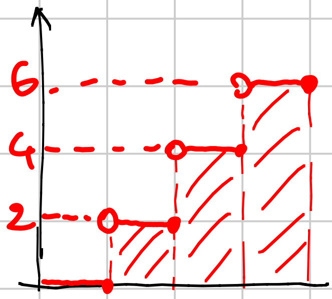
Confronto Serie - Integrale Improprio

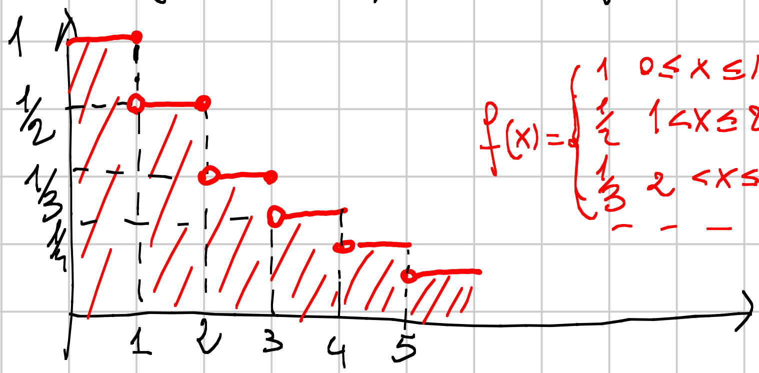
$$f(x) = \begin{cases} 0 & 0 \le x \le 1 \\ 2 & 1 < x \le 2 \\ 4 & 2 < x \le 3 \\ 6 & 3 < x \le 4 \end{cases}$$

$$\int_0^4 f(x)dx = \int_0^1 f + \int_1^2 f + \int_2^3 f + \int_3^4 f$$
$$= 0 + (2-1)\cdot 2 + (3-2)\cdot 4 + (4-3)6$$
$$= 2 + 4 + 6 = 12$$



$$\sum_{k=1}^{4} f(k) = 0 + 2 + 4 + 6 = 12$$

ovvero, quando considero $S_n = \sum_{k=1}^{n} \frac{1}{k}$, sto calcolando l'integrale su $[0,n]$ della funzione



$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ \frac{1}{2} & 1 < x \le 2 \\ \frac{1}{3} & 2 < x \le 3 \\ - & - \end{cases}$$

$$S_n = \int_0^n f(x)\,dx \quad \text{dove } f(x)$$

Questo "suggerisce" che $\sum_{k=1}^{M} 1/k = S_n$ posso essere messo in relazione con $\int_1^M f(x)dx$ con $f(x) = \frac{1}{x}$

Teorema (Confronto Serie-Integrale improprio)

Data $f:[a,+\infty[\ \to [0,+\infty[$ con $a \geq 0$

Sia f debolmente decrescente. Allora

$$\left\{ \sum_{n \geq a} f(n) \text{ converge (diverge) sse } \int_a^{+\infty} f(x)\,dx \text{ converge (diverge)} \right\}$$

dim

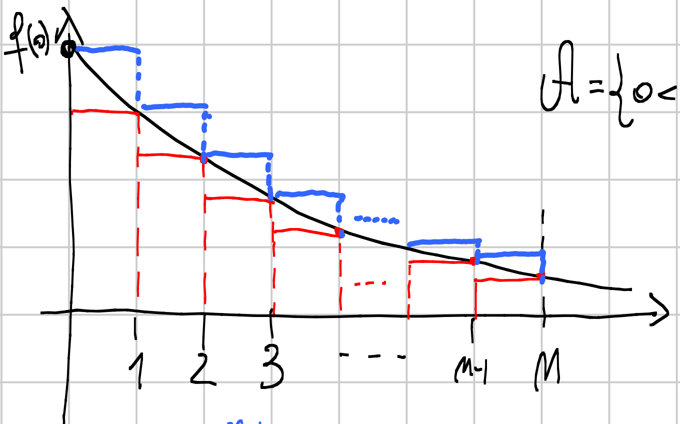
$a = 0$ per semplicità

$$\left\{ \begin{array}{l} f:[0,+\infty[\ \to [0,+\infty[\ = \searrow \text{ (deb. decrescente)} \\ \Rightarrow f \text{ è integrabile su } [0,\beta]\ \forall \beta \geq 0 \end{array} \right.$$

f.ni monotone sono integrabili

$$\left\{ \begin{array}{l} \text{Inoltre } f \geq 0 \text{ per ipotesi} \\ \Rightarrow \exists \lim_{\beta \to +\infty} \int_0^\beta f(t)\,dt \ \begin{cases} +\infty \\ \in \mathbb{R} \end{cases} \end{array} \right.$$

