2014-11-03 - Lezione 21

Lunedì h. 10.30 ÷ 12.30

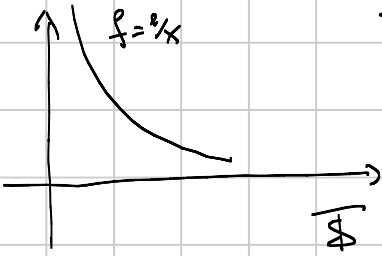
$\lim_{x\to x_0^+} f(x) = \ell \ \underset{Def}{\equiv}\ \forall U\in \mathcal{I}_\ell\ \exists\delta>0 : \forall x\in A\ \ x_0<x<x_0+\delta \Rightarrow f(x)\in U$

$\boxed{\lim_{x\to x_0} f(x) = \ell^+}\ \underset{Def}{\equiv}\ \forall\varepsilon>0\ \exists V\in\mathcal{I}_{x_0} : \forall x\in A\ \ x\in V\setminus\{x_0\} \Rightarrow \ell\le f(x)<\ell+\varepsilon$

e analogamente si definiscono $\lim_{x\to x_0^-} f(x)=\ell$ e $\lim_{x\to x_0} f(x)=\ell^-$

Esempio Data $f(x)=1/x$, provare che $\lim_{x\to+\infty} 1/x = 0^+$

dim



Devo provare che

$\forall\varepsilon>0\ \exists M=M(\varepsilon)>0 : \forall x\in\mathbb{R}\setminus\{0\}\ \ x>M \Rightarrow 0<1/x<\varepsilon$

$\forall\varepsilon>0\ \exists M=M(\varepsilon)>0 : x>M \Rightarrow 0<1/x<\varepsilon$

$\Updownarrow\ \forall\varepsilon>0\ \exists M=M(\varepsilon)>0 : x>M \Rightarrow 1/\varepsilon<x$

preso $M=1/\varepsilon$ ho la Tesi, ovvero ho verificato che $\lim_{x\to+\infty} 1/x = 0^+$ ∎

Teorema (del confronto)

$f,g: A\to\mathbb{R}$, x_0 p.d.a. per A, $f(x)\le g(x)\ \forall x\in A$

i) $\exists \lim_{x\to x_0} f(x)=\ell$, $\exists\lim_{x\to x_0} g(x)=m \Rightarrow \ell\le m$

ii) $\exists \lim_{x\to x_0} f(x)=+\infty \Rightarrow \exists\lim_{x\to x_0} g(x)=+\infty$

iii) $\exists \lim_{x\to x_0} g(x)=-\infty \Rightarrow \exists\lim_{x\to x_0} f(x)=-\infty$

dim Prendo $x_0\in\mathbb{R}$ (il caso $x_0=+\infty,-\infty$ è lasciato ai volenterosi)

i) In questo caso $\ell, m\in\mathbb{R}$ (gli altri casi rientrano in ii) e iii))

$$\text{Hp}\begin{cases}\forall\varepsilon>0\ \exists\delta_1=\delta_1(\varepsilon)>0 : \forall x\in A\ \ 0<|x-x_0|<\delta_1 \Rightarrow \ell-\varepsilon<f(x)<\ell+\varepsilon\\ \forall\varepsilon>0\ \exists\delta_2=\delta_2(\varepsilon)>0 : \forall x\in A\ \ 0<|x-x_0|<\delta_2 \Rightarrow m-\varepsilon<g(x)<m+\varepsilon\\ f(x)\le g(x)\ \forall x\in A\end{cases}$$

preso $\delta=\min\{\delta_1,\delta_2\}>0$ allora

$$\begin{cases}\forall\varepsilon>0\ \exists\delta>0 : \forall x\in A\ \ 0<|x-x_0|<\delta \Rightarrow \begin{cases}\ell-\varepsilon<f(x)<\ell+\varepsilon\\ m-\varepsilon<g(x)<m+\varepsilon\end{cases}\\ f(x)\le g(x)\ \forall x\in A\end{cases}$$

