Analisi Matematica 1 - 14 gennaio 2014

$$\int_a^b f(x)\,dx := \lim_{\beta \to b^-} \int_a^\beta f(x)\,dx .$$

Integrale di Riemann: per funzioni $f:[a,b] \to \mathbb{R}$ ove

- $[a,b]$ chiuso e limitato
- f limitata su $[a,b]$

Integrale Improprio (di Riemann): quando $f: I \to \mathbb{R}$
(f non è limitata) o (I non è chiuso e limitato)

Lo studio di un integrale improprio si fa attraverso opportuni "criteri di confronto"

Per le serie ci si confrontava con

- $\sum_n q^n$ $\quad q \in \mathbb{R}$ $\quad$ Serie Geometrica
- $\sum_n 1/n^\alpha$ $\quad \alpha \in \mathbb{R}$ $\quad$ " Armonica Generalizzata

Per gli integrali impropri è FONDAMENTALE ~~sa~~ per confrontare il problema (funzioni integrando) in esame con

$$\boxed{1/x^\alpha} \qquad \alpha \in \mathbb{R}$$

Esempio 1 Studio $\displaystyle\int_0^1 \frac{1}{x^\alpha}\,dx \quad \alpha\in\mathbb{R}$

- Se $\alpha \leq 0$ allora questo è un integrale di Riemann non improprio

- Se $\alpha > 0$ allora $f(x) = 1/x^\alpha \xrightarrow[x\to 0^+]{} +\infty$

$$\int_a^1 \frac{1}{x^\alpha}\,dx = \begin{cases} \left[\dfrac{x^{1-\alpha}}{1-\alpha}\right]_{x=a}^{x=1} & \alpha \neq 1 \\ \left[\log x\right]_{x=a}^{x=1} & \alpha = 1 \end{cases}$$

$$\alpha=1 \quad \lim_{a\to 0^+} \int_a^1 \frac{dx}{x^\alpha} = \lim_{a\to 0^+} \left[\log x\right]_a^1 = \lim_{a\to 0^+} \left[\overset{0}{\overset{\shortparallel}{\log 1}} - \log a\right] = +\infty$$

$$\alpha<1 \quad \lim_{x\to 0^+} \int_a^1 \frac{dx}{x^\alpha} = \lim_{a\to 0^+} \left[\frac{x^{1-\alpha}}{1-\alpha}\right]_a^1 = \lim_{a\to 0^+} \left[\frac{1}{1-\alpha} - \frac{a^{1-\alpha}}{1-\alpha}\right] = \frac{1}{1-\alpha}$$

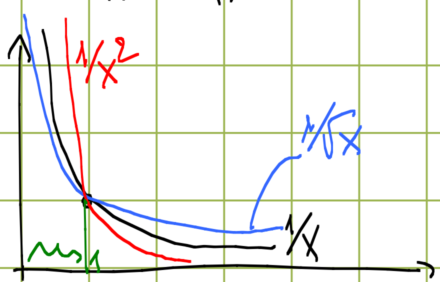
$$\boxed{1-\alpha>0}$$

$$\alpha>1 \quad \lim_{x\to 0^+} \int_a^1 \frac{dx}{x^\alpha} = \lim_{a\to 0^+} \left[\frac{x^{1-\alpha}}{1-\alpha}\right]_a^1 = \lim_{a\to 0^+} \left[\frac{a^{1-\alpha}}{\alpha-1} - \frac{1}{\alpha-1}\right]$$

$$\boxed{0<\alpha-1} \quad = +\infty - \frac{1}{\alpha-1} = +\infty$$

$$\boxed{\int_0^1 \frac{dx}{x^\alpha} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{x^\alpha} = \begin{cases} \dfrac{1}{1-\alpha} & \alpha<1 \\ +\infty & \alpha \geq 1 \end{cases}}$$

Funzione illimitata su intervallo limitato!



