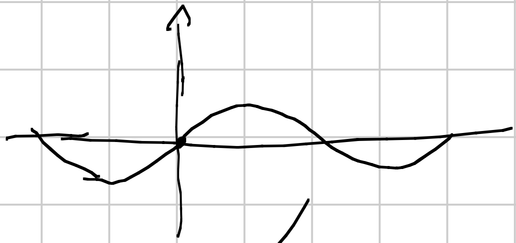
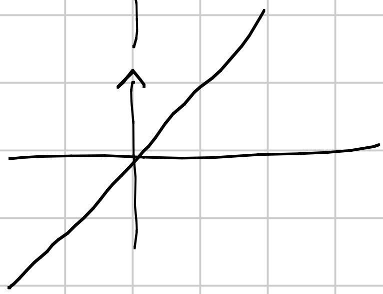
OSSERVAZIONE IMPORTANTE

$f(x) = \operatorname{sen} x$



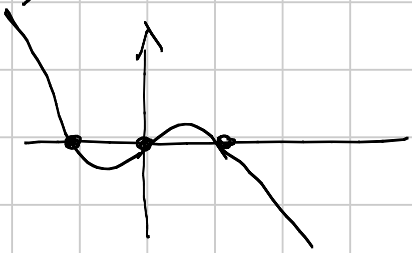
$f_1(x) = x$



$f(x) \neq f_1(x)$

ma $f(x) - f_1(x) = o(x^2, 0)$

$f_2(x) = x - x^3/6$



$f(x) \neq f_2(x)$

ma $f(x) - f_2(x) = o(x^4, 0)$

Esercizio

$\{a_n\}_n$ successione reale

1) $a_n \xrightarrow[n\to+\infty]{} \ell \Rightarrow |a_n| \xrightarrow[n\to+\infty]{} |\ell|$

2) $|a_n| \xrightarrow[n\to+\infty]{} |\ell| \not\Rightarrow a_n \xrightarrow[n\to+\infty]{} \ell$ o $a_n \xrightarrow[n\to+\infty]{} -\ell$

3) $|a_n| \xrightarrow[n\to\infty]{} 0$ ma $a_n \xrightarrow[n\to+\infty]{} 0$

dim

1) si ricorda che ① $\big||a_n| - |\ell|\big| \leq |a_n - \ell|$ (vedi proprietà $|\cdot|$)

inoltre ($\ell \in \mathbb{R}$) per ipotesi

② $\forall \varepsilon > 0\ \exists \bar{n} = \bar{n}(\varepsilon) > 0 : \forall n > \bar{n} \quad |a_n - \ell| < \varepsilon$

①+② $\forall \varepsilon > 0\ \exists \bar{n} > 0 : \forall n > \bar{n} \quad \big||a_n| - |\ell|\big| \leq |a_n - \ell| < \varepsilon$

dunque " " " $\big||a_n| - |\ell|\big| < \varepsilon$

dunque $\lim_{n\to+\infty} (|a_n| - |\ell|) = 0$ dunque $\lim_{n\to+\infty} |a_n| = |\ell|$

2) prese $\{a_n\}_n = \{(-1)^n\}_{n\in\mathbb{N}}$ io so che "$a_n \not\xrightarrow[n\to+\infty]{}$" ovvero "$\nexists \lim_{n\to+\infty} a_n$"

però $\{|a_n|\}_n = \{1\}_n$ e dunque $\exists \lim_{n\to+\infty} |a_n| = 1$

3) Nel caso in cui $\ell=0$ si ha

$$|a_n| \xrightarrow[n\to+\infty]{} \ell=0 \quad \underline{\text{sse}} \quad a_n \xrightarrow[n\to+\infty]{} 0$$

$\Leftarrow$ l'ho già provato: vedi 1)

$\Rightarrow$ Provo che $|a_n| \xrightarrow[n\to+\infty]{} 0 \Rightarrow a_n \xrightarrow[n\to+\infty]{} 0$