$f \geq 0$ allora $\exists \int_0^{+\infty} f(t)\,dt$



$$A = \{0 < 1 < 2 < \cdots < n\}$$

$$S(f, A_n) = \sum_{k=0}^{n-1} (k+1-k) \cdot \underbrace{\sup f([k,k+1])}_{f(k)} = \sum_{k=0}^{n-1} f(k)$$

$$s(f, A_n) = \sum_{k=0}^{n-1} (k+1-k) \cdot \underbrace{\inf f([k,k+1])}_{f(k+1)} = \sum_{k=0}^{n-1} f(k+1) \overset{h=k+1}{=} \sum_{h=1}^{n} f(h)$$

$$\sum_{k=1}^{n} f(k) \leq \int_0^n f(x)\,dx \leq \sum_{k=0}^{n-1} f(k)$$

passando al limite per $n \to +\infty$ scopro che

$$\sum_{k=1}^{\infty} f(k) \leq \int_0^{+\infty} f(x)\,dx \leq \sum_{k=0}^{\infty} f(k) = \sum_{k=1}^{\infty} f(k) + f(0)$$

Dunque $\int_0^{+\infty} f(x)\,dx$ e $\sum_n f(n)$ hanno lo stesso

comportamento: l'uno converge se converge l'altro
l'uno diverge se diverge l'altro ▨

Esempio Studiare la convergenza di $\sum_n 1/n^\alpha$
al variare di $\alpha \in \mathbb{R}$

dim

$f(n) = \frac{1}{n^\alpha}$ che, quando $\alpha > 0$, risulta una funzione

decrescente e dunque, per il Teorema precedente

$$\int_1^{+\infty} \frac{dx}{x^\alpha} \text{ ha lo stesso carattere di } \sum_n \frac{1}{n^\alpha} \qquad \underline{\underline{\alpha > 0}}$$

e dunque, dato che $\int_1^{+\infty} \frac{dx}{x^\alpha} \begin{cases} = +\infty & \alpha \leq 1 \\ \in \mathbb{R} & \alpha > 1 \end{cases}$

si ha che

$$\sum_n \left(\frac{1}{n}\right)^\alpha \begin{cases} \text{divergente} & 0 < \alpha \leq 1 \\ \text{convergente} & \alpha > 1 \end{cases}$$

Inoltre, quando $\alpha \leq 0$, si ha che $\left(\frac{1}{n}\right)^\alpha \not\to 0$
e dunque $\sum_n \left(\frac{1}{n}\right)^\alpha$ diverge $\forall \alpha \leq 0$

Riassumendo $\sum_n 1/n^\alpha \begin{cases} \text{converge} & \alpha > 1 \\ \text{diverge} & \alpha \leq 1 \end{cases}$ ▨

Esercizio

Studiare la convergenza di $\int_0^{+\infty} \frac{\sqrt{1+x^2}-x}{\sqrt{x}}\,dx$

dim

∘) $f(x)=\frac{\sqrt{1+x^2}-x}{\sqrt{x}} > 0 \quad \forall x>0$

∘) f è definita e continua $\forall x>0$

Quindi devo studiare cosa succede in $x=0$ $x=+\infty$

dunque debbo studiare

$$\underbrace{\int_0^3 f(x)\,dx}_{1} + \underbrace{\int_3^{+\infty} f(x)\,dx}_{2} \qquad \left(\int_0^1 f(x)dx + \int_1^{+\infty} f(x)dx\right)$$

ovvero studio una singolarità per volta

1 $\int_0^3 \frac{\sqrt{1+x^2}-x}{\sqrt{x}}\,dx$ dobbiamo studiare il comportamento di f in $x=0$