$\Downarrow$

$\Downarrow$

$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in A \quad 0 < |x - x_0| < \delta \Rightarrow \ l - \varepsilon < f(x) \leq g(x) < m + \varepsilon$

$\Downarrow$

$\forall \varepsilon > 0 \quad l - \varepsilon < m + \varepsilon$

$\Downarrow$

$\forall \varepsilon > 0 \quad l - m < 2\varepsilon$

$\Downarrow$

$l - m \leq 0$ che è la Tesi!!

ii) $\begin{cases} \forall M \in \mathbb{R} \ \exists \delta = \delta(M) > 0 : \forall x \in A, 0 < |x - x_0| < \delta \Rightarrow M < f(x) \\ f(x) \leq g(x) \quad \forall x \in A \end{cases}$

$\Rightarrow \forall M \in \mathbb{R} \ \exists \delta = \delta(M) > 0 : \forall x \in A \quad 0 < |x - x_0| < \delta \Rightarrow M < f(x) \leq g(x)$

$\Rightarrow \forall M \in \mathbb{R} \ \exists \delta = \delta(M) > 0 \ \forall x \in A \quad 0 < |x - x_0| < \delta \Rightarrow M < g(x)$

$\Rightarrow \lim_{x \to +\infty} g(x) = +\infty$ che è la Tesi

iii) è identica alla ii) ma con $g(x)$ e $f(x)$ scambiati tra loro e con $g(x) < M$ ////

Teorema (2 Carabinieri)

$f, g, h : A \to \mathbb{R} \quad x_0$ p.d.a. per A

$f(x) \leq g(x) \leq h(x) \quad \forall x \in A$

$\exists \lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = l \in \mathbb{R}$

$\Rightarrow \exists \lim_{x \to x_0} g(x) = l$

dim $x_0 \in \mathbb{R}$

$\begin{cases} \forall \varepsilon > 0 \ \exists \delta_1 = \delta_1(\varepsilon) > 0 : \forall x \in A \quad 0 < |x - x_0| < \delta_1 \Rightarrow l - \varepsilon < f(x) < l + \varepsilon \\ \forall \varepsilon > 0 \ \exists \delta_2 = \delta_2(\varepsilon) > 0 : \forall x \in A \quad 0 < |x - x_0| < \delta_2 \Rightarrow l - \varepsilon < h(x) < l + \varepsilon \\ f(x) \leq g(x) \leq h(x) \quad \forall x \in A \end{cases}$

$\Downarrow$ pongo $\delta = \min\{\delta_1, \delta_2\}$

$\begin{cases} \forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) > 0 : \forall x \in A \quad 0 < |x - x_0| < \delta \Rightarrow \begin{cases} l - \varepsilon < f(x) < l + \varepsilon & 1 \\ l - \varepsilon < h(x) < l + \varepsilon & 2 \end{cases} \\ f(x) \leq g(x) \leq h(x) \quad \forall x \in A \quad 3 \end{cases}$

$\Downarrow$

$$\Downarrow$$

$$\forall \varepsilon>0 \ \exists \delta=\delta(\varepsilon)>0 : \forall x\in A \quad 0<|x-x_0|<\delta \Rightarrow \underset{①}{\ell-\varepsilon<}\underset{③}{f(x)\le}\underset{③}{g(x)\le}\underset{②}{h(x)<}\ell+\varepsilon$$

$$\Downarrow$$

$$\forall \varepsilon>0 \ \exists \delta=\delta(\varepsilon)>0 : \forall x\in A \quad 0<|x-x_0|<\delta \Rightarrow \ell-\varepsilon<g(x)<\ell+\varepsilon$$

$$\Downarrow$$

$$\lim_{x\to x_0} g(x)=\ell \qquad ///$$

Teorema (della permanenza del segno)

