Lezione 48 – Analisi Matematica 1 – 15 gennaio 2014

Esercizio $F(x) = \int_1^x \frac{e^t}{t}\,dt$

Calcolare lo sviluppo di Taylor in un intorno di $x=1$ fino al 4° ordine

Devo ~~sviluppare~~ dire e^t/t in un intorno di $t=1$

$$\frac{e^t}{t} = \frac{e^{t-1+1}}{t-1+1} = e\cdot\frac{e^{t-1}}{1+(t-1)} \qquad \boxed{t-1=y}$$

$$= e\cdot\frac{e^y}{1+y}$$ e devo svilupparlo in un intorno di $y=0$

$$\frac{1}{1+y} = (1+y)^{-1}$$

$$= e\cdot\left(1+y+\frac{y^2}{2}+\frac{y^3}{6}+o(y^3)\right)\left(1-y+y^2-y^3+o(y^3)\right)$$

$$= e\left(1\cancel{-y}\cancel{+y^2}\cancel{-y^3}\cancel{+y}\cancel{-y^2}\cancel{+y^3}+\frac{y^2}{2}-\frac{y^3}{3}+\frac{y^3}{6}+o(y^3)\right)$$

$$= e\left(1+\frac{(t-1)^2}{2}-\frac{(t-1)^3}{6}+o\left((t-1)^3\right)\right)$$

$$F(x) = e\int_1^x \left[1+\frac{(t-1)^2}{2}-\frac{(t-1)^3}{6}+o\left((t-1)^3\right)\right]dt \qquad y=t-1$$

$$= e\int_0^{x-1}\left[1+\frac{y^2}{2}-\frac{y^3}{6}+y^3\cdot o(1)\right]dy$$

$$= e\left[y+\frac{y^3}{6}-\frac{y^4}{24}+\frac{y^4}{4}\cdot o(1)\right]_{y=0}^{y=x-1}$$

$$= e\left[(x-1)+\frac{(x-1)^3}{6}-\frac{(x-1)^4}{24}+(x-1)^4\cdot o(1)\right] \quad (x\to 1)$$

$$= e(x-1)+\frac{e}{6}(x-1)^3-\frac{e}{24}(x-1)^4+o\left((x-1)^4\right) \quad x\to 1$$

OSS: $\displaystyle\int_0^x o(t^3)\,dt = \int_0^x t^3 o(1)\,dt = \frac{x^4}{4}\cdot o(1)$

$= o(x^4)$

$(x \to 0)$

Esercizio Data $F(x) = \displaystyle\int_{\operatorname{sen}x}^{\cos x} e^t\,dt$

calcolare la retta Tg (se $\exists!$) al grafico di F

nel punto $(0, F(0))$
dim

$$F(0) = \int_{\operatorname{sen}(0)}^{\cos(0)} e^t\,dt = \Big[e^t\Big]_{t=0}^{t=1} = e-1$$

$$F(x) = \int_0^{\cos x} e^t\,dt - \int_0^{\operatorname{sen}x} e^t\,dt$$

$$F'(x) = e^{\cos x}\cdot(-\operatorname{sen}x) - e^{\operatorname{sen}x}(\cos x)$$

$$= -e^{\cos x}\cdot \operatorname{sen}x - e^{\operatorname{sen}x}\cos x$$

$$F'(0) = -e^0\cdot 1 = -1$$

$$r(x) = F(0) + F'(0)\cdot(x-0)$$

$$r(x) = e-1 - 1\cdot x \qquad \boxed{r(x) = e-1-x}$$

Esercizio $\quad F(x) = \int_1^x e^{t^2}(3+\sin t)\,dt$

1) è invertibile?

2) se lo è, allora calcolare $(F^{-1})'(0)$

dim

- $\operatorname{dom}(F) = \mathbb{R}$
- per provare l'invertibilità, essendo F continua e derivabile, proverò che $F'(x) \neq 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow F$ monotona strettamente $\Rightarrow \exists F^{-1}$ derivabile e monotona

$$F' = e^{x^2}(3+\sin x) \geq 2 \cdot e^{x^2} > 0 \quad \forall x \in \mathbb{R}$$

$\Rightarrow F$ è strettamente crescente su $\mathbb{R}$

$\Rightarrow F$ è invertibile con inversa derivabile

- Voglio $(F^{-1})'(0) = \dfrac{1}{F'(\underbrace{F^{-1}(0)})}$

$F^{-1}(0) = a$ $\quad$ se $\quad F(a) = 0 = \int_1^a e^{x^2}(3+\sin t)\,dt$

se $\boxed{a = 1}$ e dunque

$$(F^{-1})'(0) = \frac{1}{F'(F^{-1}(0))} = \frac{1}{F'(1)} = \frac{1}{e^{1^2}\cdot(3+\sin 1)}$$