|| f è continua $\forall x \in]0,3]$

||| $f(x)=\frac{\sqrt{1+x^2}-x}{\sqrt{x}}\cdot\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}+x}=\frac{1}{\sqrt{x}\,\underbrace{(\sqrt{1+x^2}+x)}_{\to 1,\ x\to 0}} \sim \frac{1}{\sqrt{x}}$

per $x \to 0$

$\left(\text{infatti } \lim_{x\to 0}\frac{f(x)}{1/\sqrt{x}}=1\right)$ e $\frac{1}{\sqrt{x}}$ è continua in $]0,3]$

||| Inoltre $\int_0^3 \frac{1}{\sqrt{x}}dx=\int_0^3\left(\frac{1}{x}\right)^{\frac{1}{2}}dx \in \mathbb{R}$ poiché $\underline{\underline{\alpha=\frac{1}{2}<1}}$

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$\Longrightarrow$ (Teorema Confronto Asintotico Integrali Impropri) $\displaystyle\int_0^3 f(x)\,dx \in \mathbb{R}$

② $\displaystyle\int_3^{+\infty} f(x)\,dx$

||| f è continua $\forall x \geq 3$

che è continua su $[3,+\infty[$

$$||| \quad f(x) = \frac{1}{\sqrt{x}\left(\sqrt{1+x^2}+x\right)} = \frac{1}{x^{3/2}} \cdot \frac{1}{\underbrace{\left(\sqrt{1+\frac{1}{x^2}}+1\right)}_{\downarrow x\to+\infty \; 2}} \sim \frac{1}{2}\cdot\frac{1}{x^{3/2}} \qquad x \to +\infty$$

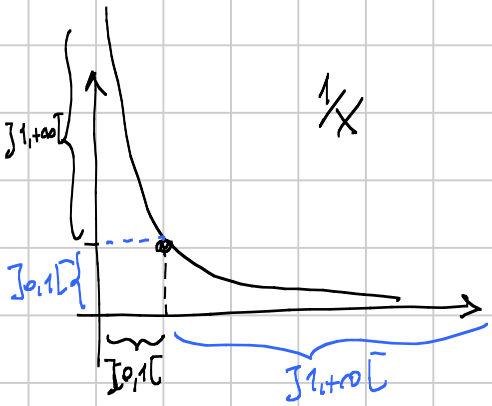
$$||| \quad \int_3^{+\infty} \frac{1}{2x^{3/2}}\,dx = \frac{1}{2}\int_3^{+\infty}\left(\frac{1}{x}\right)^{3/2} dx \in \mathbb{R}$$ in quanto $\underline{\alpha = \frac{3}{2} > 1}$

$\Longrightarrow$ (Teorema Confronto Asintotico Integrali Impropri) $\displaystyle\int_3^{+\infty} f(x)\,dx \in \mathbb{R}$

dunque $\displaystyle\int_0^3 f(x)\,dx + \int_3^{+\infty} f(x)\,dx = \int_0^{+\infty} f(x)\,dx \in \mathbb{R}$

$\boxed{\text{////}}$

Oss;



$$\int_\beta^1 \frac{dt}{t^\alpha} =$$

$\frac{1}{t} = x$; $t = 1/x$; $dt = -\frac{1}{x^2}dx$ $\qquad t=\beta \longrightarrow x = 1/\beta$

$t = 1 \longrightarrow x = 1$

$$= \int_{1/\beta}^{1} x^{\alpha} \cdot \left(-\frac{1}{x^2}\right) \cdot dx = \int_{1}^{1/\beta} \frac{1}{x^{2-\alpha}} dx$$

$$\int_{\beta}^{1} \frac{dt}{t^{\alpha}} = \int_{1}^{1/\beta} \frac{dx}{x^{2-\alpha}}$$

$$\lim_{\beta \to 0} \int_{\beta}^{1} \frac{dt}{t^{\alpha}} = \boxed{\int_{0}^{1} \frac{dt}{t^{\alpha}} = \int_{1}^{+\infty} \frac{dx}{x^{2-\alpha}}} = \lim_{\beta \to 0} \int_{1}^{1/\beta} \frac{dx}{x^{2-\alpha}}$$

