**Def**

$f : [a,b[ \to \mathbb{R}$ t.c. $\exists \int_a^\beta f(t)dt \ \forall \beta \in [a,b[$, questa si dice

- integrabile in senso improprio (generalizzato) su $[a,b[$ se $\exists \lim_{\beta \to b^-} \int_a^\beta f(t)dt = \int_a^b f(t)dt \in \mathbb{R}$

- l'integrale improprio di $f$ su $[a,b[$ diverge a $+\infty$ $(-\infty)$ se $\lim_{\beta \to b^-} \int_a^\beta f(t)dt = \int_a^b f(t)dt = +\infty$ $(-\infty)$

- l'integrale improprio di $f$ su $[a,b[$ $\nexists$ se $\nexists \lim_{\beta \to b^-} \int_a^\beta f(t)dt$

**Oss:** c'è una forte similarità tra

$\sum_n a_n$ e $\int_a^{+\infty} f(t)dt$

**Teorema (vinb[?] ieri)**

$f : [0,+\infty[ \to \mathbb{R}$, $f$ continua su $[a,+\infty[$, $\exists \lim_{x \to +\infty} f(x) = \ell$

se $\int_0^{+\infty} f(x)dx \in \mathbb{R}$ allora $\ell = 0$

**Teorema**

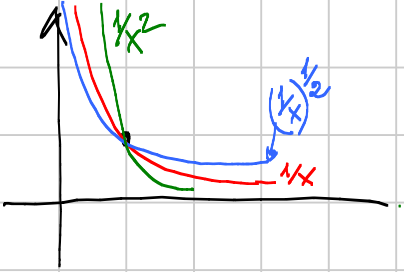
$f : [a,b[ \to \mathbb{R}$ t.c. $\exists \int_a^\beta f(t)dt$

Se $f \geq 0$ $(\leq 0)$ $\forall t \in [a,b[$ allora $\exists \int_a^b f(t)dt$ $\begin{cases} \in \mathbb{R} \\ +\infty \ (-\infty) \end{cases}$

**Esempi fondamentali**

$$\int_0^1 \frac{dt}{t^\alpha} \begin{cases} \in \mathbb{R} & \text{se } \alpha < 1 \\ +\infty & \text{se } \alpha \geq 1 \end{cases}$$

$$\int_1^{+\infty} \frac{dt}{t^\alpha} \begin{cases} \in \mathbb{R} & \text{se } \alpha > 1 \\ +\infty & \text{se } \alpha \leq 1 \end{cases}$$




$$\int_0^1 \frac{dx}{x} = +\infty$$

$$\int_1^{+\infty} \frac{dx}{x} = +\infty$$

## Teorema (confronto tra integrali impropri)

$\varphi, f : [a,b[ \to \mathbb{R}$ t.c.

1) $f$ continua su $[a,b[$
2) $|f(x)| \leq \varphi(x) \quad \forall x \in [a,b[$
3) $\displaystyle\int_a^b \varphi(x)dx = \lim_{\beta \to +\infty} \int_a^\beta \varphi(x)dx \in \mathbb{R}$

$\Rightarrow \exists \displaystyle\int_a^b f(x)dx = \lim_{\beta \to +\infty} \int_a^\beta f(x)dx \in \mathbb{R}$

### dim

$f$ continua su $[a,b[ \Rightarrow f$ integrabile su $[a,\beta] \quad \forall \beta \in [a,b[$

$\Rightarrow \exists \displaystyle\int_a^\beta f(t)dt \in \mathbb{R} \quad \forall \beta \in [a,b[$

$f(x) = f^+(x) - f^-(x) \qquad \underset{\geq 0}{f^+} = \max\{f, 0\} \qquad \underset{\geq 0}{f^-} = \max\{-f, 0\}$

$|f(x)| = f^+(x) + f^-(x)$

l'ipotesi 1 mi dice che anche $f^+$ ed $f^-$ sono continue e integrabili, quindi, su $[a,\beta] \ \forall \beta \in [a,b[$

l'ipotesi 2 mi dice $\quad 0 \leq f^+(x) + f^-(x) = |f(x)| \leq \varphi(x) \quad \forall x \in [a,b[$

$$\Rightarrow \begin{cases} 0 \leq f^+(x) \leq \varphi(x) \\ 0 \leq f^-(x) \leq \varphi(x) \end{cases} \quad \forall x \in [a,b[$$