$|a_n| \xrightarrow[n\to+\infty]{} 0$ $\underline{\text{se}}$ $\forall \varepsilon>0\ \exists \bar{n}>0 : \forall n>\bar{n}$ $\quad \big||a_n|-0\big|<\varepsilon$

se $\forall \varepsilon>0\ \exists \bar{n}>0\ \forall n>\bar{n}$ $\quad |a_n|<\varepsilon$

se $\lim_{n\to+\infty} a_n = 0$ $\quad \blacksquare$

Esercizio (limite fondamentale)

$$\lim_{n\to+\infty} q^n = \begin{cases} +\infty & 1<q \\ 1 & 1=q \\ 0 & 0\le q<1 \end{cases}$$

dim

$q>1 \Rightarrow q = 1+\varepsilon$ con $\varepsilon>0$

$$\Rightarrow \underbrace{q^n}_{b_n} = (1+\varepsilon)^n \geq 1+n\varepsilon = a_n$$

↑ Disuguaglianza Bernoulli

Ora $\begin{cases} \lim_{n\to+\infty} a_n = +\infty \\ a_n \le b_n \ \forall n \end{cases} \Rightarrow \exists \lim_{n\to+\infty} b_n = \lim_{n\to+\infty} q^n = +\infty$
(Teorema Confronto)

$q=1 \qquad q^n = 1^n = 1 \xrightarrow[n\to+\infty]{} 1$

$0<q<1 \Rightarrow \frac{1}{q} > 1 \Rightarrow \lim_{n\to+\infty} \left(\frac{1}{q}\right)^n = +\infty$
per il punto $q>1$

$$\Rightarrow \lim_{n\to+\infty} q^n = \lim_{n\to+\infty} \frac{1}{\left(\frac{1}{q}\right)^n} = \frac{1}{+\infty} = 0^+$$

$$q=0 \Rightarrow q^n = 0^n = 0 \xrightarrow[n\to+\infty]{} 0$$

$\boxed{///}$

Teorema (Criterio della radice)

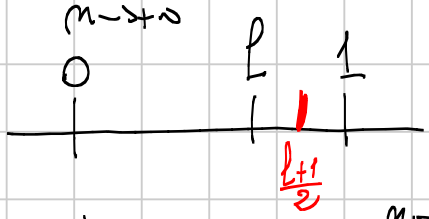
$\{a_n\}_n$ successione reale $a_n \geq 0$ t.c. $\exists \lim_{n\to+\infty} \sqrt[n]{a_n} = \ell$

1) $0 \leq \ell < 1 \Rightarrow \lim_{n\to+\infty} a_n = 0$

2) $1 < \ell \Rightarrow \lim_{n\to+\infty} a_n = +\infty$

dim

1) $0 \leq \ell < 1$



per ipotesi $\forall \varepsilon > 0\ \exists \bar{n} > 0 : \forall n > \bar{n} \quad \ell - \varepsilon < \sqrt[n]{a_n} < \ell + \varepsilon$

prendo ε in modo che $\ell + \varepsilon = \frac{1+\ell}{2}$ ovvero $\varepsilon = \frac{1-\ell}{2}$

fisso $\varepsilon = \frac{1-\ell}{2}$ $\exists \bar{n} > 0 : \forall n > \bar{n} \quad 0 \leq \sqrt[n]{a_n} < \frac{1+\ell}{2} \ (< 1)$

$$\Downarrow$$

$$\forall n > \bar{n} \quad 0 \leq a_n < \left(\frac{1+\ell}{2}\right)^n = b_n$$

1) $0 \leq a_n \leq b_n \quad \forall n > \bar{n}$

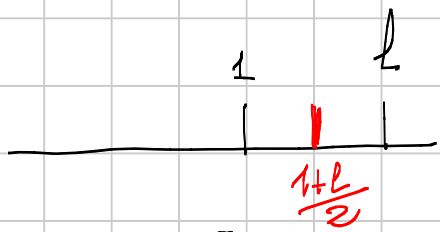
2) $\lim_{n\to+\infty} b_n = 0$ (poiché $\frac{1+\ell}{2} < 1$!!!)

