Giovedì 24 ottobre 2013 - ore 8.30 - 10.30

Analisi Matematica 1

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Ricevimento martedì dalle 11.00 - 12.30
(Fatelo sapere via e-mail)

$$\lim_{n\to+\infty} a_n = \ell \in \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$$

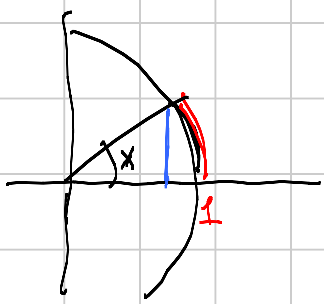
($+\infty$ l'unico punto di accumulazione per $\mathbb{N}$)

$$\forall U \in \mathcal{I}_\ell \quad \exists \bar{n} > 0 : \forall n > \bar{n} \quad a_n \in U$$

Esercizio ① $\lim_{n\to+\infty} \operatorname{sen} \frac{1}{n} = 0$ ② $\lim_{n\to+\infty} \cos \frac{1}{n} = 1$

dim

Fatto Noto: $0 \le |\operatorname{sen} x| \le |x|$



$$0 \le \operatorname{sen} \tfrac{1}{n} \le \tfrac{1}{n} \implies \lim_{n\to+\infty} \operatorname{sen} \tfrac{1}{n} = 0$$

$$\begin{array}{ccc} \downarrow{\scriptstyle n\to+\infty} & & \downarrow{\scriptstyle n\to+\infty} \\ 0 & & 0 \end{array}$$

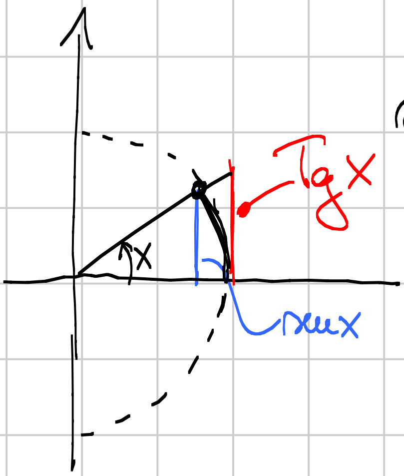
↑ Th. carabinieri

② Fatto $\cos \frac{1}{n} = \sqrt{1-\operatorname{sen}^2 \frac{1}{n}}$

Fatto $\operatorname{sen} \frac{1}{n} \xrightarrow[n\to+\infty]{} 0 \implies \operatorname{sen}^2 \frac{1}{n} \xrightarrow[n\to+\infty]{} 0$ $\Bigg\} \implies$

$$\implies \lim_{n\to+\infty} \cos \tfrac{1}{n} = \lim_{n\to+\infty} \sqrt{1-\operatorname{sen}^2 \tfrac{1}{n}} = \sqrt{1} = 1$$

Esercizio $\displaystyle \lim_{n\to+\infty} \frac{\operatorname{sen} 1/n}{1/n} = 1$



$$0 < \operatorname{sen} x \le x \le \operatorname{Tg} x \qquad \tfrac{\pi}{2} > x > 0$$

$$1 \le \frac{x}{\operatorname{sen} x} \le \frac{1}{\cos x} \qquad \tfrac{\pi}{2} > x > 0 \quad (*)$$

considero $\frac{\pi}{2} > -x > 0$

$$1 \le \frac{-x}{\operatorname{sen}(-x)} \le \frac{1}{\cos(-x)} \qquad \tfrac{\pi}{2} > -x > 0$$

$$\Downarrow$$

$$(**) \quad 1 \le \frac{+x}{+\operatorname{sen} x} \le \frac{1}{\cos x} \qquad -\tfrac{\pi}{2} < x < 0$$

(**) ∪ (*) implicano che

$$1 \geq \frac{\operatorname{sen} x}{x} \geq \cos x \qquad \forall x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\setminus \{0\}$$

$$\Downarrow$$

$$\boxed{1 \geq \frac{\operatorname{sen} 1/n}{1/n} \geq \cos 1/n \qquad \forall n = 1, 2, 3, \dots}$$

$$\downarrow n \to +\infty \qquad\qquad \downarrow n \to +\infty$$

$$1 \qquad\qquad 1$$

$\Rightarrow$ (per il Th. dei 2 Carabinieri) $\displaystyle\lim_{n \to +\infty} \frac{\operatorname{sen} 1/n}{1/n} = 1$

