Lezione 44 - Analisi Matematica 1 - 9 gennaio 2014 - h. 16.30-18.30

Teorema (Cauchy)

$f, g : [a,b] \to \mathbb{R}$

① continue su $[a,b]$

② derivabili su $]a,b[$ $\Rightarrow \exists z \in]a,b[$ $\quad \dfrac{f'(z)}{g'(z)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

③ $g'(x) \neq 0 \quad \forall x \in]a,b[$

Oss: quando $g(z) = z$ ritrovo il Teorema di Lagrange

dim

$\exists z : \quad f'(z)\left(g(b)-g(a)\right) - g'(z)\left(f(b)-f(a)\right) = 0$

Introduco la funzione ausiliaria

$h(x) = f(x)\left(g(b)-g(a)\right) - g(x)\left(f(b)-f(a)\right)$

Voglio provare che h soddisfa Teorema Rolle

① h continua su $[a,b]$ (in quanto f e g lo sono!)

② h derivabile su $]a,b[$ (" " " " " " " !)

③ $h(a) = f(a)\left(g(b)-\cancel{g(a)}\right) - g(a)\left(f(b)-\cancel{f(a)}\right) = f(a)g(b) - f(b)g(a)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$h(b) = f(b)\left(\cancel{g(b)}-g(a)\right) - g(b)\left(\cancel{f(b)} - f(a)\right) = -f(b)g(a) + f(a)g(b)$

$\Rightarrow$ (Thm Rolle) $\exists z \in]a,b[\; : \; h'(z) = 0$

ma $h'(z) = f'(z)\left(g(b)-g(a)\right) - g'(z)\left(f(b)-f(a)\right) = 0$

cioè la Tesi ↯

Teorema (Hôpital) $\left(0/0\right)$

$f, g :]a,b[\to \mathbb{R}$

1) derivabili su $]a,b[$
2) $f, g = o(1)$ per $x \to a^+$
3) $g' \neq 0$ $\forall x \in]a,b[$
4) $\exists \lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \ell \in \overline{\mathbb{R}}$

$$\Rightarrow \exists \lim_{x \to a^+} \frac{f(x)}{g(x)} = \ell$$

dim

$$\tilde{f}(x) = \begin{cases} f(x) & x \in]a,b[\\ 0 & x = a \end{cases} \qquad \tilde{g}(x) = \begin{cases} g(x) & x \in]a,b[\\ 0 & x = a \end{cases}$$

$$\frac{f(x)}{g(x)} = \frac{\tilde{f}(x) - \tilde{f}(a)}{\tilde{g}(x) - \tilde{g}(a)}$$

$\tilde{f}, \tilde{g}$ soddisfano il Teorema Cauchy su $[a, x]$

$$= \frac{\tilde{f}'(z)}{\tilde{g}'(z)} = \frac{f'(z)}{g'(z)}$$

$a < z < x$, $z \to a$, $x \to a$

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(z)}{g'(z)} = \lim_{z \to a^+} \frac{f'(z)}{g'(z)}$$

Teorema (Hôpital) $\left(\infty/\infty\right)$ (NON LO DIMOSTRO)

$f, g :]a,b[\to \mathbb{R}$

1) derivabili su $]a,b[$
2) $\lim_{x \to a^+} f = \infty$ e $\lim_{x \to a^+} g = \infty$
3) $g' \neq 0$ $\forall x \in]a,b[$
4) $\exists \lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \ell \in \overline{\mathbb{R}}$

$$\Rightarrow \exists \lim_{x \to a^+} \frac{f(x)}{g(x)} = \ell$$

Oss: se $g'(x) \neq 0$ $\forall x \in]a,b[$ e $g(x) \xrightarrow[x \to a^+]{} 0$

allora $g'(x) < 0 \Rightarrow g(x)$ decrescente $\Rightarrow g(x) \neq g(y)$ $x \neq y$ $\Rightarrow g(x) \neq 0$ $\forall x$

$\backslash \; g'(x)>0 \Rightarrow g(x)$ crescente $\Rightarrow g(x)\neq 0 \;\; \forall x$

Teorema (Taylor con il resto di Peano)