Esercizio Per quali valori di $\alpha \in \mathbb{R}$ converge

$$\int_0^{+\infty} \frac{\log(1+x^4)}{x^\alpha}\,dx$$

dim

$f(x) = \dfrac{\log(1+x^4)}{x^\alpha}$ continua $\forall x \in]0,+\infty[$

$\forall \alpha \in \mathbb{R}$

Devo studiare $\int_0^{+\infty} f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^{+\infty} f(x)\,dx$

① $\int_0^1 f(x)\,dx$

$\log(1+y) = y + o(y)$

$$f(x) = \frac{\log(1+x^4)}{x^\alpha} = \frac{x^4 + o(x^4)}{x^\alpha}$$

$$f(x) = \frac{1}{x^{\alpha-4}} \cdot (1+o(1)) \sim \frac{1}{x^{\alpha-4}} \quad (x \to 0)$$

Ma $\int_0^1 \frac{dx}{x^{\alpha-4}}$ converge se $\alpha - 4 < 1$

se $\alpha < 5$

$\Rightarrow$ (Criterio Confronto Asintotico) $\int_0^1 f(x)\,dx$ converge per $\alpha < 5$

② $\int_1^{+\infty} f(x)\,dx$

$$f(x) = \frac{\log(1+x^4)}{x^\alpha} \sim \frac{\log x^4}{x^\alpha} = \frac{4 \log x}{x^\alpha}$$

Devo studiare

$$\int_1^{+\infty} 4\,\frac{\log x}{x^\alpha}\,dx$$

Voglio provare che $\int_1^{+\infty} \frac{4\log x}{x^\alpha}$ converge $\forall \alpha > 1$

Quando $\alpha = 1$ $\int_1^{+\infty} \frac{\log x}{x} dx = \int_0^{+\infty} \frac{t}{\cancel{e^t}} \cdot \cancel{e^t} dt =$

$x = e^t$

$dx = e^t dt$

$1 \longmapsto 0$

$= \lim_{x \to +\infty} \left[\frac{t^2}{2}\right]_{t=0}^{t=x} = +\infty$

Se $\alpha < 1$ allora $\int_1^{+\infty} \frac{4\log x}{x^\alpha} dx = +\infty$ in quanto $\frac{\log x}{x^\alpha} \geq \frac{\log x}{x}$ $\forall x > 1$ $\forall \alpha < 1$

Se $\alpha > 1$ allora $\frac{4\log x}{x^\alpha} = \frac{4\log x}{x^{1-(\alpha-1)}} = \frac{4}{x^{1-\frac{\alpha-1}{2}}} \cdot \frac{\log x}{x^{\frac{\alpha-1}{2}}}$

ora $\lim_{x \to +\infty} \frac{\log x}{x^{\frac{\alpha-1}{2}}} = 0$ in quanto $\frac{\alpha-1}{2} > 0$

$\Rightarrow \exists k > 0 : \frac{\log(x)}{x^{\frac{\alpha-1}{2}}} \leq K \quad \forall x > 1$

Ne segue che $\frac{4\log x}{x^\alpha} \leq 4K \cdot \frac{1}{x^{1+\frac{\alpha-1}{2}}} \quad \forall x > 1$

Ma $\int_1^{+\infty} \frac{4 \cdot K}{x^{1+\frac{\alpha-1}{2}}} dx \in \mathbb{R} \quad \forall \alpha > 1$ (infatti $\alpha > 1 \Rightarrow \frac{\alpha-1}{2} > 0 \Rightarrow 1 + \frac{\alpha-1}{2} > 1$)

$\Rightarrow$ (Criterio Confronto Integrale Improprio) $\int_1^{+\infty} \frac{\log x}{x^\alpha} dx \in \mathbb{R} \quad \forall \alpha > 1$

$\Rightarrow$ (Criterio Confronto Asintotico Integrale Improprio) $\int_1^{+\infty} f(x) dx \in \mathbb{R} \quad \forall \alpha > 1$

$\Downarrow$ Tirando le somme

$\int_0^1 f(x) dx$ converge $\forall \alpha < 5$ & $\int_1^{+\infty} f(x) dx$ $\forall \alpha > 1$

