Esercizio $f(x) = 3e^{x^2} - 2\log(1+x^2) - \frac{x}{1-2x} - 3\cos x + \operatorname{sen} x$

Determinate: Polinomio di Taylor di ordine 4 centrato in $x_0 = 0$

Ordine e la pp. di f per $x \to 0$

Esercizio $f(x) = \dfrac{3x^\alpha - \operatorname{sen} 3x}{e^{2x} - \log(1+2x) - 1}$

Calcolate $\lim_{x \to 0^+} f(x)$ al variare di $\alpha > 0$

PROBLEMA 3

Calcolate $\lim_{x \to 0} \dfrac{e^{3x}\cos(2x) + \log(1-3x) - (1-x^2)^2}{x - \operatorname{sen} x}$.

PROBLEMA 3

Calcolate al variare di $\alpha \in \mathbb{R}$ il limite

$$\lim_{x \to 0^+} x^\alpha \left(\operatorname{sen} x \cos x + \frac{x}{x^2 - 1}\right).$$

Esercizio $f(x) = 3e^{x^2} - 2\log(1+x^2) - \dfrac{x}{1-2x} - 3\cos x + \operatorname{sen} x$

Determinate: Polinomio di Taylor di ordine 4 centrato in $x_0=0$

Ordine e la p.p. di f per $x \to 0$

dim

$$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} + o(y^4)$$

$$= 1 + y + \frac{y^2}{2} + o(y^2)$$

$$3e^{x^2} = \left(1 + x^2 + \frac{x^4}{2} + o(x^4)\right) \cdot 3$$

$$\boxed{3e^{x^2} = 3 + 3x^2 + \frac{3}{2}x^4 + o(x^4;0)}$$

$$\log(1+y) = y - \frac{y^2}{2} + o(y^2)$$

$$\boxed{2\log(1+x^2) = 2x^2 - x^4 + o(x^4;0)}$$

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + y^4 + o(y^4;0)$$

$$\frac{x}{1-2x} = x\left(1 + 2x + 4x^2 + 8x^3 + o(x^3;0)\right)$$

$$\boxed{\frac{x}{1-2x} = x + 2x^2 + 4x^3 + 8x^4 + o(x^4;0)}$$

$$\boxed{\begin{aligned} &-3\cos x + \operatorname{sen} x = \\ &= -3\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4;0)\right) + x - \frac{x^3}{6} + o(x^4;0) \\ &= -3 + x + \frac{3}{2}x^2 - \frac{x^3}{6} - \frac{x^4}{8} + o(x^4;0) \end{aligned}}$$

$$f(x) = \cancel{3} + 3x^2 + \frac{3}{2}x^4 + o(x^4;0) - 2x^2 + x^4 + o(x^4;0)$$
$$- x - 2x^2 - 4x^3 - 8x^4 + o(x^4;0) \cancel{-3} + x + \frac{3}{2}x^2 - \frac{x^3}{6} - \frac{x^4}{8}$$
$$+ o(x^4;0)$$

$$= x(-1+1) + x^2\left(3 - 2 - 2 + \frac{3}{2}\right) + x^3\left(-4 - \frac{1}{6}\right) + x^4\left(\frac{3}{2} + 1 - 8 - \frac{1}{8}\right)$$
$$+ o(x^4;0)$$

$$\frac{12-56-1}{8}$$

$$\boxed{f(x) = \frac{x^2}{2} - \frac{25}{6}x^3 - \frac{45}{8}x^4 + o(x^4;0)} = \frac{x^2}{2} + o(x^2;0)$$

$$P_4(x) = \frac{x^2}{2} - \frac{25}{6}x^3 - \frac{45}{8}x^4$$

Ordine di $f(x) = 2$

parte principale $f(x) = \dfrac{x^2}{2}$

Esercizio $f(x) = \dfrac{3x^\alpha - \sin 3x}{e^{2x} - \log(1+2x) - 1}$

Calcolare $\lim\limits_{x\to 0^+} f(x)$ al variare di $\alpha > 0$

dim

Proviamo a calcolare $\lim\limits_{x\to 0^+} f$ (perché compare x^α)

quando $\alpha = 1$ $\quad f(x) = \dfrac{3x - \sin 3x}{e^{2x} - \log(1+2x) - 1}$

$$\underbrace{3x - \sin 3x} = \cancel{3x} - \left(\cancel{3x} - \frac{(3x)^3}{6} + o(x^4;0)\right)$$