$\lim_{n\to+\infty} 0 = 0$

$$\Rightarrow \lim_{n\to+\infty} a_n = 0$$

(Teorema 2 carabinieri)

2) $1 < \ell < +\infty$



per ipotesi $\forall \varepsilon > 0\ \exists \bar{n} > 0 : \forall n > \bar{n} \quad \ell - \varepsilon < \sqrt[n]{a_n} < \ell + \varepsilon$

prendo ε : $\ell - \varepsilon = \frac{1+\ell}{2}$ ovvero $\varepsilon = \frac{\ell-1}{2}$

$\Downarrow$

$\varepsilon = \frac{\ell-1}{2}$ $\exists \bar{n} > 0 : \forall n > \bar{n}$ $\quad \frac{1+\ell}{2} < \sqrt[n]{a_n}$

$\Downarrow$

$$\forall n > \bar{n} \quad \left(\frac{1+\ell}{2}\right)^n < a_n \qquad \left(\text{con } \frac{1+\ell}{2} > 1 \text{ !!}\right)$$

ma siamo quindi $\begin{cases} \left(\frac{1+\ell}{2}\right)^n < a_n \ \forall n > \bar{n} \\ \lim\limits_{n \to +\infty} \left(\frac{1+\ell}{2}\right)^n = +\infty \quad \left(\text{poiché } \frac{1+\ell}{2} > 1\right) \end{cases}$

$\Longrightarrow$ (Teorema Confronto) $\quad \exists \lim\limits_{n \to +\infty} a_n = +\infty$

$\ell = +\infty \quad \forall M \in \mathbb{R} \ \exists \bar{n} : \forall n > \bar{n} \quad M < \sqrt[n]{a_n}$

$\Downarrow$

prendo $M = 2 \quad \exists \bar{n} > 0 : \forall n > \bar{n} \quad 2 < \sqrt[n]{a_n}$

$M = 2 \quad \exists \bar{n} > 0 : \forall n > \bar{n} \quad 2^n < a_n$

e concludo osservando che $\lim\limits_{n \to +\infty} 2^n = +\infty$! $\blacksquare$

Oss : Il caso $\ell = 1$ non è decidibile

Teorema (Criterio del Rapporto)

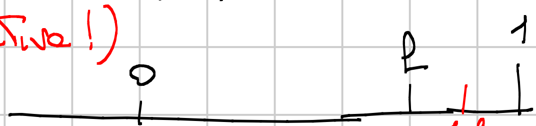
$\{a_n\}_n$ successione, $a_n > 0 \ \forall n$; $\exists \lim\limits_{n \to +\infty} \frac{a_{n+1}}{a_n} = \ell$

1) $0 \leq \ell < 1 \quad \Longrightarrow \quad \exists \lim\limits_{n \to +\infty} a_n = 0$

2) $1 < \ell \quad \Longrightarrow \quad \exists \lim\limits_{n \to +\infty} a_n = +\infty$

dim (facoltativa!)

1) $0 \leq \ell < 1$



$\frac{1+\ell}{2} = \ell + \varepsilon \Rightarrow \varepsilon = \frac{1-\ell}{2}$

Preso $\varepsilon = \frac{1-\ell}{2}$ $\exists \bar{n} : \forall n > \bar{n}$ $\quad 0 \leq \frac{a_{n+1}}{a_n} < \ell + \varepsilon = \frac{1+\ell}{2}$

$\Downarrow$

$$\forall n > \bar{n} \quad 0 \leq a_{n+1} < \left(\frac{1+\ell}{2}\right) \cdot a_n$$

$n=\bar{n} \quad 0 \le a_{\bar{n}+1} < \frac{1+\ell}{2} a_{\bar{n}}$

$n=\bar{n}+1 \quad 0 \le a_{\bar{n}+2} < \frac{1+\ell}{2} a_{\bar{n}+1} < \left(\frac{1+\ell}{2}\right)^2 a_{\bar{n}}$

$n=\bar{n}+2 \quad 0 \le a_{\bar{n}+3} < \frac{1+\ell}{2} a_{\bar{n}+2} < \dots < \left(\frac{1+\ell}{2}\right)^3 a_{\bar{n}}$

- - - -

$$n>\bar{n} \quad 0 \le a_n < \left(\frac{1+\ell}{2}\right)^{n-\bar{n}} a_{\bar{n}} = \underbrace{\boxed{\left[a_{\bar{n}} \cdot \left(\frac{2}{1+\ell}\right)^{\bar{n}}\right]}}_{=K>0} \cdot \left(\frac{1+\ell}{2}\right)^n$$

dunque

$$\begin{cases} \forall n>\bar{n} \quad 0 \le a_n < K \cdot \left(\frac{1+\ell}{2}\right)^n \\ \lim_{n\to+\infty} \left(\frac{1+\ell}{2}\right)^n = 0 \end{cases} \Rightarrow \lim_{n\to+\infty} a_n = 0$$

analogamente si procede per $1<\ell$ ▥

Teorema (Rapporto ⇒ Radice)