quindi convergono se $\boxed{\alpha < 1}$ se $-\alpha > -1$

se $\boxed{2-\alpha > 1}$

Esercizio Sia $I_n = \int_{1/4n}^{1/n} \frac{\cos x}{2\sqrt{x}} dx$

1) Calcolare $\lim_{n \to +\infty} I_n$

2) Studiare la convergenza di $\sum_n n^{\alpha} \cdot I_n$ al variare di α

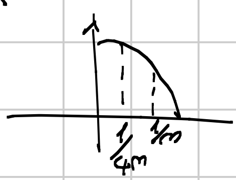
dim

$$I_n = \int_{1/4n}^{1/n} \frac{\cos x}{2\sqrt{x}} dx$$

quando $n \to +\infty$, $\left[\frac{1}{4n}, \frac{1}{n}\right] \rightsquigarrow 0$ e $\frac{\cos x}{2\sqrt{x}} \sim \frac{1}{2\sqrt{x}}$ $x \to 0$

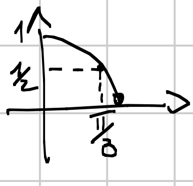
Non conosco le primitive esplicite di $\frac{\cos x}{2\sqrt{x}}$ però

$$\cos\left(\frac{1}{n}\right) \cdot \frac{1}{2\sqrt{x}} \leq \frac{\cos x}{2\sqrt{x}} \leq \cos\left(\frac{1}{4n}\right) \cdot \frac{1}{2\sqrt{x}} \quad \forall x \in]0,1[\quad \forall n \geq 1$$



$$\Downarrow$$

$$\boxed{\frac{1}{2} \cdot \frac{1}{2\sqrt{x}} \leq \frac{\cos x}{2\sqrt{x}} \leq \frac{1}{2\sqrt{x}} \quad \forall x \in]0, \tfrac{\pi}{3}[}$$



$$\frac{1}{2}\int_{1/4n}^{1/n}\frac{dx}{2\sqrt{x}} \le I_n = \int_{1/4n}^{1/n}\frac{\cos x}{2\sqrt{x}} \le \int_{1/4n}^{1/n}\frac{dx}{2\sqrt{x}} \qquad \forall n \ge 1$$

$$\frac{1}{2}\cdot\frac{1}{2}\sqrt{\frac{1}{n}} = \frac{1}{2}\left[\sqrt{x}\right]_{1/4n}^{1/n} \le I_n \le \left[\sqrt{x}\right]_{1/4n}^{1/n} = \sqrt{\frac{1}{n}} - \frac{1}{2}\sqrt{\frac{1}{n}} = \frac{1}{2}\sqrt{\frac{1}{n}}$$

$$\boxed{\frac{1}{4\sqrt{n}} \le I_n \le \frac{1}{2\sqrt{n}} \qquad \forall n}$$

$\lim\limits_{n\to+\infty} I_n = 0$ poiché $\lim\limits_{n\to+\infty}\frac{1}{2\sqrt{n}} = \lim\limits_{n\to+\infty}\frac{1}{4\sqrt{n}} = 0$!

2) Studiamo $\sum_n a_n$ con $a_n = n^\alpha I_n$ al variare di α

$n^\alpha I_n \ge 0 \quad \forall n \ge 1 \ \forall \alpha \in \mathbb{R} \Rightarrow \sum_n a_n$ converge / diverge $+\infty$

$$n^\alpha I_n \le n^\alpha \cdot \frac{1}{2\sqrt{n}} = \frac{1}{2}\left(\frac{1}{n}\right)^{\frac{1}{2}-\alpha} \qquad \forall n \ge 1 \ \forall \alpha \in \mathbb{R}$$

$\sum_n \frac{1}{2}\left(\frac{1}{n}\right)^{\frac{1}{2}-\alpha}$ converge se $\frac{1}{2}-\alpha > 1$ se $\alpha < -\frac{1}{2}$

$\Longrightarrow$ (Teorema Confronto) $\sum_n a_n$ converge se $\alpha < -\frac{1}{2}$