$f:A\to\mathbb{R}$ x_0 p.d.a. per A $\quad \exists \lim_{x\to x_0} f(x)=\ell$

Se $\ell>0$ (<0) allora $\exists V\in \mathcal{I}_{x_0} : \forall x\in (A\cap V)\setminus\{x_0\}$ $f(x)>0$ (<0)

dim ($\ell\in\mathbb{R}$)

Per ipotesi $\exists \lim_{x\to x_0} f(x)=\ell$ ovvero

$\forall \varepsilon>0 \ \exists V\in\mathcal{I}_{x_0} : \forall x\in A \ x\in V\setminus\{x_0\} \Rightarrow \ell-\varepsilon<f(x)<\ell+\varepsilon$

devo fare in modo che $\ell-\varepsilon>0$, ovvero prendere $0<\varepsilon<\ell$

$$\Downarrow$$

Fisso $\varepsilon=\ell/2$ $\exists V\in\mathcal{I}_{x_0} : \forall x\in A, x\in V\setminus\{x_0\} \Rightarrow \ell-\frac{\ell}{2}=\frac{\ell}{2}<f(x)$

$$\Downarrow$$

$$\forall x\in (A\cap V)\setminus\{x_0\} \quad f(x)>\ell/2>0 \qquad ///$$

Teorema (f ha limite finito $\Rightarrow$ f è localmente limitata)

$f:A\to\mathbb{R}$ x_0 pda per A

$\exists \lim_{x\to x_0} f(x)=\ell$, $\ell\in\mathbb{R}$ $\Rightarrow \exists V\in\mathcal{I}_{x_0} : \exists M=|\ell|+1 :$
$\forall x\in (V\cap A)\setminus\{x_0\}$ $|f(x)|\le M$

dim

per ipotesi $\forall\varepsilon>0 \ \exists V\in\mathcal{I}_{x_0} : \forall x\in A \ x\in V\setminus\{x_0\} \Rightarrow \ell-\varepsilon<f(x)<\ell+\varepsilon$

$$\Downarrow$$

prendo $\varepsilon=1$ $\exists V=V(1)\in\mathcal{I}_{x_0} : \forall x\in A \ x\in V\setminus\{x_0\} \Rightarrow \ell-1<f(x)<\ell+1$

$$\ell-1 \ge -|\ell|-1 \qquad \ell+1\le|\ell|+1$$

$$\Downarrow$$

$\exists V\in\mathcal{I}_{x_0} \ \exists M=|\ell|+1 : \forall x\in (A\cap V)\setminus\{x_0\}$ $|f(x)|<\big||\ell|+1\big|=M$ ///

Ricordo che

$\lim_{x \to x_0} f(x) = f(x_0)$ quando f continua in x_0 p.d.a. dom(f)

Teorema (permanenza segno funzioni continue)

$f: A \to \mathbb{R}$ $\quad x_0 \in A$ $\quad f$ continua in x_0

Se $f(x_0) > 0$ allora $\exists V \in \mathcal{I}_{x_0} : \quad \forall x \in V \cap A \quad f(x) > 0$

Teorema (f continua in x_0 allora f localmente limitata in x_0)

$f: A \to \mathbb{R}$ $\quad x_0 \in A$ $\quad f$ continua in x_0

allora $\exists V \in \mathcal{I}_{x_0}$ $\exists M = |f(x_0)| + 1 : \quad \forall x \in A \cap V \quad |f(x)| \le M$

COMPOSIZIONE di funzioni continue

Data $f(x) = x^2$, data $g(x) = \cos x$

queste sono continue $\forall x \in \mathbb{R}$

Posso considerare $\quad x \overset{f}{\longmapsto} x^2 \overset{g}{\longrightarrow} \cos(x^2)$

$$(g \circ f)(x) = g(f(x))$$

anche questa è continua $(g \circ f)$ per il Teorema seguente

Teorema (Composizione funzioni continue)