$$= \frac{9}{2}x^3 + o(x^4;0)$$

$$e^{2x} - \log(1+2x) - 1 = \cancel{1} + \cancel{2x} + o(x;0) - \left(\cancel{2x} + o(x;0)\right) - \cancel{1}$$

$$= o(x;0)$$

$$f(x) = \frac{\frac{9}{2}x^3 + o(x^4;0)}{o(x;0)}$$

NON posso concludere poiché non ho sufficienti informazioni sul denominatore

Devo passare almeno all'ordine 2 nel denominatore

$$e^{2x} - \log(1+2x) - 1 = \cancel{1} + \cancel{2x} + \frac{(2x)^2}{2} + o(x^2;0)$$
$$- \left(\cancel{2x} - \frac{(2x)^2}{2} + o(x^2;0)\right) - \cancel{1}$$

$$= 2x^2 + 2x^2 + o(x^2) = 4x^2 + o(x^2;0)$$

$$f(x) = \frac{3x - \operatorname{sen} 3x}{e^{2x} - \log(1+2x) - 1} = \frac{\frac{9}{2}x^3 + o(x^4;0)}{4x^2 + o(x^2;0)}$$

$$= \frac{\cancel{x^2}\ \frac{9}{2}x + o(x^2;0)}{\cancel{x^2}\ 4 + o(1;0)} \xrightarrow[x \to 0^+]{} \frac{0^+}{4} = 0^+$$

Quando $\alpha = 1$, $\lim_{x \to 0} f(x) = 0$

Considero che $\alpha > 0$

$$f(x) = \frac{3x^\alpha - \operatorname{sen} 3x}{e^{2x} - \log(1+2x) - 1} = \frac{3(x^\alpha - x) + \frac{9}{2}x^3 + o(x^3;0)}{4x^2 + o(x^2;0)}$$

$$3x^\alpha - \operatorname{sen} 3x = 3x^\alpha - \left(3x - \frac{9}{2}x^3 + o(x^3;0)\right)$$

$$= 3(x^\alpha - x) + \frac{9}{2}x^3 + o(x^3;0)$$

$$e^{2x} - \log(1+2x) - 1 = 4x^2 + o(x^2;0)$$

$$f(x) = \frac{3(x^\alpha - x) + \frac{9}{2}x^3 + o(x^3;0)}{4x^2 + o(x^2;0)} \qquad \sqrt{x} = x^{1/2}$$

$$x = o(x^{1/2};0)$$

Se $\boxed{0 < \alpha < 1}$ allora $x = o(x^\alpha;0)$

allora $f(x) = \dfrac{3x^\alpha + o(x^\alpha;0)}{4x^2 + o(x^2;0)} = \dfrac{3 + o(1;0)}{4 \cdot x^{2-\alpha} + o(x^{2-\alpha};0)}$

$$\lim_{x \to 0} f(x) = \frac{3}{0^+} = +\infty$$

Se $\alpha > 1$ allora $f(x) = \dfrac{3(x^\alpha - x) + \frac{9}{2}x^3 + o(x^3;0)}{4x^2 + o(x^2;0)}$

perché

$\alpha > 1 \Rightarrow x^\alpha = o(x)$

$$= \frac{-3x + o(x;0)}{4x^2 + o(x^2;0)} = \frac{-3 + o(1;0)}{4x + o(x;0)}$$

$$\lim_{x \to 0^+} f(x) = \frac{-3}{0^+} = -\infty$$

$$\lim_{x\to 0^+} f(x) \begin{cases} +\infty & 0<\alpha<1 \\ 0 & \alpha=1 \\ -\infty & 1<\alpha \end{cases}$$

PROBLEMA 3

Calcolate $\displaystyle\lim_{x\to 0} \frac{e^{3x}\cos(2x) + \log(1-3x) - (1-x^2)^2}{x - \operatorname{sen} x}$.

