Decomposizione in somma

$$\int (f+g)\,dx = \int f\,dx + \int g\,dx$$

Integrazione per parti

$$\int u'(x)\cdot v(x)\,dx = u(x)\cdot v(x) - \int u(x)\cdot v'(x)\,dx$$

Integrazione per sostituzione

$\varphi : I \longrightarrow J$ derivabile

$g : J \longrightarrow \mathbb{R}$ continua

$G : J \longrightarrow \mathbb{R}$ primitiva di $g(x)$

Allora $$\int (g\circ\varphi)(x)\cdot\varphi'(x)\,dx \underset{\substack{\uparrow \\ \text{dim}}}{=} \left(\int g(y)\,dy\right)_{y=\varphi(x)} = G(\varphi(x)) + c \quad c\in\mathbb{R}$$

È sufficiente fare la verifica derivando ambo i membri

$$\frac{d}{dx}\left(\int (g\circ\varphi)(x)\cdot\varphi'(x)\,dx\right) = (g\circ\varphi)(x)\cdot\varphi'(x)$$

$$\frac{d}{dx}\left((G\circ\varphi)(x) + c\right) = G'(\varphi(x))\cdot\varphi'(x) = (g\circ\varphi)(x)\cdot\varphi'(x)$$

$\blacksquare$

Nuova Tabella

Per il Teorema precedente

① $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad \forall n \neq -1 \quad \Longrightarrow$$

$$\Longrightarrow \int [\varphi(x)]^n\cdot\varphi'(x)\,dx = \left(\int y^n\,dy\right)_{y=\varphi(x)} = \frac{[\varphi(x)]^{n+1}}{n+1} + c \quad c\in\mathbb{R}$$

x esempio, se $\varphi(x) = \operatorname{sen} x$

$$\int [\operatorname{sen} x]^n \cdot \cos x \, dx = \frac{(\operatorname{sen} x)^{n+1}}{n+1} + c \qquad c \in \mathbb{R}$$

se prendo $\varphi(x) = \log x$

$$\int (\log x)^n \cdot \frac{1}{x} \, dx = \frac{(\log x)^{n+1}}{n+1} + c \qquad c \in \mathbb{R}$$

- - -

Altro esempio

Per il Teorema precedente

2) $\displaystyle\int \frac{1}{x} dx = \log|x| + c \implies$ preso $\varphi(x)$ derivabile

$$\int \frac{\varphi'(x)}{\varphi(x)} dx = \int \left[\frac{1}{\varphi(x)}\right] \varphi'(x) dx = \left(\int \frac{dy}{y}\right)_{y=\varphi(x)} = \log|\varphi(x)| + c \qquad c \in \mathbb{R}$$

$$\boxed{\varphi = \log x} \rightarrow \int \frac{(\log x)'}{\log x} dx = \int \frac{dx}{x \log x} = \log|\log x| + c \qquad c \in \mathbb{R}$$

3) $\displaystyle\int e^x dx = e^x + c \implies \varphi$ derivabile $\displaystyle\int e^{\varphi(x)} \cdot \varphi'(x) dx = e^{\varphi(x)} + c \qquad c \in \mathbb{R}$

Per il Teorema precedente

4) $\displaystyle\int \operatorname{sen} x \, dx = -\cos x + c \implies \varphi$ derivabile

$$\int \operatorname{sen}(\varphi(x)) \cdot \varphi'(x) dx = -\cos(\varphi(x)) + c \qquad c \in \mathbb{R}$$

5) $\displaystyle\int \cos x \, dx = \operatorname{sen} x + c \implies \varphi$ derivabile

$$\int \cos(\varphi(x)) \varphi'(x) dx = \operatorname{sen}(\varphi(x)) + c \qquad c \in \mathbb{R}$$

Esercizio Calcolare $\int \operatorname{tg} x \, dx$

dim

$$\int \operatorname{tg} x \, dx = -\int \frac{-\operatorname{sen} x}{\cos x} dx = -\int \frac{(\cos x)'}{\cos x} dx$$

$$= -\log|\cos x| + c \qquad c \in \mathbb{R}$$

Esercizio

Calcolare $\int x^2 \cdot \cos(x^3) \, dx$

dim

$$\int x^2 \cos(x^3) \, dx \underset{y=x^3}{=} \left(\int \cos(y) \frac{dy}{3} \right)_{y=x^3}$$