Teorema confronto integrali

$$\Rightarrow \quad 0 \leq \int_a^\beta f^+(x)dx \leq \int_a^\beta \varphi(x)dx \qquad \forall \beta \in [a,b[$$

$$0 \leq \int_a^\beta f^-(x)dx \leq \int_a^\beta \varphi(x)dx \qquad \forall \beta \in [a,b[$$

$\Rightarrow$ essendo $f^+$ ed $f^- \geq 0$ si ha che

ipotesi 3

$$\exists \lim_{\beta\to+\infty} \int_a^\beta f^+(x)dx \leq \lim_{\beta\to+\infty} \int_a^\beta \varphi(x)dx = \int_a^b \varphi(x)dx < +\infty$$

$$\exists \lim_{\beta\to+\infty} \int_a^\beta f^-(x)dx \leq \lim_{\beta\to+\infty} \int_a^\beta \varphi(x)dx = \int_a^b \varphi(x)dx < +\infty$$

$$\Rightarrow \exists \lim_{\beta\to+\infty} \int_a^\beta f(x)dx = \int_a^b f^+(x)dx - \int_a^b f^-(x)dx < +\infty$$

che è la Tesi $\blacksquare$

## Teorema (Confronto Asintotico tra integrali Impropri)

$f,g : [a,b[ \to \mathbb{R}$ t.c.

1) $f,g$ continue $\forall x \in [a,b[$

2) $f \geq 0 \ \forall x \in [a,b[$, $g > 0 \ \forall x \in [a,b[$

3) $f \sim g$ per $x \to b^-$ $\left( \lim_{x\to b^-} \frac{f(x)}{g(x)} = l \in ]0,+\infty[ \right)$

Allora

$$\left\{ \int_a^b f(x)dx \in \mathbb{R} \ (+\infty) \ \text{sse} \ \int_a^b g(x)dx \in \mathbb{R} \ (+\infty) \right\}$$

dim (sostanzialmente identica a quanto fatto per le serie a termini $\geq 0$)

Essendo $f,g$ continue $\Rightarrow \exists \int_a^\beta f(x)dx$ e $\int_a^\beta g(x)dx \ \forall \beta \in [a,b[$

" $f,g \geq 0 \Rightarrow \exists \lim_{\beta\to+\infty} \int_a^\beta f(x)dx \ {}^{<+\infty}_{\in\mathbb{R}}$

$$\exists \lim_{\beta \to +\infty} \int_a^\beta g(x)\,dx \begin{cases} +\infty \\ \in \mathbb{R} \end{cases}$$

Abbiamo utilizzato le ipotesi 1) e 2)

L'ipotesi 3) dice $f \sim g$ per $x \to b^-$, ovvero

$$\forall \varepsilon > 0\ \exists \delta > 0 : \forall x \in ]b-\delta, b[ \Rightarrow l - \varepsilon < \frac{f(x)}{g(x)} < l + \varepsilon$$

$$\Downarrow$$

$$\varepsilon = \frac{l}{2}\quad \exists \delta > 0 : \forall x \in ]b-\delta, b[ \Rightarrow \frac{l}{2} = l - \frac{l}{2} < \frac{f(x)}{g(x)} < \frac{3}{2} l = l + \frac{l}{2}$$

$$\Downarrow$$

$$\exists \delta > 0 : \forall x \in ]b-\delta, b[ \quad \left(\frac{l}{2}\right) \cdot g(x) \leq f(x) \leq \left(\frac{3}{2} l\right) g(x)$$

$\Downarrow$ ← Teorema confronto integrali def.

$$\int_{b-\delta}^{\beta} \left(\frac{l}{2}\right) g(x)\,dx \leq \int_{b-\delta}^{\beta} f(x)\,dx \leq \int_{b-\delta}^{\beta} \left(\frac{3}{2} l\right) \cdot g(x) \quad \forall \beta \in ]b-\delta, b[$$

$$\Downarrow$$

$$\exists \lim_{\beta \to b^-} \left(\frac{l}{2}\right) \int_{b-\delta}^{\beta} g(x)\,dx \leq \lim_{\beta \to b^-} \int_{b-\delta}^{\beta} f(x)\,dx \leq \lim_{\beta \to b^-} \left(\frac{3}{2} l\right) \int_{b-\delta}^{\beta} g(x)\,dx$$

$$\Downarrow$$

$$\left(\frac{l}{2}\right) \int_a^{b-\delta} g(x)\,dx + \lim_{\beta \to b^-} \left(\frac{l}{2}\right) \int_{b-\delta}^{\beta} g(x)\,dx \leq \int_a^{b-\delta} f(x)\,dx + \lim_{\beta \to b^-} \int_a^{\beta} f(x)\,dx$$

$$\leq \left(\frac{3l}{2}\right) \int_a^{b-\delta} g(x)\,dx + \lim_{\beta \to b^-} \int_{b-\delta}^{\beta} \left(\frac{3}{2} l\right) \cdot g(x)\,dx$$

$$\frac{l}{2} \int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx \leq \frac{3l}{2} \int_a^b g(x)\,dx \quad \text{e} \ \text{la tesi}$$ $\blacksquare$

