cdot \left(f^{-1}\right)'(x) = f^{-1}(x)\ !$