$$x - \operatorname{sen} x = x - \left(x - \frac{x^3}{6} + o(x^3;0)\right) = \frac{x^3}{6} + o(x^3;0)$$

$$e^{3x}\cos 2x + \log(1-3x) - (1-x^2)^2 =$$

$$= \left(1 + 3x + \frac{9}{2}x^2 + o(x^2;0)\right)\left(1 - \frac{\cancel{4}^{2}x^2}{\cancel{2}_{1}} + o(x^3;0)\right) + \left((-3x) - \frac{(-3x)^2}{2} + \frac{(-3x)^3}{3}\right.$$

$$\left. - \frac{(-3x)^4}{4} + o(x^4;0)\right) - 1 + 2x^2 - x^4$$

$$= \cancel{1} - 2x^2 + o(x^3;0) + \cancel{3x} - 6x^3 + o(x^4;0) + \frac{9}{2}x^2 - 9x^4 + o(x^5;0) + o(x^2;0)$$
$$\cancel{-3x} - \frac{9}{2}x^2 \cancel{-1} + 2x^2$$

$$= x^2\left(-2 + \frac{9}{2} - \frac{9}{2} + 2\right) + o(x^2;0) = o(x^2;0)$$

$$e^{3x}\cos 2x + \log(1-3x) - (1-x^2)^2 =$$

$$\left(1 + 3x + \frac{9}{2}x^2 + \frac{\cancel{27}^{9}}{\cancel{6}_{2}}x^3 + \frac{\cancel{81}^{27}}{\cancel{24}_{8}}x^4 + o(x^4;0)\right)\left(1 - 2x^2 + \frac{\cancel{16}^{2}}{\cancel{24}_{3}}x^4 + o(x^4;0)\right)$$

$$+\left(-3x - \frac{9}{2}x^2 - \frac{\cancel{27}^{9}}{\cancel{3}_{1}}x^3 - \frac{81}{4}x^4 + o(x^4;0)\right) - 1 + 2x^2 - x^4$$

$$= \cancel{1} \cancel{-2x^2} + \frac{2}{3}x^4 + \cancel{3x} - 6x^3 + \cancel{\frac{9}{2}x^2} - 9x^4 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + o(x^4;0)$$
$$\cancel{-3x} - \cancel{\frac{9}{2}x^2} - 9x^3 - \frac{81}{4}x^4 + o(x^4;0) \cancel{-1} + \cancel{2x^2} - x^4$$

$$= x^3\left(-6 + \frac{9}{2} - 9\right) + x^4\left(\frac{2}{3} - 9 + \frac{27}{8} - \frac{81}{4} - 1\right) + o(x^4;0)$$

$$= \frac{-21}{2}x^3 + o(x^3)$$

$$f(x) = \frac{-\frac{21}{2}x^3 + o(x^3)}{\frac{x^3}{6} + o(x^3)} = \frac{-\frac{21}{2} + o(1;0)}{\frac{1}{6} + o(1;0)} \xrightarrow{x\to 0} -\frac{21}{2}\cdot 6 = -63$$

Osservazione

$$\lim_{x\to 0} \frac{3x^3 + 5x^4 + o(x^4;0)}{-x^3 + o(x^3;0)} = \lim \frac{\text{Polin.} + o(\ )}{\text{Polin.} + o(\ )}$$

$$= \lim_{x\to 0} \frac{3 + 5x + o(x;0)}{-1 + o(1;0)} = \frac{3 + 5\cdot 0 + 0}{-1 + 0} = -3$$

Però

$$\lim_{x\to 0} \frac{3x^3 + 5x^4 + o(x^4;0)}{-x^3 + o(x^3;0)}$$

$$= \lim_{x\to 0} \frac{3x^3 + o(x^3;0)}{-x^3 + o(x^3;0)} = \lim_{x\to 0} \frac{\cancel{x^3}}{\cancel{x^3}} \cdot \frac{3 + o(1;0)}{-1 + o(1;0)}$$

$$= \frac{3+0}{-1+0} = -3$$

Quando dovete calcolare

$$\lim_{x\to x_0} \frac{f(x)}{g(x)}$$

dovete cercare di scrivere $f(x) = \text{p.p.}(f) + o(\quad)$

$g(x) = \text{p.p.}(g) + o(\quad)$

$$\lim_{x\to x_0} \frac{a\cdot|x-x_0|^\alpha + o(|x-x_0|^\alpha; x_0)}{b\,|x-x_0|^\beta + o(|x-x_0|^\beta; x_0)}$$

PROBLEMA 3

Calcolate al variare di $\alpha \in \mathbb{R}$ il limite

$$\lim_{x\to 0^+} x^\alpha \left(\operatorname{sen} x \cos x + \frac{x}{x^2-1} \right).$$