$f: A \to \mathbb{R}$ $\quad g: B \to \mathbb{R}$ $\quad \Omega = f^{-1}(f(A) \cap B) \neq \emptyset$

f continua in $x_0 \in \Omega$

g " " $f(x_0)$

Allora $(g \circ f)(x) = g(f(x))$ è continua in x_0

e si ha $(g \circ f)(x_0) = g(f(x_0))$

dim (non richiesta all'orale)

$$\begin{cases} \forall \rho>0\ \exists \delta=\delta(\rho)>0 : \forall x\in\Omega\quad |x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\rho \\ \forall \varepsilon>0\ \exists \rho=\rho(\varepsilon)>0 : \forall y\in B\quad |y-y_0|<\rho \Rightarrow |g(y)-g(y_0)|<\varepsilon \end{cases}$$

$\Omega = f^{-1}(f(A)\cap B)$ $\quad x\in\Omega \Rightarrow f(x)\in f(A)$ e $f(x)\in B$

$$\Downarrow$$

$$\begin{cases} \forall \rho>0\ \exists \delta=\delta(\rho) : \forall x\in\Omega\quad |x-x_0|<\delta \Rightarrow \begin{cases} |f(x)-f(x_0)|<\rho \\ f(x)\in B \end{cases} \\ \forall \varepsilon>0\ \exists \rho=\rho(\varepsilon)>0 : \forall y\in B\quad |y-y_0|<\rho \Rightarrow |g(y)-g(y_0)|<\varepsilon \end{cases}$$

$$\Downarrow$$

$$\forall \varepsilon>0\ \exists \rho=\rho(\varepsilon)\ \exists \delta=\delta(\rho(\varepsilon))=\delta(\varepsilon) : \forall x\in\Omega\quad |x-x_0|<\delta \Rightarrow$$

$$\Rightarrow \begin{matrix} |f(x)-f(x_0)|<\rho \\ f(x)\in B \end{matrix} \Rightarrow |g(f(x))-g(f(x_0))|<\varepsilon$$

$$\Downarrow$$

$$\forall \varepsilon>0\ \exists \delta>0 : \forall x\in\Omega\quad |x-x_0|<\delta \Rightarrow |(g\circ f)(x)-(g\circ f)(x_0)|<\varepsilon$$

ovvero $g\circ f$ continua in x_0 ▨

Teorema (Cambio variabile nei limiti)

$f: A\to\mathbb{R}$ $\quad x_0$ p.d.a. per $\Omega = f^{-1}(f(A)\cap B)$

$g: B\to\mathbb{R}$ $\quad y_0$ " " $B\cap f(A)$

$\exists \lim_{x\to x_0} f(x) = y_0$

$\exists \lim_{y\to y_0} g(x) = l$

Se vale una delle seguenti ipotesi

i) $\exists W_0\in \mathcal{I}_{x_0} : g(x)\neq y_0 \quad \forall x\in (\Omega\cap W_0)\setminus\{x_0\}$

ii) g continua in y_0

allora $\exists \lim_{x\to x_0} (g\circ f)(x) = \lim_{y\to y_0} g(y) = l$

dim (non richiesta all'esame)

① $\forall U\in \mathcal{I}_l\ \exists V\in\mathcal{I}_{y_0} : \forall y\in B \quad y\in V\setminus\{y_0\} \Rightarrow g(y)\in U$

$\forall V\in \mathcal{I}_{y_0}\ \exists W_1\in\mathcal{I}_{x_0} : \forall x\in\Omega \quad x\in W_1\setminus\{x_0\} \Rightarrow f(x)\in V$ e $f(x)\in B$

Se vale la i)

② $\forall V \in \mathcal{I}_{y_0}\ \exists W_1 \in \mathcal{I}_{x_0} : \forall x \in \Omega \quad x \in (W_1 \cap W_0) \setminus \{x_0\} \Rightarrow f(x) \in V \setminus \{y_0\}$ e $f(x) \in B$

