TUTORATO

Titolo nota 08/05/2014

ESERCIZIO:

Calcolare il valore di:

$$\omega = \frac{z^2 - \bar{z}^2}{(z+1)(\bar{z}-1)+1} \qquad \text{quando} \quad z = 1+i$$

$$z^2 - \bar{z}^2 = (1+i)^2 - (1-i)^2 = 1+2i-1-1+2i+1 = 4i$$

$$(z+1)(\bar{z}-1)+1 = (2+i)(-i)+1 = -2i+1+1 = 2-2i$$

$$\omega = \frac{4i}{2-2i} \cdot \frac{2+2i}{2+2i} = \frac{8i-8}{8} = i-1$$

$z \cdot \bar{z} = a^2+b^2$

ESERCIZI: Calcolare le potenze:

1) $(1-i\sqrt{3})^8$ 2) $\omega^6 = \left(\cos\frac{\pi}{3} - i \operatorname{sen}\frac{\pi}{6}\right)^6$

FORMULA DI DE MOIVRE

Se $z = |z|(\cos\theta + i\operatorname{sen}\theta)$,

allora $z^n = |z|^n\left(\cos(n\theta) + i\operatorname{sen}(n\theta)\right)$ $\forall n \geq 1$.

Con $|z| = \sqrt{a^2+b^2}$ e

$$\theta = \begin{cases} \operatorname{arctg}\left(\frac{b}{a}\right) & \text{se } a>0 \\ \operatorname{arctg}\left(\frac{b}{a}\right)+\pi & \text{se } a<0 \\ \frac{\pi}{2} & \text{se } a=0,\ b>0 \\ \frac{3}{2}\pi & \text{se } a=0,\ b<0 \end{cases}$$

1) $z = 1 - i\sqrt{3}$ $|z| = 2$ $\theta = \operatorname{arctg}(-\sqrt{3}) = -\frac{\pi}{3}$

$$z^8 = 2^8\left(\cos\left(-\frac{8}{3}\pi\right) + i\operatorname{sen}\left(-\frac{8}{3}\pi\right)\right) = 2^8\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -2^7(1+i\sqrt{3})$$
$$= -2^7\bar{z}$$

2) $\omega^6 = \left(\cos\frac{\pi}{3} - i\operatorname{sen}\frac{\pi}{6}\right)^6 = \left(\frac{1}{2} - \frac{i}{2}\right)^6$

$$\left.\begin{array}{l} |\omega| = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{\sqrt{2}}{2} \\ \\ \theta = \operatorname{arctg}(-1) = -\frac{\pi}{4} \end{array}\right\} \Rightarrow \omega^6 = \frac{\cancel{2^3}}{2^{\cancel{6}\,3}}\left(\cos\left(-\frac{3}{2}\pi\right) + i\operatorname{sen}\left(-\frac{3}{2}\pi\right)\right)$$
$$= \frac{i}{8}$$

ESERCIZI: Calcolare la radice n-esima dell'unità, per $n = 4$ e $n = 5$.

RADICE n-ESIMA di un numero complesso

$w \in \mathbb{C}$, $w = |w|(\cos\theta + i \operatorname{sen}\theta)$ ha, $\forall n \geq 1$, n radici n-esime w_k, con $k = 0, \dots, n-1$, date da:

$$w_k = |w|^{1/n}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \operatorname{sen}\left(\frac{\theta + 2k\pi}{n}\right)\right)$$