Trasformo $\left(\operatorname{sen} x \cos x - \dfrac{x}{x^2-1}\right)$ in un polinomio

e poi passo al limite

$$+\frac{x}{x^2-1} = -\underset{\uparrow}{x} \cdot \left(\frac{1}{1-x^2}\right) = -x\left(1+(x^2)+(x^2)^2+(x^2)^3+o(x^6;0)\right)$$

$$= -x - x^3 - x^5 - x^6 + o(x^6;0)$$

$$\operatorname{sen} x \cos x - \frac{x}{x^2-1} = \left(x - \frac{x^3}{6} + o(x^4;0)\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5;0)\right)$$

$$- x - x^3 - x^5 - x^6 + o(x^6;0)$$

$$= \cancel{x} - \frac{x^3}{2} + \frac{x^5}{24} + o(x^6;0) - \frac{x^3}{6} + \frac{x^5}{12} + \boxed{o(x^4;0)}$$

$$- \cancel{x} - x^3 + o(x^4;0)$$

$$= x^3\left(-\frac{1}{2} - \frac{1}{6} - 1\right) + o(x^3;0) \qquad \frac{-3-1-6}{6} = -\frac{10}{6}$$

$$f(x) = x^\alpha \cdot \left(-\frac{5}{3}x^3 + o(x^3;0)\right)$$

$$= -\frac{5}{3}x^{3+\alpha} + o(x^{3+\alpha};0) \qquad \text{per } \alpha \geq -3$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(-\frac{5}{3}x^{3+\alpha} + o(x^{3+\alpha};0)\right) = \begin{cases} 0 & \alpha > -3 \\ -\frac{5}{3} & \alpha = -3 \end{cases}$$

Quando $\alpha < -3$

$$\lim_{x\to 0} x^\alpha \cdot \left(-\frac{5}{3}x^3 + o(x^3;0)\right)$$

$$= \lim_{x\to 0} x^{\alpha}\cdot\left(-\frac{5}{3}x^3\right)\cdot \underbrace{\lim_{x\to 0}\frac{-\frac{5}{3}x^3+o(x^3;0)}{-\frac{5}{3}x^3}}_{=1}$$

$$= \lim_{x\to 0} -\frac{5}{3}x^{3+\alpha}\cdot 1 = -\infty$$

INTERMEZZO : Il problema del Monty-Hall

Teorema (criterio $\sqrt[n]{\ }$)

$\{a_n\}$ successione reale $a_n \geq 0$ $\quad \exists \lim_{n \to +\infty} \sqrt[n]{a_n} = l$

1) $0 \leq l < 1$ allora $a_n \xrightarrow[n \to +\infty]{} 0$

2) $1 < l$ " $a_n \to +\infty$

$(*)$

Teorema (criterio rapporto)

$\{a_n\}$ successione reale $a_n \geq 0$ $\quad \exists \lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = l$

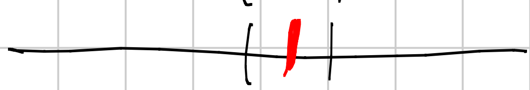
1) $0 \leq l < 1$ allora $a_n \xrightarrow[n \to +\infty]{} 0$

2) $1 < l$ " $a_n \to +\infty$

Idea della dim. nel caso $0 \le \ell < 1$ con $\ell = \lim_{n \to +\infty} \sqrt[n]{a_n}$

$0 \le \ell < 1 \qquad \sqrt[n]{a_n} \to \ell$

$$\boxed{\forall \varepsilon > 0 \ \exists \bar{n} > 0 : \forall n > \bar{n} \qquad \ell - \varepsilon < \sqrt[n]{a_n} < \ell + \varepsilon}$$



$\ell + \varepsilon < 1 \qquad \varepsilon = \frac{1-\ell}{2}$

$\varepsilon = \frac{1-\ell}{2} \quad \exists \bar{n} : \forall n > \bar{n} \qquad 0 \le \sqrt[n]{a_n} < \ell + \frac{1-\ell}{2} = \frac{1+\ell}{2}$

$\exists \bar{n} : \forall n > \bar{n} \qquad \underset{\downarrow \atop 0}{0} \le a_n \le \underbrace{\left(\frac{1+\ell}{2}\right)^n}_{\downarrow \atop 0}$

$\left(\frac{1}{2}\right)^2 > \left(\frac{1}{2}\right)^3 > \left(\frac{1}{2}\right)^4 \cdots\cdots \left(\frac{1}{2}\right)^{10{,}000}$