$\{a_n\}_n$ successione reale, $a_n>0 \ \forall n$

Se $\exists \lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \ell \in \overline{\mathbb{R}}$ allora $\exists \lim_{n\to+\infty} \sqrt[n]{a_n} = \ell$

dim (facoltativa)

$0<\ell<+\infty$

$\forall \varepsilon>0 \ \exists \bar{n}: \forall n>\bar{n} \quad \ell-\varepsilon < \frac{a_{n+1}}{a_n} < \ell+\varepsilon$

procedendo come nel criterio rapporto ottengo, prendendo $\varepsilon>0$: $\ell-\varepsilon>0$, che

$\forall \varepsilon>0 \ \exists \bar{n}: \forall n>\bar{n} \quad (\ell-\varepsilon)^{n-\bar{n}} \cdot a_{\bar{n}} < a_n < (\ell+\varepsilon)^{n-\bar{n}} \cdot a_{\bar{n}}$

ovvero

$$\forall n>\bar{n} \quad \sqrt[n]{\frac{a_{\bar{n}}}{(\ell-\varepsilon)^{\bar{n}}}} \cdot (\ell-\varepsilon) < \sqrt[n]{a_n} < (\ell+\varepsilon)\sqrt[n]{\frac{a_{\bar{n}}}{(\ell+\varepsilon)^{\bar{n}}}}$$

$$\text{ma} \quad \lim_{n\to+\infty} \sqrt[n]{\frac{a_{\bar{n}}}{(\ell-\varepsilon)^{\bar{n}}}} = \lim_{n\to+\infty} \sqrt[n]{\frac{a_{\bar{n}}}{(\ell+\varepsilon)^{\bar{n}}}} = 1$$

e dunque

$$\exists n_1 : \forall n>n_1 : \quad 1-\varepsilon < \sqrt[n]{\frac{a_{\bar{n}}}{(\ell-\varepsilon)^{\bar{n}}}} \quad e \quad \sqrt[n]{\frac{a_{\bar{n}}}{(\ell+\varepsilon)^{\bar{n}}}} < 1+\varepsilon$$

e dunque

$$\forall \varepsilon>0 \ \exists \tilde{n}=\max(n_1,\bar{n}) : \forall n>\tilde{n} \quad (1-\varepsilon)(\ell-\varepsilon) < \sqrt[n]{a_n} < (1+\varepsilon)(\ell+\varepsilon)$$

e per l'arbitrarietà di ε $\lim_{n\to+\infty} \sqrt[n]{a_n} = \ell$ ▥

Esercizio 1) provare che $\lim_{n\to+\infty} \sqrt[n]{a} = 1 \quad \forall a > 0$

2) provare che $\lim_{n\to+\infty} \sqrt[n]{n} = 1$

dim

$$a > 1 \Rightarrow \sqrt[n]{a} = 1 + b_n \quad \text{con } b_n > 0$$

$$\Rightarrow a = (1+b_n)^n \underset{\uparrow \text{Bernoulli}}{\geq} 1 + n b_n \Rightarrow$$

$$\Rightarrow \frac{a-1}{n} \geq b_n \geq 0 \quad \forall n \geq 1 \Rightarrow \exists \lim_{n\to+\infty} b_n = 0^+$$

$\frac{a-1}{n} \xrightarrow[n\to+\infty]{} 0^+$, $\quad 0 \xrightarrow[n\to+\infty]{} 0^+$

$$\Downarrow$$

$$\exists \lim_{n\to+\infty} \sqrt[n]{a} = \lim_{n\to+\infty} 1 + b_n = 1$$

$$a = 1 \quad \sqrt[n]{1} = 1 \xrightarrow[n\to+\infty]{} 1$$

$$0 < a < 1 \Rightarrow \frac{1}{a} > 1 \Rightarrow \lim_{n\to+\infty} \sqrt[n]{\frac{1}{a}} = 1$$

$$\Rightarrow \lim_{n\to+\infty} \frac{1}{\sqrt[n]{a}} = 1 \Rightarrow \lim_{n\to+\infty} \sqrt[n]{a} = 1$$