1) Radici quarte dell'unità: $z^4 = 1$

$w = 1$, $|w| = 1$, $\theta = 0$

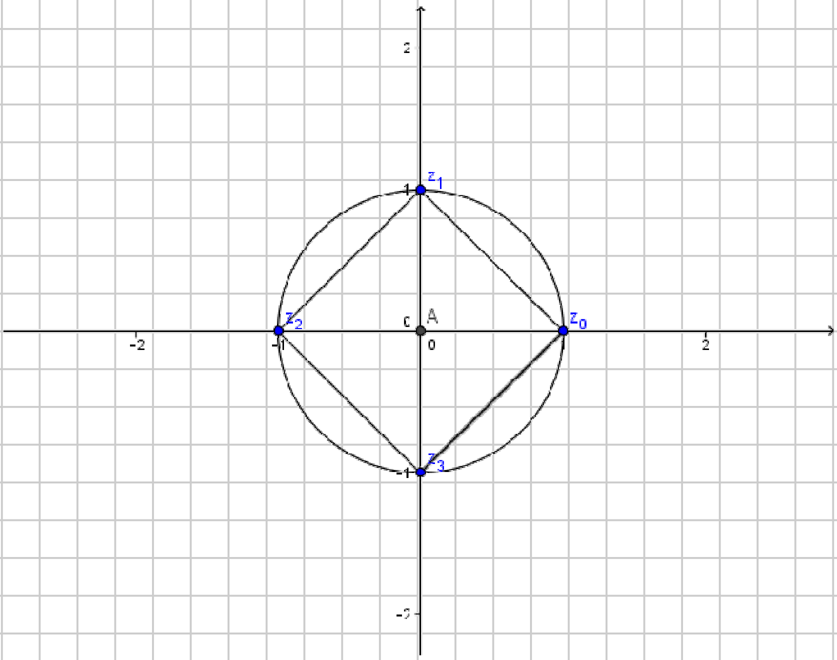
$k = 0, 1, 2, 3$

$z_0 = \cos 0 + i \operatorname{sen} 0 = 1$

$z_1 = \cos \frac{\pi}{2} + i \operatorname{sen} \frac{\pi}{2} = i$

$z_2 = \cos \pi + i \operatorname{sen} \pi = -1$

$z_3 = \cos \frac{3}{2}\pi + i \operatorname{sen} \frac{3}{2}\pi = -i$



2) Radici quinte dell'unità: $z^5 = 1$

$k = 0, 1, 2, 3, 4$

$z_0 = 1$

Non so calcolare sen e cos degli angoli $\frac{\theta + 2k\pi}{5}$ quando $k \neq 0$, quindi utilizzo un'altra strategia:

$z^5 = 1 \iff z^5 - 1 = 0$

Sappiamo, poiché $z_0 = 1$, che $z^5 - 1$ è divisibile per $z - 1$:

$z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1)$

Il secondo fattore si può riscrivere in questo modo:

$$z^4 + z^3 + z^2 + z + 1 = z^2 + z + 1 + \frac{1}{z^2} + \frac{1}{z} = z^2 + \frac{1}{z^2} + 2 - 2 + 1 + z + \frac{1}{z} =$$

$$= \left(z + \frac{1}{z} \right)^2 + \left(z + \frac{1}{z} \right) - 1$$

$$z + \frac{1}{z} \overset{\circledast}{=} z + \bar{z} \overset{z = a+ib}{=} 2a \qquad \circledast : \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \bar{z}$$

$|z| = 1$ poiché $z^5 = 1$

$$4a^2 + 2a - 1 = 0 \quad \Longleftrightarrow \quad a_{1,2} = \frac{-1 \pm \sqrt{1+4}}{4} = \begin{cases} \dfrac{-1+\sqrt{5}}{4} \\[2ex] \dfrac{-1-\sqrt{5}}{4} \end{cases}$$

$$b^2 = |z|^2 - a^2 = 1 - a^2 \quad \Longleftrightarrow \quad b = \pm\sqrt{1-a^2}$$

