Esercizio

Sia $a_n = \int_{5-\frac{16}{n^\alpha}}^{5} f(t)\,dt \quad f(t) = \frac{1}{\sqrt{5-t}} + \sqrt{5-t}$

Studiare $\sum_n a_n$ al variare di α

dim

$$a_n = \int_{5-\frac{16}{n^\alpha}}^{5} \left[(5-t)^{-1/2} + (5-t)^{1/2}\right] dt =$$

$5-t=y$ $\quad t = 5-\frac{16}{n^\alpha} \to y = \frac{16}{n^\alpha}$

$dt = -dy$ $\quad t=5 \to y=0$

$$= \int_{16/n^\alpha}^{0} \left[\frac{1}{\sqrt{y}} + \sqrt{y}\right](-1)\,dy = \int_0^{16/n^\alpha} \left(\frac{1}{\sqrt{y}} + \sqrt{y}\right) dy$$

$$= \left[2\sqrt{y} + \frac{2}{3} y^{3/2}\right]_{y=0}^{y=16/n^\alpha} = 2\sqrt{\frac{16}{n^\alpha}} + \frac{2}{3}\left(\frac{16}{n^\alpha}\right)^{3/2}$$

$$= \frac{8}{n^{\alpha/2}} + \frac{128}{3}\cdot\frac{1}{n^{3\alpha/2}}$$

$$\sum_n a_n = \sum_n \left(\frac{8}{n^{\alpha/2}} + \frac{128}{3}\left(\frac{1}{n}\right)^{\frac{3\alpha}{2}}\right)$$

$$= 8\cdot\underbrace{\sum_n \left(\frac{1}{n}\right)^{\alpha/2}}_{①} + \frac{128}{3}\underbrace{\sum_n \left(\frac{1}{n}\right)^{\frac{3\alpha}{2}}}_{②}$$

① converge se $\frac{\alpha}{2} > 1$ se $\alpha > 2$

② " se $\frac{3}{2}\alpha > 1$ se $\alpha > \frac{2}{3}$

Dunque la serie $\sum_n a_n$ converge

se $\alpha > \max\{2, 2/3\} = 2$ $\boxed{\text{VVV}}$

Esercizio

Studiare $\sum_n a_n$ dove $a_n = \int_n^{n+1/n^\alpha} t \arctg t \, dt$

al variare di $\alpha \geq 0$

dim

1ª strada (vedi l'esercizio precedente)

$$\int \underset{u'}{t} \, \underset{f}{\arctg t} \, dt = \underset{u}{\frac{t^2}{2}} \, \underset{f}{\arctg t} - \int \underset{u}{\frac{t^2}{2}} \cdot \underset{f'}{\frac{1}{1+t^2}} \, dt$$

$$= \frac{t^2}{2} \arctg t - \frac{1}{2} \int \frac{t^2+1-1}{t^2+1} dt =$$

$$= \frac{t^2}{2} \arctg t - \frac{1}{2} \int 1 \, dt + \frac{1}{2} \int \frac{dt}{1+t^2}$$

$$= \frac{t^2}{2} \arctg t - \frac{t}{2} + \frac{1}{2} \arctg t + c$$

$$a_n = \left[\frac{t^2}{2} \arctg t - \frac{t}{2} + \frac{1}{2} \arctg t \right]_n^{n+\frac{1}{n^\alpha}}$$

$$= \frac{1}{2}\left(n + \frac{1}{n^\alpha}\right)^2 \arctg\left(n + \frac{1}{n^\alpha}\right) - \frac{1}{2}\left(\cancel{n} + \frac{1}{n^\alpha}\right) + \frac{1}{2} \arctg\left(n + \frac{1}{n^\alpha}\right)$$

$$- \frac{n^2}{2} \arctg n + \frac{1}{2} \cancel{n} - \frac{1}{2} \arctg n$$

$$= \frac{1}{2}\left(n^2 + \frac{1}{n^{2\alpha}} + 2\frac{1}{n^{\alpha-1}}\right) \arctg\left(n + \frac{1}{n^\alpha}\right) + - -$$

questa strada è faticosa!