Esempio (2)
$$\int_1^{+\infty} \frac{dx}{x^\alpha}$$

$$\lim_{\beta\to+\infty} \int_1^\beta \frac{dx}{x^\alpha} = \lim_{\beta\to+\infty} \begin{cases} \left[\log x\right]_{x=1}^{x=\beta} & \alpha=1 \\ \left[\frac{x^{1-\alpha}}{1-\alpha}\right]_{x=1}^{x=\beta} & \alpha\neq 1 \end{cases}$$

$$\alpha=1 \quad \lim_{\beta\to+\infty} \int_1^\beta \frac{dx}{x^\alpha} = \lim_{\beta\to+\infty}\left[\log\beta - \log 1\right] = +\infty$$

$$\alpha<1 \quad \lim_{\beta\to+\infty} \int_1^\beta \frac{dx}{x^\alpha} = \lim_{\beta\to+\infty}\left[\frac{\beta^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha}\right] = +\infty \qquad \boxed{1-\alpha>0}$$

$$\alpha>1 \quad \lim_{\beta\to+\infty} \int_1^\beta \frac{dx}{x^\alpha} = \lim_{\beta\to+\infty}\left[\frac{1}{\alpha-1} - \frac{\beta^{1-\alpha}}{1-\alpha}\right] = \frac{1}{\alpha-1} \qquad 1-\alpha<0$$

Def $f: [a,b[\ \to \mathbb{R}$ t.c. $\exists \int_a^\beta f(x)\,dx \ \forall \beta\in[a,b[$

- se $\exists \lim_{\beta\to b} \int_a^\beta f(x)\,dx \in \mathbb{R}$ allora f è integrabile in senso improprio su $[a,b[$ e si ha (convergente)
$$\lim_{\beta\to b^-}\int_a^\beta f\,dx = \int_a^b f\,dx \in \mathbb{R}$$

- se $\exists \lim_{\beta\to b^-} \int_a^\beta f(x)\,dx = +\infty\ (-\infty)$ allora $\int_a^b f(x)\,dx = +\infty\ (-\infty)$

  e che l'integrale improprio diverge

se $\nexists \lim_{\beta\to b^-} \int_a^\beta f(x)\,dx$ allora $\nexists \int_a^b f(x)\,dx$

Le serie numeriche a termini positivi

- o convergono
- o divergono a $+\infty$

Teorema

Se $f : [a,b[\to \mathbb{R}$ continua e $f(x) \geq 0$ $\forall x \in [a,b[$

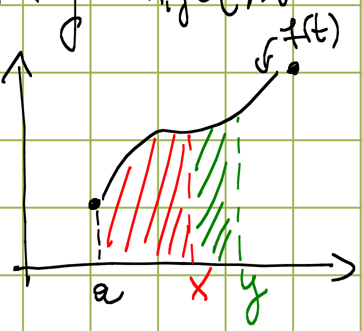
allora $\int_a^b f(x)dx$ o converge o diverge a $+\infty$

dim

$$F(x) = \int_a^x f(t)dt \qquad x \in [a,b[$$

f è continua su $[a,b[$ $\Rightarrow$ f è integrabile secondo Riemann su $[a,x]$ $\forall x \in [a,b[$

$x < y$ $\quad x,y \in [a,b[$ $\qquad F(x) = \int_a^x f(t)dt \qquad F(y) = \int_a^y f(t)dt$