2) $\sqrt[n]{n} = 1 + b_n \Rightarrow n = (1+b_n)^n \underset{\uparrow \text{Bernoulli}}{\geq} 1 + n b_n$

$b_n \geq 0$

$$\Rightarrow \frac{n-1}{n} \geq b_n \geq 0$$

non riesco a concludere poiché $\frac{n-1}{n} \xrightarrow[n\to+\infty]{} 1$

devo migliorare la stima

$$\sqrt[n]{n} = (1+b_n)^2 = 1 + b_n^2 + 2b_n \quad \text{con } \underline{b_n \geq 0}$$

$$\Downarrow$$

$$\sqrt{n} = (1+b_n)^n \underset{\uparrow \text{Bernoulli}}{\geq} 1 + n b_n$$

$$\Downarrow$$

$$\frac{\sqrt{n}-1}{n} \geq b_n \geq 0 \Rightarrow b_n \xrightarrow[n\to+\infty]{} 0$$

$\frac{\sqrt{n}-1}{n} \xrightarrow[n\to+\infty]{} 0$, $\quad 0 \xrightarrow[n\to+\infty]{} 0$

$\mathcal{M}$

Esempio ($\ell=1$ è indecidibile)

$$\lim_{n\to+\infty} \sqrt[n]{n} = 1 \quad e \quad \lim_{n\to+\infty} n = +\infty$$

Viceversa

$$\lim_{n\to+\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{n}} = 1 \quad e \quad \lim_{n\to+\infty} \frac{1}{n} = 0^+$$

$$\lim_{n\to+\infty} \frac{n+1}{n} = 1 \quad e \quad \lim_{n\to+\infty} a_n = \lim_{n\to+\infty} n = +\infty$$

$$\lim_{n\to+\infty} \frac{1/(n+1)}{1/n} = 1 \quad e \quad \lim_{n\to+\infty} a_n = \lim_{n\to+\infty} 1/n = 0$$

Esercizio (versione rapida)

1) $\sqrt[n]{a} \xrightarrow[n\to+\infty]{} 1 \quad \forall a>0$

2) $\sqrt[n]{n} \xrightarrow[n\to+\infty]{} 1$

dim

1) $\{a_n\} = \{a\}$ $\qquad a_n = a \ \forall n$ (succ. costante)

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{a}{a} = 1 \Rightarrow \lim_{n\to+\infty} \sqrt[n]{a_n} = \lim_{n\to+\infty} \sqrt[n]{a} = 1$$

2) $\{a_n\}_n = \{n\}_n$ $\qquad a_n = n \ \forall n$

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{n+1}{n} = 1 \Rightarrow \lim_{n\to+\infty} \sqrt[n]{a_n} = \lim_{n\to+\infty} \sqrt[n]{n} = 1$$

Esercizio (Limite fondamentale)

$$\lim_{n\to+\infty} \frac{q^n}{n^k} = \begin{cases} +\infty & q>1 \quad \forall k\in\mathbb{N} \\ 0 & q=1 \quad k\geq 1 \\ 0 & 0\leq q<1 \quad \forall k\in\mathbb{N} \end{cases}$$

dim

$q=1 \quad \lim_{n\to+\infty} \frac{q^n}{n^k} = \lim_{n\to+\infty} \frac{1}{n^k} = \frac{1}{+\infty} = 0^+$

$0 \leq q < +\infty \quad q \neq 1 \quad a_n = \frac{q^n}{n^k}$

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{q^{n+1}}{(n+1)^k} \cdot \frac{n^k}{q^n} = \lim_{n\to+\infty} q \cdot \frac{1}{\left(1+\frac{1}{n}\right)^k}$$

$$= q \lim_{n\to+\infty} \frac{1}{\left(1+\frac{1}{n}\right)^k} = q$$

$0 \leq q < 1 \implies \lim_{n\to+\infty} \frac{q^n}{n^k} = 0 \quad \forall k \in \mathbb{N}$