$\Downarrow$

$\int_0^{+\infty} f(x) dx$ converge se $1 < \alpha < 5$ $\lightning$

OSS: per provare che $\int_1^{+\infty} \frac{\log x}{x^\alpha}\,dx \in \mathbb{R} \quad \forall \alpha > 1$

si può osservare che $\int_1^{+\infty} \frac{\log x}{x^\alpha}\,dx \underset{\substack{x = e^t \\ dx = e^t dt}}{=} \int_0^{+\infty} \frac{\log e^t}{e^{\alpha t}} e^t\,dt$

$= \int_0^{+\infty} \frac{t}{e^{(\alpha-1)t}}\,dt$

ed ora $\forall t > 1 \quad e^{t(\alpha-1)} = \left(e^{\frac{t(\alpha-1)}{3}}\right)^3 \geq \left(1 + t\,\frac{\alpha-1}{2}\right)^3 \geq t^3 \cdot \frac{(\alpha-1)^3}{8}$

da cui segue $\quad \frac{1}{e^{t(\alpha-1)}} \leq \frac{8}{(\alpha-1)^3} \cdot \frac{1}{t^3} \quad \forall t \geq 1 \quad \forall \alpha > 1$

" " " $\quad \frac{t}{e^{t(\alpha-1)}} \leq \frac{8}{(\alpha-1)^3} \cdot \frac{1}{t^2} \quad \forall t \geq 1 \quad \forall \alpha >$

Ma $\int_1^{+\infty} \frac{8}{(\alpha-1)^3} \cdot \frac{1}{t^2}\,dt \in \mathbb{R} \quad \forall \alpha > 1$.

$\Rightarrow$ (Criterio Confronto Integrali Impropri) $\quad \int_1^{+\infty} \frac{t}{e^{t(\alpha-1)}}\,dt \in \mathbb{R} \quad \forall \alpha > 1$

ed essendo $\int_0^1 \frac{t}{e^{t(\alpha-1)}}\,dt \in \mathbb{R} \quad \forall \alpha > 1$

$\Rightarrow \int_0^{+\infty} \frac{t}{e^{t(\alpha-1)}}\,dt = \int_1^{+\infty} \frac{\log x}{x^\alpha}\,dx \in \mathbb{R} \quad \forall \alpha > 1$

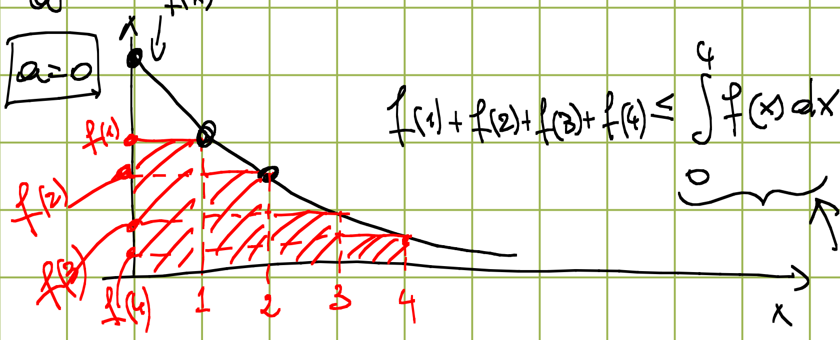
Teorema (Confronto Serie-Integrale)

$f: [a, +\infty[\longrightarrow [0, +\infty[$ debolmente decrescente

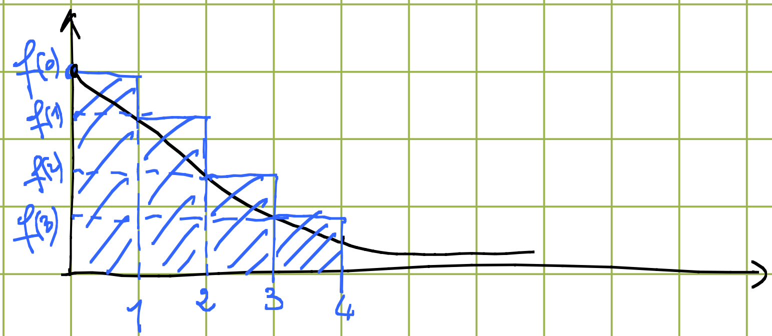
allora

$\displaystyle\int_a^{+\infty} f(t)\,dt$ converge (diverge) se $\displaystyle\sum_{n \geq [a]} f(n)$ converge (diverge)

$\boxed{a=0}$



$$f(1)+f(2)+f(3)+f(4) \leq \int_0^4 f(x)\,dx \leq \underbrace{f(0)+f(1)+f(2)+f(3)}$$