$$F(y) = \int_a^x f(t)dt + \int_x^y f(t)dt$$

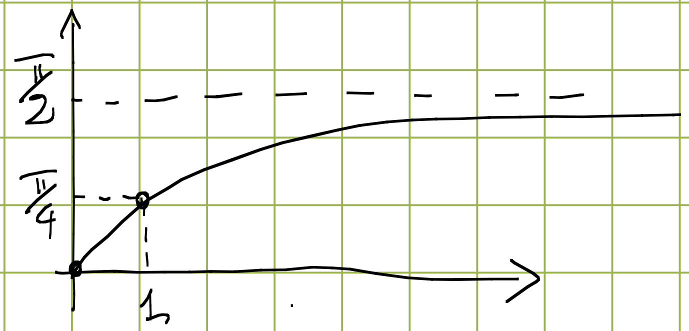
$$\geq \int_a^x f(t)dt = F(x)$$

$\Rightarrow F$ è crescente debolmente su $[a,b[$

$$\Rightarrow \exists \lim_{x \to b^-} F(x) = \int_a^b f(t)dt = \sup F([a,b[)$$

quindi l'integrale improprio esiste finito o $+\infty$ ↯

Esercizio: Studiare $\displaystyle\int_0^{+\infty} \operatorname{arctg} t^2 \, dt$

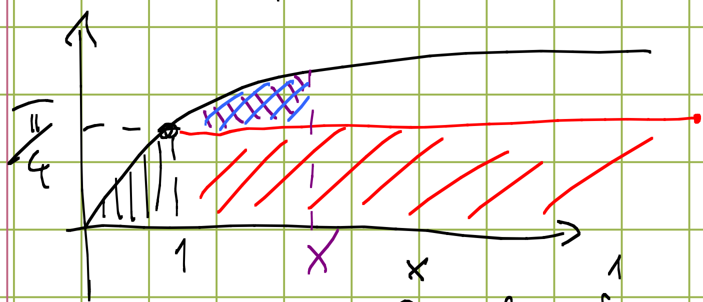


$\lim_{t\to+\infty} \operatorname{arctg} t^2 = \frac{\pi}{2} \iff \forall \varepsilon>0 \ \exists M>0 : \forall t>M$

$\frac{\pi}{2} - \varepsilon < \operatorname{arctg} t^2 < \frac{\pi}{2}+\varepsilon$

$\Rightarrow \varepsilon = \frac{\pi}{4} \quad \exists M>0 : \forall t>M \quad \frac{\pi}{4} < \operatorname{arctg} t^2$

$\Rightarrow \varepsilon = \frac{\pi}{4} \quad \exists M=1 : \forall t>M \quad \frac{\pi}{4} = \operatorname{arctg} 1 < \operatorname{arctg} t^2$



$\Rightarrow \forall x>1 \quad F(x) = \displaystyle\int_0^x \operatorname{arctg} t^2 dt = \int_0^1 \operatorname{arctg} t^2 dt + \int_1^x \operatorname{arctg} t^2 dt$

$\geq \displaystyle\int_0^1 \operatorname{arctg} t^2 dt + \int_1^x \frac{\pi}{4} dt$

$= \displaystyle\int_0^1 \operatorname{arctg} t^2 dt + (x-1)\cdot\frac{\pi}{4}$

$F(x) \geq \underbrace{\displaystyle\int_0^1 \operatorname{arctg} t^2 dt + (x-1)\frac{\pi}{4}}_{\downarrow x\to+\infty \ \ +\infty} \Rightarrow \lim_{x\to+\infty} F(x) = +\infty$

(teorema confronto limiti)

$\Rightarrow \displaystyle\int_0^{+\infty} \operatorname{arctg} t^2 dt = +\infty$

Teorema

$f: [a, +\infty[\to \mathbb{R}$ continua : $\exists \lim_{x \to +\infty} f(t) = l > 0 \; (<0)$

allora $\int_0^{+\infty} f(t)\,dt = +\infty \; (-\infty)$

ovvero

se f ha limite per $x \to +\infty$ $\neq 0$

allora $\int^{+\infty} f(t)\,dt$ diverge (a $+\infty$ o $-\infty$)

Teorema (Confronto tra Integrali Impropri)

$f, \varphi : [a,b[\to \mathbb{R}$ t.c.