Esercizio Calcolare $\displaystyle\lim_{n \to +\infty} \frac{\cos 1/n - 1}{1/n^2}$

dim.

$$\frac{\cos(1/n) - 1}{1/n^2} = \frac{\cos\left(2 \cdot \frac{1}{2n}\right) - 1}{1/n^2} = \frac{\cos^2\left(\frac{1}{2n}\right) - \operatorname{sen}^2\left(\frac{1}{2n}\right) - 1}{1/n^2}$$

$$= \frac{-\operatorname{sen}^2\left(\frac{1}{2n}\right) - \left(1 - \cos^2\left(\frac{1}{2n}\right)\right)}{\left(\frac{1}{2n}\right)^2} \cdot \frac{1}{4}$$

$$= \frac{-2\operatorname{sen}^2\left(\frac{1}{2n}\right)}{\left(\frac{1}{2n}\right)^2} \cdot \frac{1}{4} = -\frac{1}{2}\left[\frac{\operatorname{sen}\left(\frac{1}{2n}\right)}{\frac{1}{2n}}\right]^2$$

$$\lim_{n\to+\infty} \frac{\operatorname{sen} \frac{1}{n}}{\frac{1}{n}} = 1 \Rightarrow \lim_{n\to+\infty} \frac{\operatorname{sen}\left(\frac{1}{2n}\right)}{1/2n} = 1$$

$\left\{\frac{\operatorname{sen} \frac{1}{2n}}{1/2n}\right\}_{n\geq 1}$ è una sottosuccessione di $\left\{\frac{\operatorname{sen} \frac{1}{n}}{1/n}\right\}_{n\geq 1}$

$$\Rightarrow \lim_{n\to+\infty} \left[\frac{\operatorname{sen}\left(1/2n\right)}{1/2n}\right]^2 = 1$$

$$\Rightarrow \lim_{n\to+\infty} \frac{\cos\frac{1}{n} - 1}{1/n^2} = -\frac{1}{2}$$

<u>Esercizio</u> $\lim_{n\to+\infty} q^n = \begin{cases} +\infty & q>1 \\ 1 & q=1 \\ 0 & 0\leq q<1 \end{cases}$

dim

Dis Bernoulli $\quad (1+a)^n \geq 1+na \quad \forall n=0,1,2,\dots$
$\forall a > -1$

$q>1 \Rightarrow q = 1+\varepsilon \quad \varepsilon>0 \quad \Rightarrow q^n = (1+\varepsilon)^n \geq 1+n\varepsilon$
per la Disug. di Bernoulli

$$q^n \geq \underbrace{1+n\varepsilon}_{\substack{\downarrow n\to+\infty \\ +\infty}} \quad (\text{poiché } \varepsilon>0) \quad \Longrightarrow \lim_{n\to+\infty} q^n = +\infty$$

per il Teorema del confronto

$q=1$ ovvio $\qquad q^n = 1 \xrightarrow[n\to+\infty]{} 1$

$$0<q<1 \Rightarrow \frac{1}{q}>1 \Rightarrow \lim_{n\to+\infty} \left(\frac{1}{q}\right)^n = +\infty$$

$$\Rightarrow \lim_{n\to+\infty} q^n = \lim_{n\to+\infty} \frac{1}{\left(1/q\right)^n} = 0$$

$q=0$ ovvio $\qquad q^n = 0 \xrightarrow[n\to+\infty]{} 0$

IDEA del criterio radice/rapporto è

confrontare $\underline{\underline{a_n}}$ con $\underline{\underline{\boxed{q^n}}}$ $\qquad q^n \to \begin{cases} 0 & 0\le q<1 \\ +\infty & q>1 \end{cases}$

CRITERIO DELLA RAPPORTO (x successioni)

$\{a_n\}_n$ una successione reale $\quad a_n>0 \ \forall n$

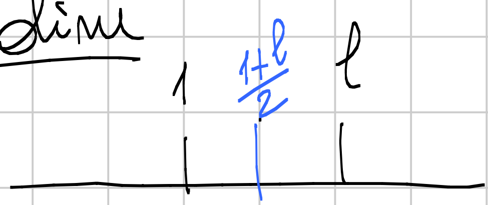
Sia $\ell = \lim_{n\to+\infty} \frac{a_{n+1}}{a_n}$