$1 < q \implies \lim_{n\to+\infty} \frac{q^n}{n^k} = +\infty \quad \forall k \in \mathbb{N}$

Criterio del rapporto

Esercizio Calcolare $\lim_{n\to+\infty} \frac{(2n)!}{2\cdot(n!)}$

Dim

$a_n = \frac{(2n)!}{2\cdot(n!)} \qquad \lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim \frac{(2n+2)!}{(n+1)!\cdot\cancel{2}} \cdot \frac{\cancel{2}\cdot n!}{(2n)!}$

$$= \lim_{n\to+\infty} \frac{(2n+1)(2n+2)}{n+1} = +\infty$$

$\implies \lim_{n\to+\infty} a_n = +\infty$

Criterio Rapporto

Teorema (limiti successioni monotone)

$\{a_n\}_n$ successione reale

$\{a_n\}_n$ debolmente crescente (decrescente)

Allora $\exists \lim_{n\to+\infty} a_n = \sup_n \{a_n\}$ $\left(\inf \{a_n\}_n\right)$

dim

$\{a_n\}_n$ debolmente crescente ovvero $a_n \leq a_{n+1}$ $\forall n \in \mathbb{N}$ (*)

$L = \sup\{a_n : n \in \mathbb{N}\} = \begin{cases} +\infty \\ \in \mathbb{R} \end{cases}$

$\boxed{L = +\infty}$ $L = \sup\{a_n\} \iff \begin{cases} \forall M \in \mathbb{R} \ \exists \bar{n} : M < a_{\bar{n}} \\ (\forall M \in \mathbb{R} \ \exists a_{\bar{n}} : M < a_{\bar{n}}) \\ + \\ \forall n > \bar{n} \quad a_{\bar{n}} \leq a_n \end{cases}$

$\Rightarrow \forall M \in \mathbb{R} \ \exists \bar{n} > 0 ; \forall n > \bar{n} \quad M < a_{\bar{n}} \leq a_n$

$\Rightarrow \forall M \in \mathbb{R} \ \exists \bar{n} > 0 : \forall n > \bar{n} \quad M < a_n$

$\exists \lim_{n\to+\infty} a_n = +\infty$

$\boxed{L = \sup\{a_n\} \in \mathbb{R}}$ $\iff \begin{cases} \forall \varepsilon > 0 \ \exists \bar{n} = \bar{n}(\varepsilon) : L - \varepsilon < a_{\bar{n}} \\ a_n \leq L \quad \forall n \in \mathbb{N} \\ + \\ \forall n > \bar{n} \quad a_{\bar{n}} \leq a_n \end{cases}$

$\Rightarrow \forall \varepsilon > 0 \ \exists \bar{n} = \bar{n}(\varepsilon) > 0 ; \forall n > \bar{n} \quad L - \varepsilon < a_{\bar{n}} \leq a_n \leq L$

$\Rightarrow \forall \varepsilon > 0 \ \exists \bar{m} > 0 : \forall m > \bar{m} \quad L - \varepsilon < Q_m < L + \varepsilon$

$\Rightarrow \exists \lim_{m \to +\infty} Q_m = L$ $\boxed{\text{||||}}$

Esercizio (IMPORTANTE !!!!)

La successione $\{e_m\}_m = \left\{\left(1+\frac{1}{m}\right)^m\right\}_m$ è convergente

e converge a $e = \lim_{m\to+\infty}\left(1+\frac{1}{m}\right)^m$ $\quad e \in]2,3[$

dim

$e_m = \left(1+\frac{1}{m}\right)^m$ Abbiamo provato che $\boxed{\forall m \geq 1 \quad e_m < e_{m+1}}$

ovvero che $\{e_m\}_m$ è strettamente crescente

$\Rightarrow \{e_m\}_m$ è debolmente crescente

$\Rightarrow \exists \lim_{m\to+\infty} e_m = e = \sup\{e_m : m \geq 1\}$

Inoltre $\quad e > 2 = e_1$

$$\left(1+\frac{1}{m}\right)^m = \sum_{k=0}^{m} \binom{m}{k} \cdot \frac{1}{m^k} \cdot 1^{m-k} = \sum_{k=0}^{m} \frac{\overbrace{m(m-1)\cdots(m-k+1)}^{k \text{ fattori}}}{k!} \frac{1}{m^k}$$