1) f continua su $[a,b[$

2) $|f|(x) \leq \varphi(x) \quad \forall x \in [a,b[$

3) $\int_a^b \varphi(x)\,dx \in \mathbb{R}$

$\Rightarrow \int_a^b f(x)\,dx \in \mathbb{R}$

Oss

2) $\lim_{x \to b^-} \frac{f(x)}{\varphi(x)} = 0$ con f, φ continue su $[a,b[$

$\Rightarrow \exists M > 0 : \left| \frac{f(x)}{\varphi(x)} \right| \leq M \quad \forall x \in [a,b[$

$\Rightarrow |f(x)| \leq M \varphi(x) \quad \forall x \in [a,b[$

Teorema (Confronto Asintotico)

$f, g : [a, b[\to \mathbb{R}$

1) f, g continue su $[a, b[$

2) $f, g \geq 0$ su $[a, b[$

3) $\lim_{x \to b^-} \frac{f(x)}{g(x)} = l \in]0, +\infty[$

$$\Downarrow$$

$\int_a^b f(x)dx$ converge (diverge a $+\infty$) se $\int_a^b g(x)dx$ converge (diverge a $+\infty$)

Esempio Studiare l'integrale improprio

$$\int_0^1 \left(\frac{1}{\operatorname{sen} t} - \frac{1}{t}\right) dt$$

dim

Che sia improprio segue dal fatto che

$$f(t) = \frac{1}{\operatorname{sen} t} - \frac{1}{t} \text{ è continua su }]0, 1]$$

ma per $t \to 0$ trovo che $f(t) \to +\infty - \infty$

ovvero una forma indeterminata

Devo studiare il comportamento di $f(t)$ per $t \to 0$

(il che equivale a studiare $\lim_{x \to 0} \int_x^1 f(t)dt$!)

$$f(t)=\frac{t-\operatorname{sen} t}{t \operatorname{sen} t}\geq 0 \quad \forall t\in]0,1]$$

quindi $\displaystyle\int_0^1 f(t)\,dt$ o converge o diverge

$$f(t)=\frac{\cancel{t}-\left(\cancel{t}-\frac{t^3}{6}+o(t^4)\right)}{t\left(t-\frac{t^3}{6}+o(t^4)\right)}=\frac{\frac{t^3}{6}+o(t^4)}{t^2-\frac{t^4}{6}+o(t^5)}$$

$$=\frac{\frac{t^3}{6}+o(t^4)}{t^2+o(t^3)}=\frac{\frac{t^3}{6}}{t^2}\cdot\frac{1+o(t)}{1+o(t)} \quad (t\to 0)$$

$$\Downarrow$$

$$f(t)=\frac{t}{6}\cdot\frac{1+o(t)}{1+o(t)}\sim\boxed{\frac{t}{6}} \quad \text{per } t\to 0^+$$

$$\left(\lim_{t\to 0^+}\frac{f(t)}{t/6}=1\ !\right)$$

Ma $\displaystyle\int_0^1 \frac{t}{6}\,dt$ esiste finito.

$\Rightarrow$ (Criterio Confronto asintotico Integrali Impropri) $\displaystyle\int_0^1 f(t)\,dt$ esiste finito

Esercizio Studiare la convergenza di $\displaystyle\int_0^{+\infty} \frac{\sqrt{1+x^2}-x}{\sqrt{x}}\,dx$

dim

$$f(t) = \frac{\sqrt{1+t^2}-t}{\sqrt{t}} \quad \text{è continua su }]0,+\infty[.$$

$$\lim_{t\to 0^+} f(t) = +\infty \qquad \lim_{t\to+\infty} f(t) = \frac{\infty}{\infty}\ ... = \lim_{t\to\infty} \frac{1}{\sqrt{t}\left(\sqrt{1+t^2}+t\right)} = 0^+$$

$$\int_0^{+\infty} f(t)\,dt = \underbrace{\int_0^1 f(t)\,dt}_{1} + \underbrace{\int_1^{+\infty} f(t)\,dt}_{2}$$