2° modo

$$a_n = \int_n^{n+\frac{1}{n^\alpha}} \underbrace{t\,\mathrm{arctg}\,t}_{f(t)}\,dt$$

$$f'(t) = \mathrm{arctg}\,t + t\cdot\frac{1}{1+t^2} > 0 \quad \forall t>0 \Rightarrow f \nearrow \ \forall t>0$$

$$\Rightarrow \min f\left(\left[n, n+\frac{1}{n^\alpha}\right]\right) = f(n) = n\,\mathrm{arctg}\,n$$

$$\max f\left(\left[n, n+\frac{1}{n^\alpha}\right]\right) = f\left(n+\frac{1}{n^\alpha}\right) = \left(n+\frac{1}{n^\alpha}\right)\cdot \mathrm{arctg}\left(n+\frac{1}{n^\alpha}\right)$$

$$\text{e } n\,\mathrm{arctg}\,n \le f(x) \le \left(n+\frac{1}{n^\alpha}\right)\mathrm{arctg}\left(n+\frac{1}{n^\alpha}\right) \quad \forall x\in\left[n, n+\frac{1}{n^\alpha}\right]$$

Teorema confronto integrali

$$\Rightarrow \left(\cancel{n}+\frac{1}{n^\alpha}-\cancel{n}\right)\cdot n\,\mathrm{arctg}\,n \le \underbrace{\int_n^{n+1/n^\alpha} t\,\mathrm{arctg}\,t\,dt}_{= a_n} \le \left(\cancel{n}+\frac{1}{n^\alpha}-\cancel{n}\right)\left(n+\frac{1}{n^\alpha}\right)\mathrm{arctg}\left(n+\frac{1}{n^\alpha}\right)$$

$$\forall (n>1) \qquad \frac{1}{n^{\alpha-1}}\cdot\frac{\pi}{4} \qquad\qquad \wedge\!|\quad \frac{1}{n^\alpha}\cdot\left(n+\frac{1}{n^\alpha}\right)\cdot\frac{\pi}{2}$$

$$\Rightarrow b_n = \frac{\pi}{4}\cdot\frac{1}{n^{\alpha-1}} \le a_n \le \frac{1}{n^{\alpha-1}}\cdot\pi = c_n$$

Osservo ora che $a_n \geq 0 \ \forall n$

ed inoltre $\sum_n c_n = \sum_n \pi\cdot\left(\frac{1}{n}\right)^{\alpha-1}$ converge

se $\alpha-1>1$ se $\alpha>2$

Quindi se $\alpha>1$ allora $\sum_n a_n$ converge

Inoltre $\sum_n b_n = \sum_n \frac{\pi}{4}\cdot\left(\frac{1}{n}\right)^{\alpha-1}$ diverge

se $\alpha-1\le 1$ se $\alpha\le 2$

Ne segue che se $\alpha \leq 2$ allora $\sum_n a_n$ diverge

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Quindi riassumendo

$$\sum_n a_n \begin{cases} \text{converge se } \alpha > 2 \\ \text{diverge se } \alpha \leq 2 \end{cases}$$

Esercizio data $f(x) = \int_{\operatorname{sen} x}^{\cos x} e^t\,dt$

1) Calcolare $f'(0)$ (se esiste)

2) calcolare l'equazione della retta Tg in $(0, f(0))$

dim

$$f(x) = \int_{\operatorname{sen} x}^{\cos x} e^t\,dt = \int_{\operatorname{sen} x}^{0} e^t\,dt + \int_{0}^{\cos x} e^t\,dt$$

$$= \int_{0}^{\cos x} e^t\,dt - \int_{0}^{\operatorname{sen} x} e^t\,dt$$

$$= G(\cos x) - G(\operatorname{sen} x)$$

dove $G(y) = \int_0^y e^t\,dt$

$$F'(x) = G'(\cos x)\cdot(\cos x)' - G'(\operatorname{sen} x)\cdot(\operatorname{sen} x)'$$

$$= e^{\cos x}\cdot(-\operatorname{sen} x) - e^{\operatorname{sen} x}\cdot \cos x$$

$$= -\left[e^{\cos x}\operatorname{sen} x + e^{\operatorname{sen} x}\cos x\right]$$

$$F'(0) = -\left[e^1\cdot \emptyset + e^0\cdot 1\right] = -1$$

L'eq. della retta Tg cercata è $\quad y - F(0) = F'(0)\cdot(x-0)$ (*)

ma $F(0) = \int_{\text{sen}(0)}^{\cos(0)} e^t\,dt = \int_0^1 e^t dt = \left[e^t\right]_0^1 = e-1$ $(\bullet)$

$F'(0) = -1$ $(\bullet)$

$$\boxed{y-(e-1) = -x}$$ eq. retta Tg.