$y = x^3$

$\frac{dy}{dx} = 3x^2$

$dy = 3x^2 dx$

$\frac{dy}{3} = x^2 dx$

$$= \left(\frac{1}{3} \operatorname{sen} y + c \right)_{y=x^3} = \frac{1}{3} \operatorname{sen}(x^3) + c \qquad c \in \mathbb{R}$$

Verifica: $\left(\frac{1}{3} \operatorname{sen}(x^3) + c \right)' = \frac{1}{\cancel{3}} \cos(x^3) \cdot \cancel{3} x^2$ ✓

Esercizio Calcolare $\int \sqrt{1-x^2} \, dx$

dim

$$\int \sqrt{1-x^2} \, dx \underset{x = \operatorname{sen} t}{=} \left(\int \sqrt{1-\operatorname{sen}^2 t} \, \frac{dx}{dt} \, dt \right)_{x=\operatorname{sen} t}$$

$x = \operatorname{sen} t$

$\frac{dx}{dt} = \cos t$

$$= \left(\int |\cos t| \cdot \cos t \, dt \right)_{x = \operatorname{sen} t} =$$

ho bisogno che $\frac{dx}{dt} \neq 0$

dunque, per esempio, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

ma quando $t \in]-\frac{\pi}{2}, \frac{\pi}{2}[$
si ha che $\cos t > 0$

$$= \left(\int \cos^2 t \, dt \right)_{x = \operatorname{sen} t} =$$

$$= \left(\frac{1}{2} \left(t + \operatorname{sen} t \cos t \right) + c \right)_{x = \operatorname{sen} t} =$$

$$= \frac{1}{2} \operatorname{arcsen} x + \frac{1}{2} \operatorname{sen}(\operatorname{arcsen} x) \cdot \cos(\operatorname{arcsen} x) + c \qquad c \in \mathbb{R}$$

$$= \frac{1}{2} \operatorname{arcsen} x + \frac{1}{2} x \cdot \sqrt{1 - [\operatorname{sen}(\operatorname{arcsen} x)]^2} + c \qquad c \in \mathbb{R}$$

$$= \frac{1}{2} \left(\operatorname{arcsen} x + x \sqrt{1 - x^2} \right) + c \qquad c \in \mathbb{R}$$

Verifica: $\left(\frac{1}{2} \operatorname{arcsen} x + \frac{1}{2} x \sqrt{1 - x^2} + c \right)' =$

$$= \frac{1}{2} \frac{1}{\sqrt{1 - x^2}} + \frac{1}{2} \sqrt{1 - x^2} + \frac{1}{\cancel{2}} x \cdot \frac{1}{2\sqrt{1 - x^2}} \cdot (-\cancel{2}x)$$

$$= \frac{1}{2} \frac{1}{\sqrt{1 - x^2}} + \frac{1}{2} \sqrt{1 - x^2} + \frac{1 - x^2 - 1}{2\sqrt{1 - x^2}}$$

$$= \cancel{\frac{1}{2} \frac{1}{\sqrt{1 - x^2}}} + \frac{1}{2} \sqrt{1 - x^2} + \frac{1}{2} \frac{1 - x^2}{\sqrt{1 - x^2}} - \cancel{\frac{1}{2} \frac{1}{\sqrt{1 - x^2}}}$$

$$= \sqrt{1 - x^2} \qquad \checkmark$$

$$\int \operatorname{tg} x \, dx = x \operatorname{tg} x - \int \frac{x}{\cos^2 x} \, dx$$

$$\| $$

$$-\log|\cos x| + c \qquad \Rightarrow \qquad \int \frac{x}{\cos^2 x} dx = x \operatorname{tg} x - \log|\cos x| + c$$

Esercizio Data $f(x) = \operatorname{sen} x$, calcolare $g'(x)$ dove $g(x) = f^{-1}(x)$

dim

$f(x)$ è strettamente crescente e derivabile $\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$

$$\Rightarrow \exists \left(f^{-1}\right)'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{\cos(\operatorname{arcsen}(y_0))} =$$

$$= \frac{1}{\sqrt{1 - \operatorname{sen}^2(\operatorname{arcsen} y_0)}} = \frac{1}{\sqrt{1 - y_0^2}} \qquad \forall y_0 \in]-1, 1[$$

