TUTORATO

Titolo nota 03/04/2014

ESERCIZIO: Dimostrare che $\forall n \in \mathbb{N}$, $n \geq 3$ si ha $(n+1)^n \leq n^{n+1}$

Dim. CASO BASE: $n = 3$. $4^3 = 64 \leq 81 = 3^4$

PASSO INDUTTIVO: Sia vero l'asserto per n.

Proviamolo per $n+1$.

$(n+2)^{n+1} \leq (n+1)^{n+2}$?

$$(a+b)^k = \sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i}$$ BINOMIO DI NEWTON

$$(n+2)^{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} (n+1)^i \cdot 1^{n+1-i} = \sum_{i=0}^{n+1} (n+1) \binom{n+1}{i} \underbrace{(n+1)^{i-1}}_{\leq n^i \text{ HP induttiva}}$$

$$\leq (n+1)(n+1)^{n+1} = (n+1)^{n+2} \qquad \square$$

OPPURE:

Considero la funzione $f(x) = (x+1)^x - x^{(x+1)}$ e vado a vedere dove è ≤ 0, per $x \geq 3$, $x \in \mathbb{R}$.

$x^x = e^{\log x^x} = e^{x \log x}$; $x > 0$

$e^{x \log(x+1)} - e^{(x+1)\log x} \leq 0 \iff \dfrac{e^{x \log(x+1)}}{e^{x \log x} \cdot x} \leq 1$

$g(x) = \dfrac{e^{x \log\left(\frac{x+1}{x}\right)}}{x}$. Quando $g(x) \leq 1$?

Su $[3, +\infty)$ è continua e derivabile

$g(3) = \dfrac{64}{81} \leq 1$

$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} \dfrac{e^{\frac{\log\left(1+\frac{1}{x}\right)}{1/x}}}{x} = 0$ (l'esponente tende a 1)

Per concludere la dimostrazione mi serve g decrescente.

Derivata: $g'(x) = -\frac{1}{x^2} e^{x\log\left(\frac{x+1}{x}\right)} + \frac{1}{x} e^{x\log\left(\frac{x+1}{x}\right)}\left(\log\left(\frac{x+1}{x}\right) + \frac{x^2}{x+1}\cdot\left(-\frac{1}{x^2}\right)\right) = \underbrace{\frac{e^{x\log\left(\frac{x+1}{x}\right)}}{x^2}}_{>0}\left[x\log\left(\frac{x+1}{x}\right) - \frac{x}{x+1} - 1\right]$

Mi chiedo quando g' è ≤ 0, ovvero:

Quando è che $x\log\left(\frac{x+1}{x}\right) - \frac{2x+1}{x+1} \leq 0$?

$\log\left(\frac{x+1}{x}\right) - \frac{2x+1}{x^2+x} \leq 0$?

Derivo: $\frac{x}{x+1}\cdot\left(-\frac{1}{x^2}\right) - \frac{2x^2+2x-(2x+1)^2}{x^2(x+1)^2} =$

$= \frac{-x^2-x-2x^2-2x+4x^2+4x+1}{x^2(x+1)^2} = \frac{x^2+x+1}{x^2(x+1)^2}$

$x^2+x+1 > 0 \quad \forall x \in \mathbb{R} \Rightarrow g'(x) \nearrow \quad \forall x \geq 3$

Inoltre $\lim_{x\to+\infty} g'(x) = 0$

Quindi $g'(x) \leq 0 \quad \forall x \in [3,+\infty) \Rightarrow g(x) \searrow \quad \forall x \in [3,+\infty)$

$\Rightarrow g(3) = \frac{64}{81}$ massimo di g su $[3,+\infty) \Rightarrow g(x) \leq 1$

$\forall x \in [3,+\infty)$ □

TERZO METODO:

Sia $x_n = \frac{(n+1)^n}{n^n} = \left(1+\frac{1}{n}\right)^n \quad \forall n > 0$

- Se $\exists \lim_{n\to+\infty} x_n = \ell \Rightarrow \ell = e$, infatti: $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e < 3$

- $x_n \nearrow$:

$\frac{x_n}{x_{n-1}} = \left(\frac{n+1}{n}\right)^n \cdot \left(\frac{n-1}{n}\right)^{n-1} = \left(\frac{n^2-1}{n^2}\right)^n \cdot \left(\frac{n}{n-1}\right)$

Disuguaglianza di Bernoulli: $(1+x)^n \geq 1+nx \quad \forall x \geq -1 \quad \forall n \in \mathbb{N}$

↑ per induzione

$$\geq \left[1 + x\left(-\frac{1}{n^2}\right)\right] \cdot \left(\frac{n}{n-1}\right) = \frac{n-1}{n} \cdot \frac{n}{n-1} = 1$$

$x = -\frac{1}{n^2}$

- Quindi $\exists \lim_{n \to +\infty} x_n$ ed è < 3

  Inoltre $x_1 = 2$

  $\Rightarrow x_n < 3 \quad \forall n > 0 \quad \Leftrightarrow \quad x_n \leq n \quad \forall n \geq 3$