qui f è illimitata su intervallo limitato

f è limitata su intervallo illimitato

1 $\displaystyle\int_0^1 f(t)\,dt \qquad f:]0,1] \to \mathbb{R}$ è continua

$$f(t) = \frac{\sqrt{1+t^2}-t}{\sqrt{t}} \cdot \frac{\sqrt{1+t^2}+t}{\sqrt{1+t^2}+t} = \frac{1+\cancel{t^2}-\cancel{t^2}}{\sqrt{t}\left(t+\sqrt{1+t^2}\right)}$$

$$= \frac{1}{\sqrt{t}} \cdot \frac{1}{t+\sqrt{1+t^2}} \sim \frac{1}{\sqrt{t}} \qquad t\to 0$$

$$\left(\text{ovvero } \lim_{t\to 0^+} \frac{f(t)}{1/\sqrt{t}} = 1 \,!\right)$$

$\displaystyle\int_0^1 \frac{dt}{\sqrt{t}}$: questo integrale è del tipo $\displaystyle\int_0^1 \frac{dx}{x^\alpha}$ con $\alpha<1$

$\Rightarrow \displaystyle\int_0^1 \frac{dt}{\sqrt{t}}$ converge $\xRightarrow{\text{criterio confronto generalizzato}}$ $\displaystyle\int_0^1 f(t)\,dt$ converge

② $\int_1^{+\infty} f(t)\,dt$

$$f(t) = \frac{1}{\sqrt{t}\left(t+\sqrt{1+t^2}\right)} = \frac{1}{t\sqrt{t}} \cdot \frac{1}{1+\sqrt{1+1/t^2}}$$

$$f(t) \sim \frac{1}{2}\,\frac{1}{t\sqrt{t}} = \frac{1}{2t^{3/2}} \qquad t \to +\infty$$

$\frac{1}{2}\int_1^{+\infty} \frac{dt}{t^{3/2}}$ è del tipo $\frac{1}{2}\int_1^{+\infty} \frac{dt}{t^{\alpha}}$ con $\alpha > 1$

e quindi $\frac{1}{2}\int_1^{+\infty} \frac{dt}{t^{3/2}}$ converge

$\Longrightarrow$ (Criterio Confronto Integrali impropri asintotico) $\int_1^{+\infty} f(t)\,dt$ converge

$\Downarrow$

$\int_0^1 f(t)\,dt \in \mathbb{R}$ e $\int_1^{+\infty} f(t)\,dt \in \mathbb{R}$.

$\Downarrow$

$\int_0^{+\infty} f(t)\,dt$ converge $\left(\int_0^{+\infty} f(t)\,dt < +\infty\right)$ ↯

Esercizio: Calcolare $\displaystyle\lim_{x\to+\infty} \frac{1}{x}\int_2^x \frac{dt}{\log t}$

$$\lim_{x\to+\infty} \frac{\overbrace{\int_2^x \frac{dt}{\log t}}^{\text{dim}\;=F(x)}}{\underbrace{x}_{=G(x)}} \overset{H}{\underset{\downarrow}{=}} \lim_{x\to+\infty}\frac{\frac{1}{\log x}}{1} = 0^+$$

$F, G : [2,+\infty[\to \mathbb{R}$ ok

$\quad$ " $\quad$ continue $\quad$ ok

$\exists \displaystyle\lim_{x\to+\infty} \frac{F'(x)}{G'(x)} = 0$ $\quad$ ok

$\dfrac{\infty}{\infty}$? $\quad$ certamente $G(x) = x \xrightarrow[x\to+\infty]{} +\infty$, e per $F(x)$ si ha

$$F(x) = \int_2^x \frac{dt}{\log t} \qquad \underline{\text{Tb}}: \lim_{x\to+\infty} F(x) = +\infty \;?$$

$e^x \geq 1+x$ $\qquad\qquad$ $\log t \leq t-1 \;\; \forall t>0$

$\log t \leq t-1 \;\; \forall t>0 \Rightarrow \log t \leq t \;\; \forall t \geq 2$

$$\Rightarrow \frac{1}{\log t} \geq \frac{1}{t} \quad \forall t \geq 2$$

$$\Rightarrow \int_2^x \frac{dt}{\log t} \geq \int_2^x \frac{dt}{t} = \underbrace{\log x - \log 2}_{\downarrow\, x\to+\infty \atop +\infty} \Rightarrow \int_2^{+\infty} \frac{dt}{\log t} = +\infty$$