Se $\ell>1$ allora $a_n \xrightarrow[n\to+\infty]{} +\infty$

Se $0\le \ell<1$ allora $a_n \xrightarrow[n\to+\infty]{} 0$

<u>dim</u>

$1<\ell<+\infty$



$\rightarrow$ $\forall \varepsilon>0\ \exists \bar{n} : \forall n \geq \bar{n} \quad l-\varepsilon < \frac{a_{n+1}}{a_n} < l+\varepsilon \qquad l \in \mathbb{R}$

Voglio $l-\varepsilon = \frac{1+l}{2} \iff \varepsilon = \frac{l-1}{2}$

fisso $\varepsilon = \frac{l-1}{2}$ $\exists \bar{n} : \forall n \geq \bar{n} \quad \frac{1+l}{2} < \frac{a_{n+1}}{a_n}$

$\Updownarrow$

$\forall n \geq \bar{n} \quad \left(\frac{1+l}{2}\right) \cdot a_n < a_{n+1}$

$n=\bar{n} \qquad \left(\frac{1+l}{2}\right) a_{\bar{n}} < a_{\bar{n}+1}$

$\left(\frac{1+l}{2}\right)^2 a_{\bar{n}} < \left(\frac{1+l}{2}\right) a_{\bar{n}+1} < a_{\bar{n}+2}$

$\left(\frac{1+l}{2}\right)^3 a_{\bar{n}} < \left(\frac{1+l}{2}\right) a_{\bar{n}+2} < a_{\bar{n}+3}$

$- - - -$

$\left(\frac{1+l}{2}\right)^{n-\bar{n}} \cdot a_{\bar{n}} < a_n \qquad \underline{\forall n > \bar{n}}$

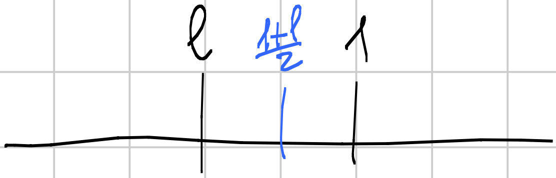
$\left(\frac{1+l}{2}\right)^n \cdot \boxed{\frac{a_{\bar{n}}}{\left(\frac{1+l}{2}\right)^{\bar{n}}}} < a_n \qquad \forall n \geq \bar{n}$

n.ro finito
che dipende da $\bar{n}$, $a_{\bar{n}}$ ed l

$\lim_{n\to+\infty} \left(\frac{1+l}{2}\right)^n = +\infty \implies \lim_{n\to+\infty} a_n = +\infty$

per il teorema
confronto

$0 \le \ell < 1$



$\boxed{\frac{1+\ell}{2} < 1}$

$\varepsilon = \frac{1-\ell}{2}$ $\quad \exists \bar n \quad \forall n \ge \bar n \quad 0 < \frac{a_{n+1}}{a_n} < \ell+\varepsilon = \ell + \frac{1-\ell}{2} = \frac{1+\ell}{2}$

$$\Downarrow$$

$$\forall n > \bar n \quad 0 < \frac{a_{n+1}}{a_n} < \frac{1+\ell}{2}$$

$$\forall n > \bar n \quad 0 < a_{n+1} < \left(\frac{1+\ell}{2}\right) \cdot a_n$$

$n = \bar n \qquad 0 < a_{\bar n+1} < \left(\frac{1+\ell}{2}\right) a_{\bar n}$

$$- - - -$$

$$0 < a_n < \left(\frac{1+\ell}{2}\right)^{n-\bar n} \cdot a_{\bar n} = \left(\frac{1+\ell}{2}\right)^{n} \cdot \frac{a_{\bar n}}{\left(\frac{1+\ell}{2}\right)^{\bar n}}$$

$$\lim_{n\to+\infty} \left(\frac{1+\ell}{2}\right)^{n} = 0$$

$\Longrightarrow$ per il Teorema dei carabinieri $\quad \lim_{n\to+\infty} a_n = 0$