①+② $\forall U \in \mathcal{I}_\ell\ \exists V \in \mathcal{I}_{y_0}\ \exists W_1 \in \mathcal{I}_{x_0}\ \forall x \in \Omega \quad x \in (W_1 \cap W_0) \setminus \{x_0\} \Rightarrow$

$f(x) \in V \setminus \{y_0\}$ e $f(x) \in B \Rightarrow g(f(x)) \in U$

ovvero

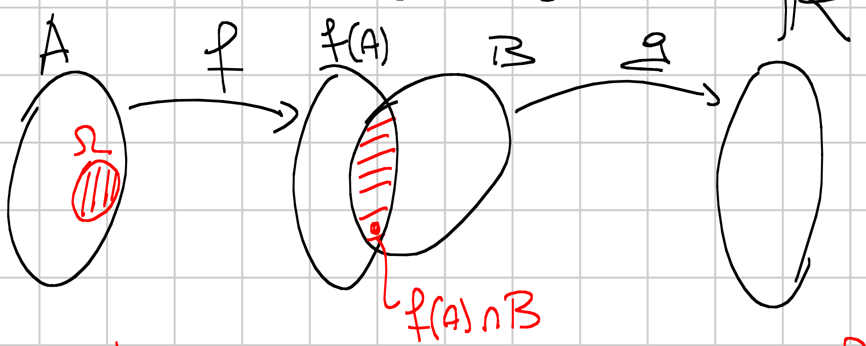
$\forall U \in \mathcal{I}_\ell\ \exists W = W_1 \cap W_0 \in \mathcal{I}_{x_0} : \forall x \in \Omega \quad x \in W \setminus \{x_0\} \Rightarrow f(x) \in U$

cioè $\lim_{x \to x_0} (g \circ f)(x) = \lim_{y \to y_0} g(y) = \ell$

Quando vale la ii), $y \in B \cap V \Rightarrow g(y) \in U$ e quindi si conclude ∎

Esempio

$$\lim_{x \to 0} \frac{\operatorname{sen} 3x}{3x} \overset{?}{=} \lim_{y \to 0} \frac{\operatorname{sen} y}{y}$$



$\Omega = f^{-1}(f(A) \cap B) = \{x \in A : f(x) \in f(A) \cap B\}$

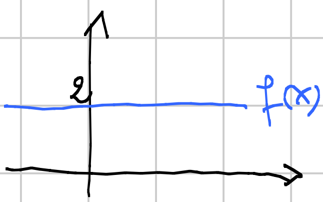
x_0 p.d.a. per Ω

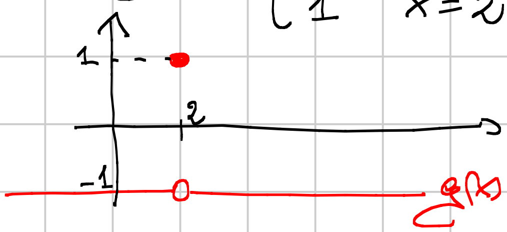
y_0 " " $f(A) \cap B$

Controesempio (al cambio di variabile nei limiti)

$f(x) = 2 \quad \forall x \in \mathbb{R}$

$$g(x) = \begin{cases} -1 & x \neq 2 \\ 1 & x = 2 \end{cases}$$





$\lim_{x \to 0} f(x) = 2 \qquad \lim_{y \to 2} g(x) = -1$ poiché $y=2$ è escluso dal test di limite

Domanda: sarà vero che
$$\lim_{x\to 0}(g\circ f)(x) = \lim_{y\to 2} g(x) = -1 \quad ????$$

NO!

infatti $(g\circ f)(x) = g(f(x)) \overset{f(x)=2\ \forall x!!}{=} g(2) = 1 \quad \boxed{\forall x \in \mathbb{R}}$

dunque $g\circ f$ è una funzione costantemente $=1$

" $\lim_{x\to 0}(g\circ f)(x) = 1$

In questo esempio
- $g(x)$ è discontinua in $y_0 = 2$

e inoltre
- $\forall x \in \mathbb{R}$ $f(x) = 2$

quindi sono contraddette entrambe le ipotesi 1) e 2) del Teorema nel cambio di variabili che in questo caso non vale!!!