(Teorema Confronto Integrali Riemann)

$$\Downarrow$$

$$\lim_{x\to+\infty} F(x) = +\infty$$

c) $\displaystyle\int_0^{+\infty} \frac{arctgx}{(x^2+1)^\alpha \cdot x^{3\alpha}}dx$

Studiare, al variare di $\alpha>0$, la convergenza di c)

OSS se $\alpha \leq 0$ allora $\displaystyle\int_0^{+\infty} \frac{arctg\,x}{(x^2+1)^\alpha x^{3\alpha}} dx = +\infty$

(se $\alpha \leq 0$ allora $\displaystyle\lim_{x\to+\infty} \frac{arctg\,x}{(x^2+1)^\alpha x^{3\alpha}} \neq 0$!!!)

$$\int_0^1 f(x)dx + \int_1^{+\infty} f(x)dx \qquad f(x) = \frac{arctg\,x}{(x^2+1)^\alpha \cdot x^{3\alpha}}$$

① $\displaystyle\int_0^1 f(x)dx$ $\quad f:]0,1] \to \mathbb{R}$ continua

$$\frac{arctg\,x}{(x^2+1)^\alpha \cdot x^{3\alpha}} = \frac{x+o(x)}{(1+o(1))^\alpha \cdot x^{3\alpha}} = \frac{1}{x^{3\alpha-1}} \cdot \frac{1+o(1)}{(1+o(1))^\alpha} \quad (x\to 0)$$

$$f(x) \sim \frac{1}{x^{3\alpha-1}} \quad x\to 0 \quad \left(\lim_{x\to0^+} \frac{f(x)}{1/x^{3\alpha-1}} = 1 !\right)$$

$\displaystyle\int_0^1 \frac{dx}{x^{3\alpha-1}}$ converge se $3\alpha - 1 < 1$

se $3\alpha < 2$

se $\alpha < 2/3$

$$\boxed{\int_0^1 f(x)dx \text{ converge se } \alpha < 2/3}$$

→ (Criterio Confronto Asintotico Integrali Generalizzati)

(2) $\displaystyle\int_1^{+\infty} f(x)\,dx$

$x \to +\infty$

$$f(x) = \frac{\operatorname{arctg} x}{(x^2+1)^{\alpha} \cdot x^{3\alpha}} \sim \frac{\pi/2}{x^{3\alpha} \cdot x^{2\alpha}} = \frac{\pi}{2 \cdot x^{5\alpha}}$$

$$\left(\lim_{x \to +\infty} \frac{f(x)}{\frac{\pi}{2x^{5\alpha}}} = 1 \; !! \right)$$

Adesso $\displaystyle\frac{\pi}{2}\int_1^{+\infty} \frac{dx}{x^{5\alpha}} \in \mathbb{R}$ se $5\alpha > 1$ se $\alpha > 1/5$

$\Rightarrow$ (Criterio Confronto Asintotico Integrali Impropri) $\displaystyle\int_1^{+\infty} f(x)\,dx$ converge se $\alpha > 1/5$

$\Downarrow$ Riassumendo

$$\int_0^{+\infty} f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^{+\infty} f(x)\,dx \text{ converge se}$$

$$\underset{\int_1^{+\infty} f \in \mathbb{R}}{\frac{1}{5} <} \alpha \underset{\int_0^1 f \in \mathbb{R}}{< \frac{2}{3}}$$