$\blacksquare$

Oss: Allo stesso modo posso provare che, presa $f(x) = \cos(x)$, si ha

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{-\operatorname{sen}(\arccos(y_0))} =$$

$$= \frac{1}{-\sqrt{1 - \cos^2(\arccos(y_0))}}$$

$$= -\frac{1}{\sqrt{1 - y_0^2}} \qquad \forall y_0 \in]-1, 1[$$

$\blacksquare$

Integrazione delle funzioni razionali

Una funzione razionale è del tipo $f(x) = \frac{P(x)}{Q(x)}$

dove $P(x)$ e $Q(x)$ sono due polinomi nella variabile x

Noi considereremo solo grado $(P(x)) \leq 2$

Teorema (non lo dimostriamo)

Data $f(x) = \frac{P(x)}{Q(x)}$ funzione razionale con P e Q di ordine $\forall$

Allora $\exists$ $F(x)$, primitiva di f, ed è esprimibile in termini di funzioni elementari

1° Caso grado $Q(x) = 1$

Esercizio Calcolare $\int \frac{x^3 - 3x^2 + 1}{x-2}\, dx = \int f(x)\, dx$

div.

1° passo Se grado $(P(x)) \geq$ grado $(Q(x))$, allora si divide

Nel caso in esame $3 = \text{grado}(P(x)) > \text{grado}(Q(x)) = 1$

$$
\begin{array}{rrrr|l}
x^3 & -3x^2 & & +1 & x-2 \\
\hline
x^3 & -2x^2 & & & x^2 - x - 2 \\
\hline
// & -x^2 & & +1 & \\
 & -x^2 & +2x & & \\
\hline
 & // & -2x & +1 & \\
 & & -2x & +4 & \\
\hline
 & & // & -3 &
\end{array}
$$

$$x^3 - 3x^2 + 1 = (x^2 - x - 2)(x-2) - 3$$

$$f(x) = \frac{(x^2-x-2)\cancel{(x-2)}}{\cancel{x-2}} - \frac{3}{x-2}$$

$$\int f(x)\,dx = \int (x^2-x-2)\,dx - 3\int \frac{dx}{x-2}$$

$$= \frac{x^3}{3} - \frac{x^2}{2} - 2x - 3\log|x-2| + c \quad c\in\mathbb{R}$$

OSS: $\frac{1}{2}\int \frac{2\,dx}{2x+1} = \frac{1}{2}\log|2x+1| + c$

grado di $Q(x) = 2$

Radici Reali $\neq$

Esempio Calcolare $\int \frac{x^2}{x^2+4x+1}\,dx = \int f(x)\,dx$

1° passo dividere x^2+4x+1 per x^2

$$f(x) = \frac{x^2 \cancel{+4x+1}}{x^2 \cancel{+4x+1}} - \frac{4x+1}{x^2+4x+1} = 1 - \frac{4x+1}{x^2+4x+1}$$

2 passo

$$\frac{4x+1}{x^2+4x+1} = \frac{A}{x-x_1} + \frac{B}{x-x_2}$$

dove $x_{1,2} = -2 \pm \sqrt{4-1}$ ovvero $x_1 = -2-\sqrt{3}$, $x_2 = -2+\sqrt{3}$

$$\underbrace{\frac{4x+1}{x^2+4x+1}} = \frac{A}{x+2+\sqrt{3}} + \frac{B}{x+2-\sqrt{3}} = \underbrace{\frac{Ax+(2-\sqrt{3})A+Bx+B(2+\sqrt{3})}{x^2+4x+1}}$$

$$\Downarrow$$

$$4x+1 = (A+B)x + (2-\sqrt{3})A + (2+\sqrt{3})B$$

$$\Downarrow$$

$$\begin{cases} 4 = A+B \\ 1 = 2(A+B) - \sqrt{3}(A-B) \end{cases} \qquad \begin{cases} A = 4-B \\ 1 = 8 - \sqrt{3}(4-2B) \end{cases}$$

$$\Rightarrow \begin{cases} A = 4 - B \\ -\sqrt{3}\cdot 4 + 2\sqrt{3}\cdot B = -7 \end{cases} \qquad \begin{cases} A = \dfrac{8\sqrt{3} - 4\sqrt{3} + 7}{2\sqrt{3}} \\ B = \dfrac{4\sqrt{3} - 7}{2\sqrt{3}} \end{cases}$$

$$\int f(x)dx = \int 1 \cdot dx - \int \left(\left(2 + \frac{7}{2\sqrt{3}}\right) \cdot \frac{1}{x+2+\sqrt{3}} + \left(2 - \frac{7}{2\sqrt{3}}\right) \frac{1}{x+2-\sqrt{3}} \right) dx$$

$$= x - \left(2 + \frac{7}{2\sqrt{3}}\right) \cdot \log|x+2+\sqrt{3}| - \left(2 - \frac{7}{2\sqrt{3}}\right) \cdot \log|x+2-\sqrt{3}| + c$$

$$= x - 2\log(x^2+4x+1) + \frac{7}{2\sqrt{3}} \log\left|\frac{x+2-\sqrt{3}}{x+2+\sqrt{3}}\right| + c \qquad c \in \mathbb{R}$$