$\ell = +\infty \quad \forall M > 0 \ \exists \bar n \ \forall n > \bar n \quad M < \frac{a_{n+1}}{a_n}$

Preso $M > 1$ si conclude come nel caso $\ell > 1$

Teorema (CRITERIO RADICE)

$\{a_n\}$ $\quad a_n \geq 0$ successione reale

$$l = \lim_{n\to+\infty} \sqrt[n]{a_n}$$

Se $l > 1$ allora $\lim\limits_{n\to+\infty} a_n = +\infty$

Se $0 \leq l < 1$ allora $\lim\limits_{n\to+\infty} a_n = 0$

Teorema (Legame tra criteri radice/rapporto)

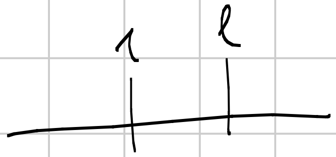
$\{a_n\}_n$ succ. reale $\quad a_n > 0$

Se $\exists \lim\limits_{n\to+\infty} \dfrac{a_{n+1}}{a_n} = l$ allora $\exists \lim\limits_{n\to+\infty} \sqrt[n]{a_n} = l$

Idea dim criterio radice $\quad l \in \mathbb{R}$

$\forall \varepsilon > 0 \ \exists \bar{n} : \forall n > \bar{n} \quad l-\varepsilon < \sqrt[n]{a_n} < l+\varepsilon$

$l > 1$



$\Downarrow$

$\varepsilon = \dfrac{l-1}{2}$ $\quad \exists \bar{n} : \forall n \geq \bar{n} \quad l-\varepsilon = \dfrac{1+l}{2} < \sqrt[n]{a_n}$

" " " $\quad \left(\dfrac{1+l}{2}\right)^n < a_n$

Il caso $l=1$ (che non è decidibile utilizzando i criteri della radice / del rapporto)

① $a_n = n$ $\lim_{n\to+\infty} a_n = +\infty$ $\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim \frac{n+1}{n}$
$$\| \\ 1$$

$b_n = 1/n$ $\lim_{n\to+\infty} b_n = 0$ $\lim_{n\to+\infty} \frac{b_{n+1}}{b_n} =$

$= \lim_{n\to+\infty} \frac{1/(n+1)}{1/n}$

$= \lim_{n\to+\infty} \frac{n}{n+1} = 1$

<u>Esercizio</u> Calcolare $\lim_{n\to+\infty} \frac{q^n}{n^k}$ $q \geq 0$ $k=1,2,3,\dots$

<u>Sol.</u>

$a_n = \frac{q^n}{n^k}$ $\frac{a_{n+1}}{a_n} = \frac{q^{\cancel{n+1}}}{(n+1)^k} \cdot \frac{n^k}{\cancel{q^n}}$

$= q \cdot \left(\frac{n}{n+1}\right)^k \xrightarrow[n\to+\infty]{} q$

Se $0 \leq q < 1$ allora $\lim_{n\to+\infty} \frac{q^n}{n^k} = 0$ $\forall k$

Se $q > 1$ allora $\lim_{n\to+\infty} \frac{q^n}{n^k} = +\infty$ $\forall k$

Se $q = 1$ allora $\lim_{n\to+\infty} 1/n^k = 0$

$$\frac{P(n)}{Q(n)} \xrightarrow[n\to+\infty]{} ?$$

Esercizio $$\lim_{n\to+\infty} \frac{n^5 - n^3 + \sin n - 1/n}{3n^7 + 4n^2 - 1/n^2}$$ sono inessenziali

dim

$$\lim_{n\to+\infty} \frac{n^5}{n^7} \cdot \boxed{\frac{1 - \frac{1}{n^2} + \frac{\sin n}{n^5} - \frac{1}{n^6}}{3 + 4/n^5 - \frac{1}{n^9}}}$$

$$\lim_{n\to+\infty} \frac{n^5}{n^7} \cdot \frac{1 + (\text{quantità che tende a } 0 \text{ per } n\to+\infty)}{3 + (\qquad\qquad)}$$

$$= \frac{1}{3} \cdot \lim_{n\to+\infty} \frac{1}{n^2} = 0$$

Esercizio $$\lim_{n\to+\infty} \frac{\overline{3n^2 - n^4 + n - \cos n}^{\,0}}{-5n^3 + 6n^4 - n^2}$$

dim

$$\lim_{n\to+\infty} \frac{\cancel{n^4}^{\,1}}{\cancel{n^4}_{\,1}} \frac{-1 + 3/n^2 + \frac{1}{n^3} - \frac{\cos n}{n^4}}{6 - 5/n - 1/n^2}$$

$$\lim_{n\to+\infty} \frac{-1 + (\text{Termini} \to 0)}{6 + (\text{Termini} \to 0)} = -\frac{1}{6}$$