2° modo $F(x) = \int_{\text{sen}x}^{\cos x} e^t\,dt = \left[e^t\right]_{t=\text{sen}x}^{t=\cos x} = e^{\cos x} - e^{\text{sen}x}$

$F(0) = e^1 - e^0 = e-1$

$F'(x) = \left(e^{\cos x} - e^{\text{sen}x}\right)' = e^{\cos x}\cdot(-\text{sen}x) - e^{\text{sen}x}\cdot(\cos x)$

$= -\left(e^{\cos x}\cdot \text{sen}x + e^{\text{sen}x}\cos x\right)$

$F'(0) = -\left(e^1\cdot 0 + e^0\cdot 1\right) = -1$

Pb: $f(x) = (\log x)^{\log x}$ $\qquad f'(x) \overset{?}{=}$

$f = e^{\log x \cdot \log(\log x)}$

è definita per x t.c. $\log x > 0$

ovvero per $\boxed{x>1}$!!!

$f' = \left(e^{\log x\cdot\log(\log x)}\right)'$

$= e^{\log x\cdot\log(\log x)}\cdot\left(\frac{1}{x}\log(\log x) - \cancel{\log x}\cdot\frac{1}{\cancel{\log x}}\cdot\frac{1}{x}\right) =$

$$= (\log x)^{\log x} \cdot \frac{1}{x}\left(\log(\log x) - 1\right)$$

$$\left[(\operatorname{sen} x)^{\operatorname{sen} x}\right]' \overset{?}{=}$$

Esercizio per quali α converge $\displaystyle\int_0^{+\infty} \underbrace{\frac{\log(1+x^4)}{x^\alpha}}_{f(x)}\,dx$

dim

$f(x)$ = rapporto tra funzioni continue $\Rightarrow$ f continua $\forall x > 0\ \forall \alpha \geq 0$

$f(x) > 0\ \forall x > 0\ \forall \alpha \geq 0$

Dunque $\displaystyle\int_0^{+\infty} f(x)\,dx$ esiste $\begin{cases} \text{finito} \\ \circ \\ +\infty \end{cases}$

Inoltre dobbiamo studiare $x = 0$ e $x = +\infty$, quindi studieremo $\displaystyle\int_0^{e} f(x)\,dx$ e $\displaystyle\int_e^{+\infty} f(x)\,dx$ (prendiamo e perché ci torna comodo, ma posso prendere un $\forall$ numero $\in\,]0, +\infty[$)

Ⓐ Studio $\displaystyle\int_0^{e} f(x)\,dx$. Dobbiamo studiare f per $x \to 0$

$$f(x) = \frac{\log(1+x^4)}{x^\alpha} = \frac{x^4 + o(x^7)}{x^\alpha} \sim \frac{x^4}{x^\alpha} = \frac{1}{x^{\alpha-4}} \quad x \to 0$$

$$\left(\text{infatti } \lim_{x\to 0^+} \frac{f(x)}{1/x^{\alpha-4}} = \lim_{x\to 0^+} \frac{x^4 + o(x^7)}{x^\alpha} \cdot x^{\alpha-4} = \right.$$

$$\left. = \lim_{x \to 0^+} \frac{1 + o(x^3)}{1} = 1 \right)$$

Ma $\displaystyle\int_0^{e} \frac{1}{x^{\alpha-4}}\,dx \in \mathbb{R}$ sse $\alpha - 4 < 1$ sse $\alpha < 5$

e dunque, per il criterio del confronto asintotico

$\displaystyle\int_0^{e} f(x)\,dx \in \mathbb{R}$ sse $\alpha < 5$

(B) $\int_e^{+\infty} f(x)dx$ Dobbiamo studiare il comportamento asintotico di $f(x)$ per $x \to +\infty$

$$f(x) = \frac{\log(1+x^4)}{x^\alpha} \sim \frac{\log x^4}{x^\alpha} = 4\frac{\log x}{x^\alpha} \text{ per } x \to +\infty$$

1° passo per $\alpha \le 1$ $\int_e^{+\infty} \frac{\log x}{x^\alpha} dx = +\infty$

infatti: $\log x \ge \log e = 1 \ \forall x \ge e \Rightarrow \frac{\log x}{x^\alpha} \ge \frac{1}{x^\alpha} \ \forall x \ge e \ \forall \alpha > 0$