**Esempio**

Studiare la convergenza di $\displaystyle\int_0^{+\infty} \frac{\operatorname{sen} x + 1}{x^2+1}\,dx$

**dim**

- l'integranda $f(x) = \dfrac{\operatorname{sen} x + 1}{x^2+1}$ è continua e limitata $\forall x \geq 0$

$\Rightarrow \exists \displaystyle\int_0^\beta f(x)\,dx \in \mathbb{R} \quad \forall \beta \geq 0$

- Inoltre $\displaystyle\lim_{x\to+\infty} f(x) = 0$

Unico pb: il limite per $\beta \to +\infty$ di $\int^\beta f(x)\,dx$ esiste certamente poiché $f(x) \geq 0$, ma non sappiamo se

$$\int_0^\beta f(x)\,dx \xrightarrow[\beta\to+\infty]{} \begin{cases} +\infty \\ \in \mathbb{R} \end{cases} ?$$

A tal fine si osserva che

$$|f(x)| = f(x) = \frac{1+\operatorname{sen} x}{1+x^2} \leq \frac{2}{1+x^2} = \varphi(x)$$

e osservo che $\displaystyle\int_0^\beta \varphi(x)\,dx = 2\left[\operatorname{arctg} x\right]_{x=0}^{x=\beta} = 2\operatorname{arctg}\beta \xrightarrow[\beta\to+\infty]{} \pi$

$\overset{\text{(Teorema Confronto)}}{\Longrightarrow} \displaystyle\int_0^{+\infty} \frac{1+\operatorname{sen} x}{1+x^2}\,dx \in \mathbb{R}$

# LEZIONI CONCLUSIVE

| | | | |
|---|---|---|---|
| Lunedì 22 | 10,30 ÷ 12.30 | Aula P | M. Belloni |
| | 14 30 ÷ 16.30 | Aula H | Tutoraggio (L. Amadosi) |
| Martedì 23 | 8,30 ÷ 10 30 | Aula P | Tutoraggio (D. Fioreni) |
| Giovedì 8 gennaio | 10.30 ÷ 12.30 | Aula P | M. Belloni |
| | 16.30 ÷ 18,30 | | Tutoraggio |
| Venerdì 9 gennaio | 10.30 ÷ 12.30 | Aula G | M. Belloni |
| | 14. 30 ÷ 16.30 | " | Tutoraggio |

## Esercizio

Calcolare $\displaystyle\lim_{x\to+\infty} \frac{1}{x}\int_2^x (\log t)^{-1}\,dt$

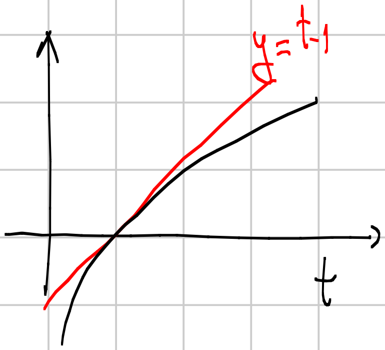
**dim**

$$\lim_{x\to+\infty} \frac{\int_2^x (\log t)^{-1}\,dt}{x} \overset{H}{=} \lim_{x\to+\infty} \frac{(\log x)^{-1}}{1} = 0^+$$

Se ci fermiamo qui, sbagliamo: abbiamo applicato il Teorema dell'Hopital supponendo che ci troviamo nella forma $\infty/\infty$

ma chi ci dice che $\displaystyle\int_2^{+\infty} \frac{dt}{\log t} = +\infty$ ?