Esercizio: Studiare, al variare di $\alpha>0$, la convergenza di $\displaystyle\int_2^{+\infty} \frac{dt}{\sqrt{e^t-e^2}\,(t-2)^\alpha}$

dim

$$f(t)=\frac{1}{\sqrt{e^t-e^2}\,(t-2)^\alpha} \qquad f:]2,+\infty[\to \mathbb{R} \text{ continua}$$

$$\int_2^{+\infty} \frac{dt}{e^{t/2}\sqrt{1-e^{2-t}}\,(t-2)^\alpha} \underset{\substack{t-2=y\\ dt=dy}}{=} \int_0^{+\infty} \frac{dy}{e^{\frac{y+2}{2}}\sqrt{1-e^{-y}}\cdot y^\alpha}$$

$$=\frac{1}{e}\int_0^{+\infty} \frac{dy}{e^{y/2}\sqrt{1-e^{-y}}\cdot y^\alpha} = \frac{1}{e}\underbrace{\int_0^{+\infty} \frac{dy}{\sqrt{e^y-1}\cdot y^\alpha}}$$

$$\int_2^3 f(t)\,dt + \int_3^{+\infty} f(t)\,dt \qquad f(t)=\frac{1}{\sqrt{e^t-e^2}\cdot(t-2)^\alpha}$$

1) $\displaystyle\int_2^3 f(t)\,dt \qquad f(t)=\frac{1}{e\sqrt{e^{t-2}-1}\,(t-2)^\alpha}$

$t\to 2$

$$f=\frac{1}{e\sqrt{\cancel{1}+(t-2)+o(t-2)-\cancel{1}}\cdot(t-2)^\alpha}=$$

$$=\frac{1}{e\cdot\sqrt{1+o(1)}\cdot(t-2)^{\alpha+\frac{1}{2}}} \qquad t\to 2$$

$$\sim \frac{1}{e\cdot(t-2)^{\alpha+\frac{1}{2}}} \qquad t\to 2$$

$$\left(\text{Infatti, } \lim_{t\to 2} \frac{f(t)}{\frac{1}{e\cdot(t-2)^{\alpha+\frac{1}{2}}}} = 1\right)$$

Si ha quindi

$$\int_2^3 \frac{dt}{e\cdot(t-2)^{\alpha+\frac{1}{2}}} \quad \text{converge se} \quad \alpha+\tfrac{1}{2}<1$$

$$\text{se} \quad \boxed{\alpha<\tfrac{1}{2}}$$

$\Rightarrow$ (Criterio confronto asintotico dell'integrale improprio)

$$\boxed{\int_2^3 f(t)\,dt \text{ converge se } 0<\alpha<\tfrac{1}{2}}$$

② $\displaystyle\int_3^{+\infty} f(t)\,dt$

$$f(t) = \frac{1}{\sqrt{e^t-e^2}\,(t-2)^\alpha} = \frac{1}{\sqrt{e^t}\sqrt{1-e^{2-t}}\,(t-2)^\alpha}$$

$$\sim \frac{1}{e^{t/2}\cdot t^\alpha} \qquad t\to+\infty$$

Ma $e^{t/2} = \left(e^{t/4}\right)^2 \geq \left(1+\frac{t}{4}\right)^2 \geq \frac{t^2}{16} \quad \forall t\geq 3$

$$\Rightarrow \frac{1}{e^{t/2}} \leq \frac{16}{t^2} \quad \forall t\geq 3$$

$$\Rightarrow \frac{1}{e^{t/2}\cdot t^\alpha} \leq \frac{16}{t^{2+\alpha}} \quad \forall t\geq 3 \qquad \text{ma} \quad \int_3^{+\infty} \frac{16}{t^{2+\alpha}}\,dt \in \mathbb{R} \quad \forall \alpha>0$$

$\Rightarrow$ (Criterio confronto) $\displaystyle\int_3^{+\infty} \frac{dt}{e^{t/2}\,t^\alpha}\,dt \in \mathbb{R} \quad \forall \alpha>0$

$\Rightarrow$ (Criterio confronto Integrali Impropri) $\displaystyle\int_3^{+\infty} f(t)\,dt \in \mathbb{R} \quad \forall \alpha > 0$