$\mathcal{A} = \{0, 1, 2, 3, 4\}$

$$s(f, \mathcal{A}) = f(1)+f(2)+f(3)+f(4) \leq \int_0^4 f(x)\,dx \leq \underbrace{f(0)+f(1)+f(2)+f(3)}_{S(f, \mathcal{A})}$$

$\mathcal{A}_n = \{0, 1, 2, \dots, n\}$

$$s(f, \mathcal{A}_n) = f(1)+f(2)+\dots+f(n) \leq \int_0^n f(x)\,dx \leq \underbrace{f(0)+f(1)\dots+f(n-1)}_{S(f, \mathcal{A}_n)}$$

$$\underbrace{s(f,\mathcal{A}_m) = \sum_{k=1}^{m} f(k) \le \int_0^m f(x)\,dx \le \sum_{k=0}^{m-1} f(k) = S(f,\mathcal{A}_m)}$$

Se $\int_0^{+\infty} f(x)\,dx \in \mathbb{R}$ allora $\sum_{k=1}^{\infty} f(k) \in \mathbb{R}$ ovvero la serie converge

Se $\int_0^{+\infty} f(x)\,dx = +\infty$ allora $\sum_{k=0}^{+\infty} f(k) = +\infty$ ovvero la serie diverge

Se $\sum_{k=0}^{+\infty} f(k) \in \mathbb{R}$ allora $\int_0^{+\infty} f(x)\,dx \in \mathbb{R}$ ovvero l'integrale converge

Se $\sum_{k=0}^{\infty} f(k) = +\infty$ allora $\int_0^{+\infty} f(x)\,dx = +\infty$ ovvero l'integrale diverge

$$\boxed{\text{Serie}} \le \text{integrale} \le \boxed{\text{Serie}}$$

Esercizio $\sum_n 1/n^\alpha$ $\alpha > 0$ studiare il carattere

$f(x) = 1/x^\alpha$ questa è decrescente $\forall \alpha > 0$

Allora

$\sum_{n \ge 1} f(n)$ converge se $\int_1^{+\infty} f(x)\,dx$ converge se $\int_1^{+\infty} \frac{dx}{x^\alpha}$ converge se $\alpha > 1$ ↯

Esercizio $\lambda > 0$ sia $f_\lambda = \lambda x + 1/x$ per $x>0$

Sia x_λ il punto di minimo di f_λ su $]0,+\infty[$

Calcolare $\displaystyle\int_{1/2x_\lambda}^{2x_\lambda} f_\lambda \, dt$

dim

$\lambda x + 1/x$

per esempio $f(x) = x + 1/x$

$\lim_{x\to 0^+} f = +\infty \qquad \lim_{x\to+\infty} f = +\infty$

$f' = (x + 1/x)' = 1 - 1/x^2 = 0$

$\boxed{x = 1}$

$f' = \begin{cases} <0 & x<1 \\ >0 & x>1 \end{cases}$

$$\frac{d}{dx}\left(\lambda x + \frac{1}{x}\right) = \lambda - \frac{1}{x^2} = 0 \qquad x = \pm\sqrt{\lambda}$$

$x_\lambda = \sqrt{\lambda}$ ($-\sqrt{\lambda}$ non lo considero poiché studio il caso $x>0$!)

$f'(x) \begin{cases} <0 & 0<x<x_\lambda \\ >0 & x_\lambda < x \end{cases}$

$$f(x_\lambda) = \lambda \cdot \sqrt{\lambda} + \frac{1}{\sqrt{\lambda}} = \frac{\lambda^2+1}{\sqrt{\lambda}}$$

$x_\lambda = \sqrt{\lambda}$ punto di min relativo interno

si ha che $\sqrt{\lambda}$ è punto di minimo assoluto

$$\int_{1/2\sqrt{\lambda}}^{2\sqrt{\lambda}} f_\lambda(t)\,dt = \int_{1/2\sqrt{\lambda}}^{2\sqrt{\lambda}} \left(\lambda t + 1/t\right) dt =$$

$$= \left[\lambda \frac{t^2}{2} + \log t\right]_{t=1/2\sqrt{\lambda}}^{2\sqrt{\lambda}} = \lambda \frac{4\lambda}{2} + \log(2\sqrt{\lambda}) - \lambda \cdot \frac{1}{8\lambda} - \log\frac{1}{2\sqrt{\lambda}}$$

$$= 2\lambda^2 + \log 2\sqrt{\lambda} - \frac{1}{8} + \log 2\sqrt{\lambda}$$

$$= 2\lambda^2 - \frac{1}{8} + 2\log(2\sqrt{\lambda})$$