Si osserva che $\log t \leq t-1 \quad \forall t>0$




$$\Rightarrow \frac{1}{\log t} \geq \frac{1}{t-1} \quad \forall t \geq 2$$

$$\Rightarrow \int_2^\beta \frac{dt}{\log t} \geq \int_2^\beta \frac{dt}{t-1} \quad \forall \beta \geq 2$$

$$\overset{y=t-1}{\|}$$

$$\int_1^{\beta-1} \frac{dy}{y} = \log(\beta-1) \quad \forall \beta \geq 2$$

ho confuso con $1/y$

scritto meglio $\displaystyle\int_2^\beta \frac{dt}{\log t} \geq \log(\beta-1) \quad \forall \beta \geq 2$

$$\Rightarrow \int_2^{+\infty} \frac{dt}{\log t} \geq \lim_{\beta\to+\infty} \log(\beta-1) = +\infty$$

∎

## Esercizio

Studiare, al variare di $\alpha \in \mathbb{R}$, l'integrale $\displaystyle\int_2^{+\infty} \frac{dt}{t(\log t)^\alpha}$

**dim**

Noi sappiamo che $\displaystyle\int_2^{+\infty} \frac{dt}{t} = +\infty$

come pure $\displaystyle\int_2^{+\infty} \frac{dt}{t^\alpha} \in \mathbb{R} \quad \forall \alpha > 1$

il caso $\alpha = 1$ è un caso particolare poiché divide {convergente} da {non convergente}

$$\int_2^{\beta} \frac{dt}{t\cdot(\log t)^\alpha} = \int_{\log 2}^{\log\beta} \frac{\cancel{e^y}\,dy}{\cancel{e^y}(y)^\alpha} = \int_{\log 2}^{\log\beta} \frac{dy}{y^\alpha} \xrightarrow{\beta\to+\infty} \int_{\log 2}^{+\infty} \frac{dy}{y^\alpha} \begin{cases} +\infty & \alpha\leq 1 \\ \in\mathbb{R} & \alpha>1 \end{cases}$$

$y = \log t$

$t = e^y$

$dt = e^y\,dy$

$t = 2 \longrightarrow y = \log 2$

$t = \beta \longrightarrow y = \log\beta$

$$\Rightarrow \int_2^{+\infty} \frac{dt}{t(\log t)^\alpha} \begin{cases} +\infty & \alpha\leq 1 \\ \in\mathbb{R} & \alpha>1 \end{cases}$$

$\boxed{M}$

**Per casa**

Studiare, al variare di $\alpha \in \mathbb{R}$, $\displaystyle\int_{e^2}^{+\infty} \frac{dt}{t\cdot\log t\cdot(\log(\log t))^\alpha}$

Esercizio

1) Calcolare $\int x \cdot 2^{-x^2} dx$

2) posto $a_n = \int_{\sqrt{n}}^{\sqrt{2n}} x\, 2^{-x^2} dx$, calcolare $\sum_{n=1}^{\infty} a_n$

dim

1) $$\int x \cdot 2^{-x^2} dx = \int x \cdot \left(e^{\log 2}\right)^{-x^2} dx = \int x \cdot e^{-\log 2 \cdot x^2} dx$$

$$= \frac{1}{-2\log 2} \int \left(-2\log 2 \cdot x\right) e^{-x^2 \cdot \log 2} dx = \frac{1}{-2\log 2} \left( \int e^y dy \right)_{y = -x^2 \log 2}$$

$y = (-x^2 \log 2)$

$dy = (-2x \log 2)\, dx$

$$= -\frac{1}{2\log 2} \cdot \left(e^y + c\right)_{y = -x^2 \log 2} = -\frac{1}{2\log 2} \cdot e^{-x^2 \log 2} + c \quad c \in \mathbb{R}$$

$$= -\frac{1}{2\log 2} \cdot 2^{-x^2} + c \qquad c \in \mathbb{R}$$

2) $$a_n = \int_{\sqrt{n}}^{\sqrt{2n}} x\, 2^{-x^2} dx = \left[ -\frac{1}{2\log 2} \cdot 2^{-x^2} \right]_{x=\sqrt{n}}^{x=\sqrt{2n}}$$

$$= -\frac{1}{2\log 2} \left( 2^{-2n} - 2^{-n} \right) = -\frac{1}{2\log 2} \left( \frac{1}{4^n} - \frac{1}{2^n} \right)$$

$$\sum_{n=\boxed{1}}^{\infty} a_n = -\frac{1}{2\log 2} \sum_{n=1}^{\infty} \left( \frac{1}{4^n} - \frac{1}{2^n} \right) =$$

$$= \frac{1}{2\log 2} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n - \frac{1}{2\log 2} \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$$

$$= \frac{1}{2\log 2} \left( \frac{1}{1-\frac{1}{2}} - 1 \right) - \frac{1}{2\log 2} \left( \frac{1}{1-\frac{1}{4}} - 1 \right)$$

$$= \frac{1}{2\log 2} - \frac{1}{3} \cdot \frac{1}{2\log 2} = \frac{\cancel{2}^{\,1}}{3} \cdot \frac{1}{\cancel{2}\log 2} = \frac{1}{\log 8}$$

$\blacksquare$