Ma $\int_e^{+\infty} \frac{dx}{x^\alpha} = +\infty \ \forall \alpha \le 1 \Rightarrow$ (Teorema Confronto) $\int_e^{+\infty} \frac{\log x}{x^\alpha} dx = +\infty \ \forall \alpha \le 1$

2° passo $\int_e^{+\infty} \frac{\log x}{x^\alpha} dx \in \mathbb{R} \ \forall \alpha > 1$

Essendo $\alpha > 1$ (0, 1, α)

$$\left.\begin{aligned} &\alpha = 1 + \frac{\alpha-1}{2} + \frac{\alpha-1}{2} \\ &\frac{\alpha-1}{2} > 0 \\ &1 + \frac{\alpha-1}{2} > 1 \end{aligned}\right\}$$

e quindi

$$\frac{\log x}{x^\alpha} = \frac{1}{x^{1+\frac{\alpha-1}{2}}} \cdot \frac{\log x}{x^{\frac{\alpha-1}{2}}} \le \frac{1}{x^{1+\frac{\alpha-1}{2}}}$$

$\forall x \ge M$: si veda (*)

(*) infatti $\lim_{x \to +\infty} \frac{\log x}{x^{\frac{\alpha-1}{2}}} = 0^+ \Rightarrow \overset{\text{fissato}}{\varepsilon = 1} \ \exists M > 0 : \forall x \ge M \quad 0 \le \frac{\log x}{x^{\frac{\alpha-1}{2}}} < 1$

$$\Rightarrow \text{(Teorema confronto)} \int_e^{+\infty} \frac{\log x}{x^\alpha} dx = \int_e^{M} \frac{\log x}{x^\alpha} dx + \int_M^{+\infty} \frac{\log x}{x^\alpha} \le \underbrace{\int_e^{M} \frac{\log x}{x^\alpha} dx}_{\in \mathbb{R}} + \int_M^{+\infty} \frac{dx}{x^{1+\frac{\alpha-1}{2}}}$$

ed essendo $\int_M^{+\infty} \frac{dx}{x^{1+\frac{\alpha-1}{2}}} dx \in \mathbb{R}$ se $1 + \frac{\alpha-1}{2} > 1$

se $\frac{\alpha-1}{2} > 0$

se $\alpha > 1$

Si ha, per il Teorema Confronto,

$$\int_e^{+\infty} \frac{\log x}{x^\alpha} dx \in \mathbb{R} \quad \text{se} \quad \alpha > 1$$

Riassumendo, dai passi 1 e 2 si ha $\int_e^{+\infty} \frac{\log x}{x^\alpha} dx \in \mathbb{R}$ se $\alpha > 1$ e dunque, per il criterio del confronto asintotico,

$$\int_e^{+\infty} f(x) dx \in \mathbb{R} \quad \text{se } \alpha > 1$$

$$(A) + (B) \Rightarrow \int_0^{+\infty} f(x)\,dx \in \mathbb{R} \text{ se } 1<\alpha<5 \quad \boxed{\text{ciò}}$$

Oss: Abbiamo provato che $\int_e^{+\infty} \frac{\log x}{x^\alpha}\,dx \in \mathbb{R}$ se $\alpha>1$

$$\Rightarrow \int_e^{+\infty} \frac{\log x}{x^\alpha}\,dx \underset{\substack{y=\log x\\ x=e^y\\ dx=e^y dy}}{=} \int_1^{+\infty} \frac{y}{e^{\alpha y}}\cdot e^y dy = \int_1^{+\infty} \frac{y}{e^{y(\alpha-1)}}\,dy$$

Ne segue che $\int_1^{+\infty} \frac{y}{e^{y(\alpha-1)}}\,dy \in \mathbb{R}$ se $\alpha>1$