Teorema (limitata per infinitesimo = infinitesimo)

$f, g : A \to \mathbb{R}$, x_0 p.d.a. per A

i) $\exists \lim_{x\to x_0} f(x) = 0$

ii) $\exists M>0$: $|g(x)| \leq M$ $\forall x \in A$

$$\Rightarrow \exists \lim_{x\to 0}(f\cdot g)(x) = 0$$

dim

$$\begin{cases} \forall \varepsilon>0\ \exists V \in \mathcal{J}_{x_0} : \forall x \in A \quad x \in V\setminus\{x_0\} \Rightarrow |f(x)| < \varepsilon \\ \forall x \in A \quad |g(x)| \leq M \end{cases}$$

$\Downarrow$

$$\forall \varepsilon>0\ \exists V\in\mathcal{J}_{x_0} : \forall x\in A \quad x\in V\setminus\{x_0\} \Rightarrow |f(x)\cdot g(x)| \leq |f(x)|\cdot M < \varepsilon\cdot M$$

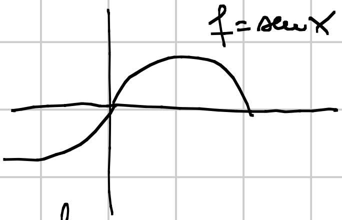
$\Downarrow$

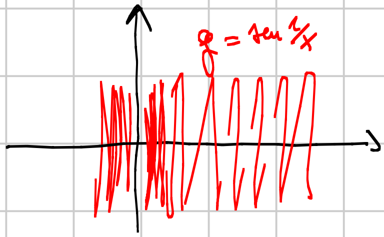
$$\lim_{x\to x_0}(f\cdot g)(x) = 0 \qquad \blacksquare$$

Esempio Calcolare $\lim_{x\to 0} \operatorname{sen} x \cdot \operatorname{sen} \frac{1}{x}$

($\operatorname{sen} x$ ↑ $f(x)$, $\operatorname{sen}\frac{1}{x}$ ↑ $g(x)$)

dim





$\operatorname{sen}\frac{1}{x}$ non è definita in $x=0$!

$\lim_{x\to 0} \operatorname{sen} x = 0$ $\qquad \left|\operatorname{sen}\frac{1}{x}\right| \le 1$

$\Rightarrow$ (per il Teorema precedente) $\lim_{x\to 0} (\operatorname{sen} x)\cdot\left(\operatorname{sen}\frac{1}{x}\right) = 0$ ∎

Algebra in $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$

$c>0$

$c \cdot +\infty \overset{①}{=} +\infty \qquad c\cdot(-\infty) \overset{②}{=} -\infty$

$c/{+\infty} \overset{③}{=} 0^+ \qquad c/{-\infty} \overset{④}{=} 0^-$

$\frac{+\infty}{c} \overset{⑤}{=} +\infty \qquad \frac{-\infty}{c} \overset{⑥}{=} -\infty$

$c+(+\infty) \overset{⑦}{=} +\infty \qquad c+(-\infty) \overset{⑧}{=} -\infty$

$c<0$

$c \cdot +\infty \overset{①'}{=} -\infty \qquad c\cdot(-\infty) \overset{②'}{=} +\infty$

$c/{+\infty} \overset{③'}{=} 0^- \qquad c/{-\infty} \overset{④'}{=} 0^+$

$\frac{+\infty}{c} \overset{⑤'}{=} -\infty \qquad \frac{-\infty}{c} \overset{⑥'}{=} +\infty$

$c+(+\infty) \overset{⑦'}{=} +\infty \qquad c+(-\infty) \overset{⑧'}{=} -\infty$

Teorema (prova di ①)