Esercizio Dimostrare che $\left(1+\frac{1}{n}\right)^n$ è strettamente crescente

dim

$n=1$ è vero $\qquad a_1 = (1+1)^1 = 2 < \left(1+\frac{1}{2}\right)^2 = a_2$

Valga per n $\qquad a_n = \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} = a_{n+1}$

Devo provare $\qquad a_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} < \left(1+\frac{1}{n+2}\right)^{n+2} = a_{n+2} \iff$

$$\iff \left(\frac{n+2}{n+1}\right)^{n+1} < \left(\frac{n+3}{n+2}\right)^{n+2}$$

$$\iff \frac{n+1}{n+2} \cdot \left(\frac{n+2}{n+1}\right)^{n+2} < \left(\frac{n+3}{n+2}\right)^{n+2}$$

$$\iff \frac{n+1}{n+2} < \left(\frac{n+3}{n+2}\right)^{n+2} \left(\frac{n+1}{n+2}\right)^{n+2}$$

$$\iff \frac{n+1}{n+2} < \left(1+\frac{1}{n+2}\right)^{n+2} \left(1-\frac{1}{n+2}\right)^{n+2} = \left(1-\frac{1}{(n+2)^2}\right)^{n+2}$$

Ma questo è vero poiché

$$\left(1-\frac{1}{(n+2)^2}\right)^{n+2} \overset{\text{Bernoulli}}{>} 1-\frac{n+2}{(n+2)^2} = 1-\frac{1}{n+2} = \frac{n+1}{n+2}$$

<u>Esercizio</u> $\left(1+\frac{1}{n}\right)^{n+1}$ è decrescente

<u>dim</u>

$$a_n = \left(1+\frac{1}{n}\right)^{n+1} \qquad a_2 = \left(1+\frac{1}{2}\right)^3 < (1+1)^2 = a_1$$

Suppongo valga per n $\quad a_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+2} < \left(1+\frac{1}{n}\right)^{n+1} = a_n$

Devo provare $\Leftrightarrow$ $a_{n+2} = \left(1+\frac{1}{n+2}\right)^{n+3} < \left(1+\frac{1}{n+1}\right)^{n+2} = a_{n+1}$

$$\Leftrightarrow \left(\frac{n+3}{n+2}\right)^{n+3} < \left(\frac{n+2}{n+1}\right)^{n+3} \frac{n+1}{n+2}$$

$$\Leftrightarrow \frac{n+2}{n+1} < \left(\frac{n+2}{n+1}\right)^{n+3} \cdot \left(\frac{n+2}{n+3}\right)^{n+3}$$

$$\Leftrightarrow \frac{n+2}{n+1} < \left(\frac{(n+2)^2}{(n+1)(n+3)}\right)^{n+3} = \left(1+\frac{1}{(n+1)(n+3)}\right)^{n+3}$$

x Bernoulli

$$\left(1+\frac{1}{(n+1)(n+3)}\right)^{n+3} > 1+\frac{\cancel{n+3}^{\,1}}{(n+1)\cancel{(n+3)}} = \frac{n+2}{n+1}$$

$$2 \le \left(1+\frac{1}{n}\right)^{n} \le \left(1+1\right)^{2}$$

$$\Rightarrow \exists \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n} = e$$

$$\sqrt[n]{\frac{n!}{n^n}} \xrightarrow[n\to+\infty]{}$$

$$a_n = \frac{n!}{n^n}$$

$$\lim_{n\to+\infty}\frac{a_{n+1}}{a_n} = \ell \,?$$

$$\Downarrow$$

$$\lim_{n\to+\infty}\sqrt[n]{a_n} = \ell \,?$$