  $\Rightarrow (n+1)^n \leq n^{n+1} \quad \forall n \geq 3$ □

ESERCIZIO: Studiare la funzione

$$f(x) = (1-x^2)\int_0^x e^{-t^2}\,dt \qquad \text{nell'intervallo } [0,1]$$

Buona def: e^{-t^2} è continua in $[0,1]$

Segno: $f(x) \geq 0 \iff 1-x^2 \geq 0 \iff x^2 \leq 1 \iff -1 \leq x \leq 1$

ovvero sempre in $[0,1]$. $\left(\int_0^x e^{-t^2}\,dt \geq 0 \quad \forall x \in [0,1]\right.$

$\left.\text{poiché } x \geq 0 \text{ e } e^{-t^2} > 0 \ \forall t\right)$

Monotonia: $f'(x) = -2x\int_0^x e^{-t^2}\,dt + (1-x^2)e^{-x^2} \geq 0$?

$$f''(x) = -2\int_0^x e^{-t^2}\,dt - 2xe^{-x^2} - 2xe^{-x^2} - 2x(1-x^2)e^{-x^2} =$$

$$= -2\int_0^x e^{-t^2}\,dt - 6xe^{-x^2} + 2x^3e^{-x^2}$$

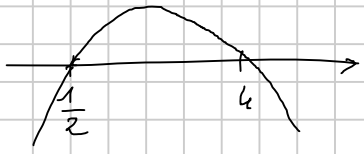
$$f'''(x) = -2e^{-x^2} - 6e^{-x^2} + 12x^2e^{-x^2} + 6x^2e^{-x^2} - 4x^4e^{-x^2}$$

$$= -4x^4e^{-x^2} + 18x^2e^{-x^2} - 8e^{-x^2} =$$

$$= 2e^{-x^2}(-2x^4 + 9x^2 - 4)$$

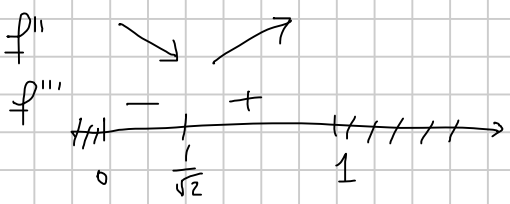
$t = x^2 \qquad -2t^2 + 9t - 4 = 0$

$$t_{1,2} = \frac{-9 \pm \sqrt{81-32}}{-4} = \begin{cases} 4 \\ \frac{1}{2} \end{cases}$$



$$\Rightarrow \frac{1}{2} \leq t \leq 4 \iff \begin{cases} x^2 \geq \frac{1}{2} \\ x^2 \leq 4 \end{cases} \iff \frac{1}{\sqrt{2}} \leq x \leq 1$$

(↑ siamo in $[0,1]$)



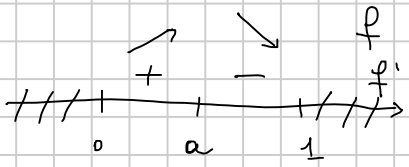
$f''\left(\frac{1}{\sqrt{2}}\right) < 0 \qquad f''(0) = 0$

$f''(1) < 0$

$\Rightarrow f''(x) \leq 0 \quad \forall x \in [0,1]$

$\Rightarrow f'(x) \searrow \ \forall x \in [0,1] \Rightarrow \exists! a \in [0,1]$ t.c. $f'(a) = 0$