IMPORTANTE

$$\int_1^{+\infty} \frac{\log x}{x^\alpha}\,dx \in \mathbb{R} \text{ se } \alpha>1$$

$$\int_2^{+\infty} \frac{dx}{x^\alpha \log x}\,dx \in \mathbb{R} \text{ se } \alpha>1$$

mentre

$$\int_2^{+\infty} \frac{dx}{x(\log x)^\alpha} \in \mathbb{R} \text{ se } \alpha>1$$

$$\int_{e^2}^{+\infty} \frac{dx}{x \log x\,[\log(\log x)]^2} \in \mathbb{R} \text{ se } \alpha>1$$

$$\int_{e^{e^2}}^{+\infty} \frac{dx}{x \log x\, \log(\log x)\,[\log(\log(\log x))]^\alpha} \in \mathbb{R} \text{ se } \alpha>1$$

Esercizio

Sia $f_\lambda(x) = \lambda x + \frac{1}{x}$ definita per $x > 0$

Sia $x_\lambda = \min f_\lambda(]0,+\infty[)$

1) Calcolare x_λ (se esiste)

2) Calcolare $\int_{1/e x_\lambda}^{2 x_\lambda} f_\lambda(t)\,dt$

Sol.

$f_\lambda = \lambda t + \frac{1}{t} \qquad \lambda > 0$

$\lim_{t \to 0^+} f_\lambda(t) = +\infty$

$\lim_{t \to +\infty} f_\lambda(t) = +\infty$

$\Rightarrow$ per il Corollario di Weierstrass $\exists\ x_\lambda = \min f_\lambda$

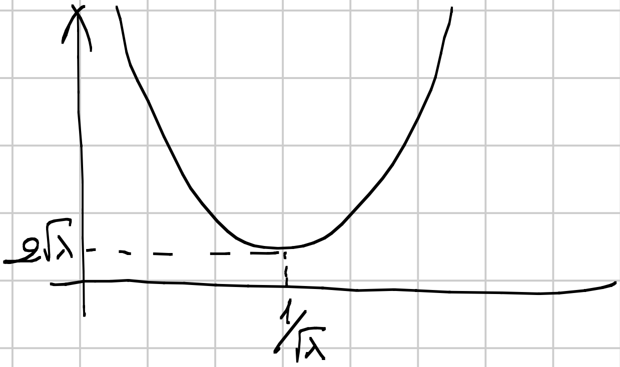
$f'_\lambda = \lambda - \frac{1}{t^2} = 0$ se $t^2 = 1/\lambda$ se $t = \frac{1}{\sqrt{\lambda}}$

($t = -1/\sqrt{\lambda}$ non è accettabile)

essendo x_λ l'unico punto stazionario

in $]0,+\infty[$, e dovendo esistere un minimo, allora necessariamente x_λ è p.to di minimo assoluto

$$f_\lambda(x_\lambda) = \lambda \cdot \frac{1}{\sqrt{\lambda}} + \sqrt{\lambda} = 2\sqrt{\lambda}$$



$$I_\lambda = \int_{1/2x_\lambda}^{2x_\lambda} f_\lambda(t)\,dt = \left[\lambda \frac{t^2}{2} + \log t\right]_{t=\frac{\sqrt{\lambda}}{2}}^{t=2/\sqrt{\lambda}}$$

$$= \frac{\cancel{\lambda}}{\cancel{2}_{1}} \cdot \left(\frac{\cancel{4}^{\,2}}{\cancel{\lambda}}\right) + \log \frac{2}{\sqrt{\lambda}} - \frac{\lambda}{2}\left(\frac{\lambda}{4}\right) - \log \frac{\sqrt{\lambda}}{2}$$

$$= 2 - \frac{\lambda^2}{8} + \log \frac{2}{\sqrt{\lambda}} \cdot \frac{2}{\sqrt{\lambda}}$$

$$= 2 - \frac{\lambda^2}{8} + \log \frac{4}{\lambda} = I_\lambda$$

Si ha $\lim_{\lambda \to 0^+} I_\lambda = +\infty$ $\quad \lim_{\lambda \to +\infty} I_\lambda = -\infty$

e $\frac{d}{d\lambda} I_\lambda = -\frac{\lambda}{4} + \frac{\lambda}{4} \cdot \left(-\frac{4}{\lambda^2}\right) = -\frac{\lambda}{4}\left(\frac{4}{\lambda^2} + 1\right) < 0$

$\forall \lambda > 0$

$\Rightarrow I_\lambda$ strettamente decrescente $\Rightarrow \exists!\, \bar{\lambda} > 0 : I_{\bar{\lambda}} = 0$

e si vede che quando $\bar{\lambda} = 4$ $I_4 = 0$

FINE CORSO

FIN

THE END