$\Downarrow$ Tirando le somme

$$\int_2^3 f(t)\,dt \in \mathbb{R} \qquad 0 < \alpha < \tfrac{1}{2}$$

$$\int_3^{+\infty} f(t)\,dt \in \mathbb{R} \qquad \forall \alpha > 0$$

$\Downarrow$

$\displaystyle\int_2^{+\infty} f(t)\,dt$ converge $\forall \alpha \in \left]0, \tfrac{1}{2}\right[$

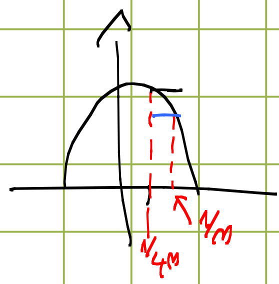
Esercizio $\quad I_n = \int_{1/4n}^{1/n} f(t)\,dt \qquad f(t) = \frac{\cos t}{2\sqrt{t}}$

1) calcolare $\lim_{n\to+\infty} I_n$

2) Per quali $\alpha > 0$ converge $\sum_n n^\alpha I_n$

dim

$$\cos\frac{1}{n}\int_{1/4n}^{1/n}\frac{dt}{2\sqrt{t}} \le \int_{1/4n}^{1/n}\frac{\cos t}{2\sqrt{t}} \le \cos\frac{1}{4n}\int_{1/4n}^{1/n}\frac{dt}{2\sqrt{t}}$$



$$\left(\cos\frac{1}{n}\right)\cdot\left[\sqrt{t}\right]_{t=1/4n}^{t=1/n} \le I_n \le \left(\cos\frac{1}{4n}\right)\cdot\left[\sqrt{t}\right]_{t=1/4n}^{t=1/n}$$

$$\cos\left(\frac{1}{n}\right)\cdot\left(\frac{1}{\sqrt{n}}-\frac{1}{2\sqrt{n}}\right) \le I_n \le \left(\cos\frac{1}{4n}\right)\cdot\left(\frac{1}{\sqrt{n}}-\frac{1}{2\sqrt{n}}\right)$$

$$\frac{\cos 1/n}{2\sqrt{n}} \le I_n \le \frac{\cos 1/4n}{2\sqrt{n}}$$

$$\downarrow_{n\to+\infty} \qquad\qquad \downarrow$$

$$0^+ \qquad\qquad 0^+$$

$\Longrightarrow$ (Teorema Carabinieri) $\quad I_n \xrightarrow[n\to+\infty]{} 0^+$

(2) $\quad a_n = n^\alpha I_n$

$$n^\alpha \frac{\cos 1/n}{\sqrt{n}} \le n^\alpha I_n \le n^\alpha \cdot \frac{\cos 1/4n}{\sqrt{n}}$$

$$a_n = \frac{\cos(1/n)}{n^{\frac{1}{2}-\alpha}} \qquad b_n = \frac{\cos(1/4n)}{n^{\frac{1}{2}-\alpha}}$$

• $a_n \sim \frac{1}{n^{\frac{1}{2}-\alpha}}$ $n \to +\infty$ e si ha

$\sum_n \frac{1}{n^{\frac{1}{2}-\alpha}}$ converge se $\frac{1}{2}-\alpha > 1$ se $-\frac{1}{2} > \alpha$

$\Rightarrow$ (Criterio Confronto asintotico per le serie) $\sum_n a_n$ converge se $\alpha < -\frac{1}{2}$

Analogamente si prova

$b_n \sim \frac{1}{n^{\frac{1}{2}-\alpha}}$ $n \to +\infty$

$\sum_n \frac{1}{n^{\frac{1}{2}-\alpha}}$ converge se $\frac{1}{2}-\alpha > 1$ se $\alpha < -\frac{1}{2}$

$\Rightarrow$ (Criterio Confronto Asintotico Serie)

$\Downarrow$ Riassumendo

$$\sum_n n^\alpha I_n \text{ converge se } \alpha < -\tfrac{1}{2}$$