$\blacksquare$

Caso delle radici reali e coincidenti

Esempio Calcolare $\int \frac{x^3-x+1}{x^2-4x+4}\,dx = \int f(x)\,dx$

divi

Eseguo la divisione poiché $\text{grado}(Q(x)) < \text{grado}(P(x))$

$$\begin{array}{rrrr|l} x^3 & & -x & +1 & x^2-4x+4 \\ \cline{5-5} x^3 & -4x^2 & +4x & & x+4 \\ \hline // & +4x^2 & -5x & 1 & \\ & 4x^2 & -16x & +16 & \\ \hline & // & +11x & -15 & \end{array}$$

$$x^3-x+1 = (x^2-4x+4)(x+4) + 11x - 15$$

$$\int f(x)\,dx = \int \frac{(x+4)\cancel{(x^2-4x+4)}^{1}}{\cancel{x^2-4x+4}_{1}}\,dx + \int \frac{11x-15}{x^2-4x+4}\,dx$$

Devo individuare A e B t.c.

$$\frac{11x-15}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} = \frac{Ax-2A+B}{(x-2)^2}$$

$$\Downarrow$$

$$11x-15 = Ax - (2A-B)$$

$$\Downarrow$$

$$\begin{cases} 11 = A \\ -15 = -(2A-B) \end{cases} \Rightarrow \begin{cases} A = 11 \\ B-2A = -15 \end{cases} \quad \begin{cases} A = 11 \\ B = 7 \end{cases}$$

$$\Downarrow$$

$$\int f(x)dx = \int (x+4)\,dx + \int \left(\frac{11}{x-2} + \frac{7}{(x-2)^2}\right)dx$$

$$= \frac{x^2}{2} + 4x + 11\int \frac{dx}{x-2} + 7\int \frac{dx}{(x-2)^2}$$

$$= \frac{x^2}{2} + 4x + 11\log|x-2| + 7 \cdot \frac{(x-2)^{-2+1}}{-2+1} + c \quad c \in \mathbb{R}$$

$$= \frac{x^2}{2} + 4x + 11\log|x-2| - \frac{7}{(x-2)} + c \quad c \in \mathbb{R}$$

Radici Complesse e Coniugate

Esempio Calcolare $\int \frac{x^3-2x^2+x}{x^2+2x+3}\,dx = \int f(x)dx$

dim

$$\begin{array}{l|l} x^3 - 2x^2 + x & x^2+2x+3 \\ \hline x^3 + 2x^2 + 3x & x-4 \\ \hline // \ -4x^2 - 2x & \\ \quad -4x^2 - 8x - 12 & \\ \hline \quad // \ 6x + 12 & \end{array}$$

$$x^3-2x^2+x = (x^2+2x+3)(x-4) + 6x + 12$$

$$\int f(x)dx = \int (x-4)\,dx + 3\int \frac{2x+4}{x^2+2x+3}\,dx$$

$$= \frac{x^2}{2} - 4x + 3\int \frac{2x+2}{x^2+2x+3}\,dx + 6\int \frac{dx}{x^2+2x+3}$$

$$= \frac{x^2}{2} - 4x + 3\log(x^2+2x+3) + 6\int \frac{dx}{(x+1)^2+2}$$

$$= \frac{x^2}{2} - 4x + 3\log(x^2+2x+3) + \frac{\cancel{6}}{\cancel{2}}\int \frac{dx}{1+\left(\frac{x+1}{\sqrt{2}}\right)^2}$$

$$= \qquad // \qquad + 3 \cdot \left(\int \frac{\sqrt{2}\,dy}{1+y^2}\right)_{y=\frac{x+1}{\sqrt{2}}}$$

$$= \frac{x^2}{2} - 4x + 3\log(x^2+2x+3) + 3\sqrt{2} \cdot \left(\operatorname{arctg} y + c\right)_{y=\frac{x+1}{\sqrt{2}}}$$

$$= \frac{x^2}{2} - 4x + 3\log(x^2+2x+3) + 3\sqrt{2}\operatorname{arctg}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

$c \in \mathbb{R}$

$\blacksquare$