$f: A \to \mathbb{R}$ $\quad x_0$ p.d.a per A

$\exists \lim_{x\to x_0} f(x) = +\infty \qquad c>0 \ (<0)$

$\Rightarrow \exists \lim_{x\to x_0} c\cdot f(x) = +\infty \ (-\infty)$

dim

per ipotesi $\begin{cases} \forall M>0 \ \exists V \in \mathcal{I}_{x_0} : \forall x \in A \ \ x \in V \setminus \{x_0\} \Rightarrow M < f(x) \\ c > 0 \end{cases}$

$\Rightarrow \forall M>0 \ \exists V \in \mathcal{I}_{x_0} : \forall x \in A \ \ x \in V \setminus \{x_0\} \Rightarrow c \cdot M < c \cdot f(x)$

$\Rightarrow \lim_{x \to x_0} c f(x) = +\infty$

ovvero

per ipotesi $\begin{cases} \forall M>0 \ \exists V \in \mathcal{I}_{x_0} : \forall x \in A \ \ x \in V \setminus \{x_0\} \Rightarrow M < f(x) \\ c < 0 \end{cases}$

$\Rightarrow \forall M>0 \ \exists V \in \mathcal{I}_{x_0} : \forall x \in A \ \ x \in V \setminus \{x_0\} \Rightarrow c \cdot M > c \cdot f(x)$

$\Rightarrow \lim_{x \to x_0} c f(x) = -\infty$ ∎

FORME INDETERMINATE

1. $+\infty + (-\infty)$;
2. $0 \cdot (\pm\infty)$;
3. $\pm\infty / \pm\infty$;
4. $0/0$;
5. 0^0
6. ∞^0
7. 1^∞ ← importante: è legato alla definizione di e^x (e del numero $e = \lim_{n \to +\infty} (1+1/n)^n$)

Problema $f: A \to \mathbb{R}$, x_0 p.d.a. per A $\exists \lim_{x \to x_0} f(x) = l$

$\overset{?}{\Rightarrow} \exists \lim_{x \to x_0} |f(x)| = |l|$?

SI' : $\big||f(x)| - |l|\big| \leq |f(x) - l|$, e quindi

$\lim_{x\to x_0} f(x) = \ell$ → $\forall \varepsilon>0\ \exists V \in \mathcal{I}_{x_0} : \forall x\in A\ \ x\in V\setminus\{x_0\} \Rightarrow |f(x)-\ell|<\varepsilon$

inoltre sappiamo che (disug. del modulo!)

$\big||f(x)|-|\ell|\big| \le |f(x)-\ell|$

$\Rightarrow \forall\varepsilon>0\ \exists V\in\mathcal{I}_{x_0} : \forall x\in A\ \ x\in V\setminus\{x_0\} \Rightarrow \big||f(x)|-|\ell|\big|<\varepsilon$

Problema

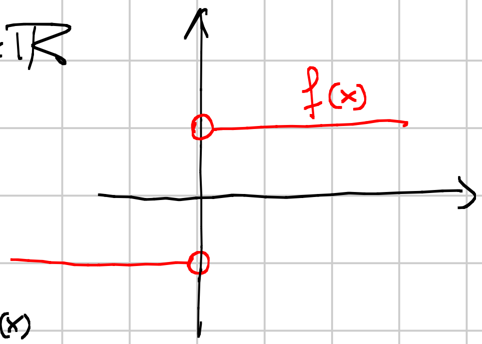
$f: A\to\mathbb{R}$ x_0 p.d.a per A $\exists \lim_{x\to x_0} |f(x)| = |\ell|$

$\overset{??}{\Longrightarrow}$ $\exists \lim_{x\to x_0} f(x) = \ell$??

NO: infatti preso $f(x) = \begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases}$

si ha che $\lim_{x\to 0} |f(x)| = 1$

poiché $|f(x)| = 1$ $\forall x \in\mathbb{R}$



però $\nexists \lim_{x\to 0} f(x)$

in quanto $\lim_{x\to 0^-} f(x) = -1 \neq 1 = \lim_{x\to 0^+} f(x)$