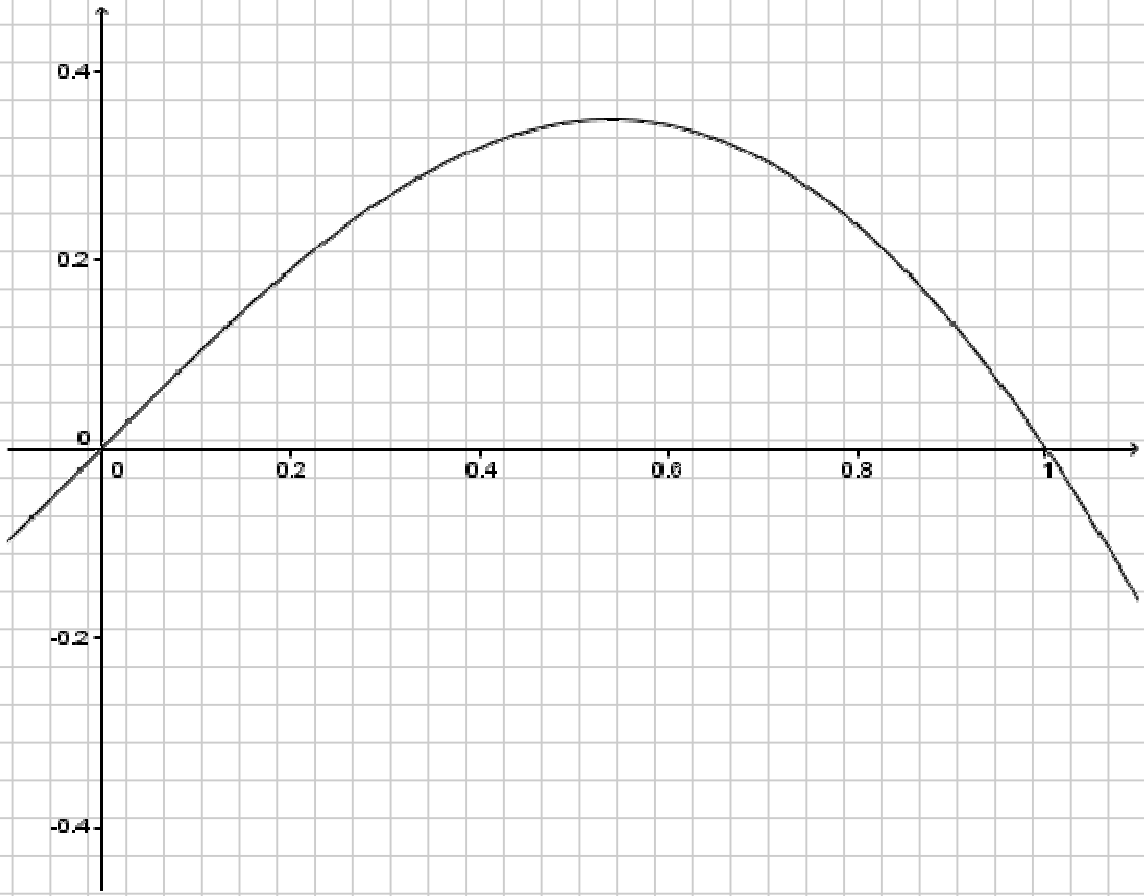
$f'(0) = 1$, $f'(1) < 0 \Rightarrow$



$\Rightarrow a$ è di massimo per f

f è concava in $[0,1]$

Grafico:



ESERCIZIO: Derivare

$$\int_{-x}^{1} e^{t^2}\,dt \quad , \quad \int_{\sin x}^{\cos x} \frac{1}{t^2-1}\,dt$$

- $$\int_{-x}^{1} e^{t^2}\,dt = -\int_{1}^{-x} e^{t^2}\,dt$$

$$x \xrightarrow{-} -x \xrightarrow{F} -\int_{1}^{-x} e^{t^2}\,dt = \int_{1}^{-x} (-e^{t^2})\,dt$$

Da cui: $$\frac{d}{dx}\left(-\int_{1}^{-x} e^{t^2}\,dt\right) = -F'(-x) = -f(-x) =$$

$$= -f(x) = e^{x^2} = F'(x) = \frac{d}{dx}\left(\int_{1}^{x} e^{t^2}\,dt\right)$$

$f(x) = -e^{x^2}$ è pari

- $$\int_{\sin x}^{\cos x} \frac{1}{t^2-1}\,dt = \int_{0}^{\cos x} \frac{1}{t^2-1}\,dt - \int_{0}^{\sin x} \frac{1}{t^2-1}\,dt$$

$$x \xrightarrow{\cos} \cos x \xrightarrow{F} \int_{0}^{\cos x} \frac{1}{t^2-1}\,dt$$

$$x \xrightarrow{\sin} \sin x \xrightarrow{G} \int_{0}^{\sin x} \frac{1}{1-t^2}\,dt$$

$$\frac{d}{dx}\left(\int_{\sin x}^{\cos x} \frac{1}{t^2-1}\,dt\right) = F'(\cos x)\cdot(-\sin x) + G'(\sin x)\cos x =$$

$$= -f(\cos x)\sin x + g(\sin x)\cos x = -\frac{\sin x}{\cos^2 x - 1} + \frac{\cos x}{1-\sin^2 x} =$$

$$= \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{1}{\sin^2 x \cos^2 x} = \frac{4}{\sin^2 2x}$$

ESERCIZIO: Calcolare

$$\int \frac{1}{\sqrt{x^2+4x+7}}\,dx =$$

Oss: $x^2+4x+7 = x^2+4x+4+3 = (x+2)^2+3$

Sostituzione: $t = x+2$, $dt = dx$

$$= \int \frac{1}{\sqrt{t^2+3}}\,dt =$$

Sostituzione: $t = \sqrt{3}\,\text{senh}(y)$; $dt = \sqrt{3}\cosh(y)\,dy$

$$y = \text{settsenh}\left(\frac{t}{\sqrt{3}}\right)$$

$$= \int \frac{1}{\sqrt{3\,\text{senh}^2(y)+3}} \cdot \sqrt{3}\cosh(y)\,dy =$$

$$= \int \frac{1}{\cancel{\sqrt{3}}\,\underbrace{\sqrt{\text{senh}^2(y)+1}}_{=\cosh^2(y)}} \cdot \cancel{\sqrt{3}}\cosh(y)\,dy = y + c =$$

$$= \text{settsenh}\left(\frac{t}{\sqrt{3}}\right) + c = \text{settsenh}\left(\frac{x+2}{\sqrt{3}}\right) + c =$$

$$= \log\left(\frac{x+2}{\sqrt{3}} + \sqrt{\frac{(x+2)^2}{3}+1}\right) + c =$$

$$= \log\left(x+2+\sqrt{x^2+4x+7}\right) \underbrace{-\log\sqrt{3} + c}_{=k}$$