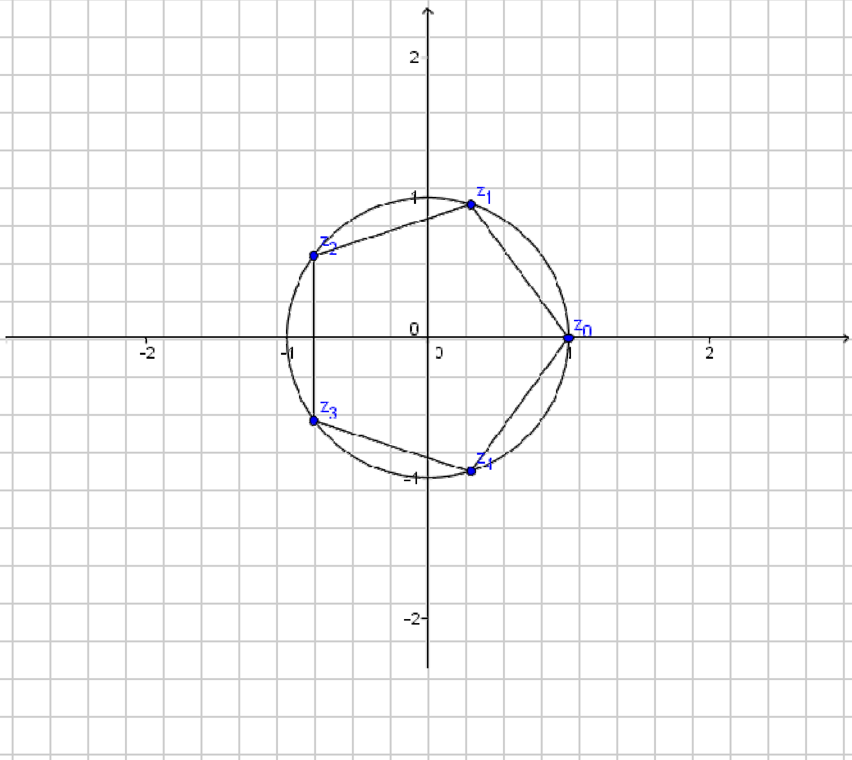
$$b_1 = \sqrt{1 - \left(\frac{-1+\sqrt{5}}{4} \right)^2} = \sqrt{1 - \frac{1 - 2\sqrt{5} + 5}{16}} = \frac{\sqrt{2(5-\sqrt{5})}}{4}$$

$$b_2 = \sqrt{1 - \left(\frac{-1-\sqrt{5}}{4} \right)^2} = \sqrt{1 - \frac{1 + 2\sqrt{5} + 5}{16}} = \frac{\sqrt{2(5+\sqrt{5})}}{4}$$

$$b_3 = -\frac{\sqrt{2(5-\sqrt{5})}}{4}$$

$$b_4 = -\frac{\sqrt{2(5+\sqrt{5}}}{4}$$

$$\Longrightarrow \quad \begin{aligned} z_1 &= a_1 + i b_1 \\ z_2 &= a_2 + i b_2 \\ z_3 &= a_2 + i b_3 \\ z_4 &= a_1 + i b_4 \end{aligned}$$



ESERCIZI: Trovare le soluzioni di:

1) $z^6 + 4z^3 + 8 = 0$

2) $(2z+3)^2 = -27i$

3) $\begin{cases} z^2 + w = 1 \\ 2\bar{w} + iz = 2 \end{cases}$

1) Ponendo $w = z^3$ si ottiene: $w^2 + 4w + 8 = 0$

$$\Longleftrightarrow \quad w_{1,2} = -2 \pm \sqrt{4-8} = \begin{cases} -2+2i \\ -2-2i \end{cases}$$

$\underset{1}{z^3 = w_1}$, $\underset{2}{z^3 = w_2}$ (otteniamo 6 soluzioni)

1: $z = \sqrt[3]{-2+2i}$ $\quad |w_1| = 2\sqrt{2}$, $\vartheta = \pi + \operatorname{arctg}(-1) = \frac{3}{4}\pi$, $n = 3$

$$z_0 = 2^{1/3}(\sqrt{2})^{1/3}\left(\cos\frac{\frac{3}{4}\pi}{3} + i\operatorname{sen}\frac{3/4\,\pi}{3}\right) =$$

$$= \underset{|z|}{\sqrt{2}}\left(\cos\underbrace{\frac{\pi}{4}}_{\alpha_0} + i\operatorname{sen}\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1+i$$

$$\alpha_1 = \frac{\frac{3}{4}\pi + 2\pi}{3} = \frac{11}{12}\pi \;, \quad \alpha_2 = \frac{3/4\,\pi + 4\pi}{3} = \frac{19}{12}\pi$$