$$= \sum_{k=0}^{m} \frac{m}{m} \cdot \frac{m-1}{m} \cdot \frac{m-2}{m} \cdot \ldots \cdot \frac{m-k+1}{m} \cdot \frac{1}{k!}$$

$$= \sum_{k=0}^{m} \frac{m}{m} \cdot \left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\left(1-\frac{3}{m}\right) \cdots \left(1-\frac{k-1}{m}\right) \cdot \frac{1}{k!}$$

$$< \sum_{k=0}^{m} \frac{1}{k!} = 1 + \sum_{k=1}^{m} \frac{1}{k!} \overset{(**)}{\leq} 1 + \sum_{k=1}^{m} \frac{1}{2^{k-1}}$$

$$= 1 + \sum_{k=0}^{m-1} \frac{1}{2^k} = 1 + \frac{1-\left(\frac{1}{2}\right)^m}{1-\frac{1}{2}}$$

$\Downarrow$

$$e_m < 1 + \frac{1-\left(\frac{1}{2}\right)^m}{1-\frac{1}{2}} \Rightarrow \lim_{m\to+\infty} e_m \leq 1 + \lim_{m\to+\infty} \frac{1-\left(\frac{1}{2}\right)^m}{1-\frac{1}{2}}$$

$$\Rightarrow \lim_{n\to+\infty} e_n = e \leq 1+2 = 3$$

Ma noi ci voleva $e<3$: con un poco di fatica in più si ha

$$e_n = \sum_{k=0}^{n} \binom{n}{k}\cdot\frac{1}{n^k} = 1+\binom{n}{1}\cdot\frac{1}{n}+\binom{n}{2}\cdot\frac{1}{n^2}+\binom{n}{3}\cdot\frac{1}{n^3}+\sum_{k=4}^{n}\binom{n}{k}\frac{1}{n^k}$$

$$= 1+1+\frac{n(n-1)}{2}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{6}\cdot\frac{1}{n^3}+\sum_{k=4}^{n}\binom{n}{k}\frac{1}{n^k}$$

$$\leq 1+1+\frac{1}{2}+\frac{1}{6}+\sum_{k=4}^{n}\frac{1}{k!}$$

$$\leq \frac{\cancel{16}^{\,8}}{\cancel{6}_{\,3}}+\sum_{k=4}^{n}\frac{1}{2^{k-1}} = \frac{8}{3}+\frac{1}{2}\sum_{k=4}^{n}\frac{1}{2^k}$$

$$\leq \frac{8}{3}+\frac{1}{2}\sum_{k=4}^{\infty}\frac{1}{2^k} = \frac{8}{3}+\frac{1}{2}\cdot\left(\sum_{k=0}^{\infty}\frac{1}{2^k}-\sum_{k=0}^{3}\frac{1}{2^k}\right)$$

$$= \frac{8}{3}+\frac{1}{2}\left(2-\frac{1-\left(\frac{1}{2}\right)^4}{1-\frac{1}{2}}\right) = \frac{8}{3}+\frac{1}{2}\left(2-\frac{15}{\cancel{16}_{8}}\cdot\cancel{2}^{1}\right)$$

$$= \frac{8}{3}+1-\frac{15}{16} = \frac{128+48-45}{48} = \frac{131}{48} = \frac{144-13}{48} = 3-\frac{13}{48}$$

e dunque

$$\forall n\geq 1\quad e_n \leq 3-\frac{13}{48} \Rightarrow \lim_{n\to+\infty} e_n = e \leq 3-\frac{13}{48} < 3 \quad \blacksquare$$

(**)
$$\boxed{\frac{1}{k!}\leq\frac{1}{2^{k-1}}\qquad \forall k\geq 1}$$

ovvero $2^{k-1}\leq k!\quad \forall k\geq 1$

$\boxed{k=1}$ $\quad 2^0=1\leq 1!=1$

$\boxed{k=n}$ $\quad 2^{n-1}\leq n!$ (Hip. induttiva)

Devo provare $2^n\leq(n+1)!$

$$(n+1)! = (n+1)\cdot n! \overset{\text{Hip. Ind.}}{\geq} (n+1)\cdot 2^{n-1} \overset{\forall n\geq 1}{\geq} 2^n$$