$\blacksquare$

Oss: $I_n = \int_{1/4n}^{1/n}\frac{\cos x}{2\sqrt{x}}\,dx = \left(\frac{1}{n}-\frac{1}{4n}\right)\cdot\frac{\cos z_n}{2\sqrt{z_n}}$ con $z_n \in \left[\frac{1}{4n}, \frac{1}{n}\right]$

Teorema media integrale

e dunque $\quad 0 \le I_n = \frac{3}{4n}\cdot\frac{\cos z_n}{2\sqrt{z_n}} \le \frac{3}{4n}\cdot\frac{\overset{\cos(0)}{1}}{2\sqrt{\frac{1}{4n}}} = \frac{3}{4\sqrt{n}}$

da cui segue (essendo $\lim\limits_n \frac{3}{4\sqrt{n}} = 0$) che $\lim\limits_{n\to+\infty} I_n = 0$

Inoltre $a_n = n^\alpha I_n \le n^\alpha\cdot\frac{3}{4\sqrt{n}} = \frac{3}{4}\cdot\left(\frac{1}{n}\right)^{\frac{1}{2}-\alpha} = b_n$

e quindi si conclude (essendo $\sum_n b_n$ convergente $\forall \alpha < -\frac{1}{2}$) che $\sum_n a_n$ converge $\forall \alpha < -\frac{1}{2}$

Esercizio

Studiare $\sum_n a_n$ dove $a_n = \int_n^{n+\frac{1}{n^\alpha}} t \arctan t \, dt$

dim

$$a_n = \int_n^{n+\frac{1}{n^\alpha}} t \arctan t \, dt \overset{\text{Teorema Media}}{=} \frac{1}{n^\alpha} \cdot \left(z_n \cdot \arctan(z_n)\right)$$

$z_n \in \left[n, n+\frac{1}{n^\alpha}\right]$

$f(t) = t \arctan t \uparrow$ poiché $g(t) = t \uparrow$ e $h(t) = \arctan t \uparrow$

$\Downarrow$

$$a_n = \frac{1}{n^\alpha}\left(z_n \cdot \arctan(z_n)\right) \leq \frac{1}{n^\alpha} \cdot \left(n + \frac{1}{n^\alpha}\right) \cdot \frac{\pi}{2}$$

$\Downarrow$

$$a_n \leq \frac{\pi}{2}\left(\frac{1}{n^{\alpha-1}} + \frac{1}{n^{2\alpha}}\right) = b_n + c_n \quad \text{con}$$

$$b_n = \frac{\pi}{2}\left(\frac{1}{n}\right)^{\alpha-1}$$

$$c_n = \frac{\pi}{2}\left(\frac{1}{n}\right)^{2\alpha}$$

Ora $\sum_n b_n$ converge se $\alpha - 1 > 1$ se $\alpha > 2$

$\sum_n c_n$ " se $2\alpha > 1$ se $\alpha > \frac{1}{2}$

dunque $\sum_n b_n$ e $\sum_n c_n$ convergono se $\alpha > 2$

dunque se $\alpha > 2$ allora $\sum_n a_n$ converge

Quando $\alpha \leq 2$ si ha che, posto $f(t) = t \arctan t$

$$a_n \overset{\text{Teorema media integrale}}{=} \frac{1}{n^\alpha} z_n \cdot \arctan z_n \quad \text{con } z_n \in \left[n, n+\frac{1}{n^\alpha}\right]$$

$$\underset{f\uparrow}{\geq} \frac{1}{n^\alpha} \cdot n \cdot \arctan n \geq \left(\frac{1}{n}\right)^{\alpha-1} \cdot \frac{\pi}{4} \quad \forall n \geq 1$$

ovvero $a_n \geq \left(\frac{1}{n}\right)^{\alpha-1} \cdot \frac{\pi}{4} = b_n \quad \forall \alpha \ \forall n \geq 1$ **9**

Ma $\sum_n b_n = \sum_n \frac{\pi}{4} \cdot \left(\frac{1}{n}\right)^{\alpha-1}$ diverge se $\alpha \leq 2$

$\Longrightarrow \sum_n a_n$ diverge se $\alpha \leq 2$