2 $\quad |w_2| = 2\sqrt{2} \Rightarrow |z| = \sqrt{2}$

$$\vartheta = \pi + \operatorname{arctg} 1 = \frac{5}{4}\pi \Rightarrow \alpha_3 = \frac{5}{12}\pi \;, \quad \alpha_4 = \frac{5/4\,\pi + 2\pi}{3} = \frac{13}{12}\pi$$

$$\alpha_5 = \frac{5/4\,\pi + 4\pi}{3} = \frac{7}{4}\pi$$

2) $(2z+3)^3 = -27i \iff 2z+3 = \sqrt[3]{-27i} = -3\sqrt[3]{i}$

Calcoliamo $\sqrt[3]{i} = w \quad : \quad |w| = 1 \qquad \theta = \frac{\pi}{2}$

$$\alpha_0 = \frac{\pi/2}{3} = \frac{\pi}{6} \implies w_0 = \frac{\sqrt{3}}{2} + \frac{i}{2}$$

$$\alpha_1 = \frac{\pi/2 + 2\pi}{3} = \frac{5}{6}\pi \implies w_1 = -\frac{\sqrt{3}}{2} + \frac{i}{2}$$

$$\alpha_2 = \frac{\pi/2 + 4\pi}{3} = \frac{\cancel{9}^3}{\cancel{6}_2}\pi \implies w_2 = -i$$

$\implies 2z_k + 3 = -3w_k \quad , \quad k = 0,1,2 \qquad$ (otteniamo 3 soluzioni)

$$2z_0 + 3 = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i \iff z_0 = \frac{-3\sqrt{3} - 6}{4} - \frac{3}{4}i$$

$$2z_1 + 3 = \frac{3\sqrt{3}}{2} - \frac{3}{2}i \iff z_1 = \frac{3\sqrt{3} - 6}{4} - \frac{3}{4}i$$

$$2z_2 + 3 = 3i \iff z_2 = -\frac{3}{2} + \frac{3}{2}i$$

3) $\begin{cases} z^2 + w = 1 \\ 2\bar{w} + iz = 2 \end{cases} \qquad \begin{cases} w = 1 - z^2 \\ 2\overline{(1-z^2)} + iz = 2 \end{cases} \quad (*)$

$(*)$: $2(1-\bar{z}^2) + iz = 2 \iff \cancel{2} - 2\bar{z}^2 + iz = \cancel{2} \iff iz - 2\bar{z}^2 = 0$

$z = a + ib \quad , \quad iz = -b + ia \quad , \quad \bar{z}^2 = (a - ib)^2 = a^2 - 2abi - b^2$

$\implies -b + ia - 2a^2 + 4iab + 2b^2 = 0$

$\quad 2b^2 - 2a^2 - b + i(4ab + a) = 0$

$\iff \begin{cases} 2b^2 - 2a^2 - b = 0 \\ 4ab + a = 0 \end{cases} \qquad \begin{cases} 2b^2 - 2a - b = 0 \\ a(4b+1) = 0 \end{cases}$

$$\begin{cases} a = 0 \\ 2b^2 - b = 0 \end{cases} \iff b(2b-1) = 0 \quad \begin{cases} b = 0 \\ b = \dfrac{1}{2} \end{cases}$$

$$\begin{cases} b = -\dfrac{1}{4} \\ \cancel{2} \cdot \dfrac{1}{\cancel{16}_{8}} - 2a^2 + \dfrac{1}{4} = 0 \iff 2a^2 = \dfrac{3}{8} \iff a = \pm \dfrac{\sqrt{3}}{4} \end{cases}$$

$$z_0 = 0 \implies w_0 = 1$$

$$z_1 = \frac{i}{2} \implies 2\bar{w}_1 + i\left(\frac{i}{2}\right) = 2 \iff \bar{w}_1 = \frac{5}{4} \iff w_1 = \frac{5}{4}$$

$$z_2 = \frac{\sqrt{3}}{4} - \frac{i}{4} \implies 2\bar{w}_2 + i\left(\frac{\sqrt{3}}{4} - \frac{i}{4}\right) = 2$$

$$\bar{w}_2 = 1 - i\frac{\sqrt{3}}{8} - \frac{1}{8} = \frac{7}{8} - i\frac{\sqrt{3}}{8}$$

$$w_2 = \frac{7}{8} + i\frac{\sqrt{3}}{8}$$

$$z_3 = -\frac{\sqrt{3}}{4} - \frac{i}{4} \implies \bar{w}_2 = 1 + i\frac{\sqrt{3}}{8} - \frac{1}{8} = \frac{7}{8} + i\frac{\sqrt{3}}{8}$$

$$w_2 = \frac{7}{8} - i\frac{\sqrt{3}}{8}$$