$f:]a,b[\to \mathbb{R} \qquad x_0 \in]a,b[$

f derivabile $(n-1)$ volte in $]a,b[$

f " n volte in x_0

Posto $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

Allora $f(x) - P_n(x) = o\left((x-x_0)^n\right) \quad x \to x_0$

dim

Tesi $\lim_{x\to x_0} \frac{f(x)-P_n(x)}{(x-x_0)^n} = 0$

$$\begin{cases} P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\ P_n(x_0) = f(x_0) \end{cases}$$

$$\lim_{x\to x_0} \frac{f(x)-P_n(x)}{(x-x_0)^n} \overset{H}{=} \lim_{x\to x_0} \frac{f'(x)-P_n'(x)}{n\cdot(x-x_0)^{n-1}}$$

$$\begin{cases} P_n'(x) = f'(x_0) + f''(x_0)\cdot(x-x_0) + \frac{f'''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{(n-1)!}(x-x_0)^{n-1} \\ P_n'(x_0) = f'(x_0) \end{cases}$$

$$\lim_{x\to x_0} \frac{f'(x)-P_n'(x)}{n(x-x_0)^{n-1}} \overset{H}{=} \lim_{x\to x_0} \frac{f''(x)-P_n''(x)}{n\cdot(n-1)(x-x_0)^{n-2}}$$

$$\overset{H}{=} \underset{n-1 \text{ volte}}{\dots} \overset{H}{=} \lim_{x \to x_0} \frac{f^{(n-1)}(x) - P_n^{(n-1)}(x)}{n!\,(x-x_0)}$$

$$= \lim_{x \to x_0} \frac{f^{(n-1)}(x) - \left[f^{(n-1)}(x_0) + f^{(n)}(x_0)(x-x_0)\right]}{n!\,(x-x_0)} =$$

$$= \frac{1}{n!}\left\{ \underbrace{\lim_{x \to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{x-x_0}} - f^{(n)}(x_0) \right\}$$

$$= \frac{1}{n!}\left\{ f^{(n)}(x_0) - f^{(n)}(x_0) \right\} = 0$$

Oss: $\sin x = x + o(x^2)$ per $x \to 0$

però $\sup_{x \in \mathbb{R}} \sin x = 1$ $\quad \sup_{x \in \mathbb{R}} (x) = +\infty$

quindi $o(x^2)$ è infinitesimo "per $x \to 0$"

Teorema (Taylor con il resto di Lagrange)

$f:]a,b[\to \mathbb{R}$ $\quad x_0 \in]a,b[$

f derivabile $n+1$ volte in $]a,b[$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

Allora $\quad f(x) - P_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-x_0)^{n+1}$

z compreso tra x_0 e x

$$(f+g)' = f' + g'$$

$$(f \cdot g)' = f'g + f g' \longrightarrow \text{Integrazione per parti}$$

$$(f \circ g)' = g'(f) \cdot f' \longrightarrow \text{" " sostituzione}$$

Ricerca delle Primitive

La derivata è unica
mentre
Le primitive di una funzione sono ∞

Due primitive di f differiscono per una costante
$\Downarrow$
Se f ha F come primitiva allora $\int f(x)dx = F(x) + c$
$c \in \mathbb{R}$

Esempio $\int \cos x \, dx = \operatorname{sen} x + c \quad c \in \mathbb{R}$

$$(u \cdot v)' = u'v + uv'$$

$$u'v = (u \cdot v)' - uv'$$

$$\Downarrow$$

$$\int u'v \, dx = \int (uv)' dx - \int uv' dx$$

$$\int u'v \, dx = u \cdot v - \int uv' dx$$

Esempio $\int x e^x \, dx$

dim

$$\int \underset{u'}{x}\, \underset{v}{e^x} \, dx = \underset{u}{\frac{x^2}{2}} \cdot \underset{v}{e^x} - \int \underset{u}{\frac{x^2}{2}} \cdot \underset{v'}{e^x} dx$$

$$\int \underset{u}{x}\, \underset{v'}{e^x} \, dx = \underset{u}{x} \cdot \underset{v}{e^x} - \int \underset{u'}{1} \cdot \underset{v}{e^x} dx$$

$$= \underbrace{x e^x - e^x + c}\quad c \in \mathbb{R}$$

Esempio Calcolare $\int \log x \, dx$

dim

$$\int \underset{u}{\log x} \cdot \underset{v'}{1} \, dx = \underset{u}{\log x} \cdot \underset{v}{x} - \int \underset{u'}{\frac{1}{x}} \cdot \underset{v}{x} \, dx$$

$$= \log x \cdot x - x + c \quad c \in \mathbb{R}$$

Esempio Calcolare $\int \operatorname{sen}^2 x \, dx$

dim

$$\int \underset{u'}{\operatorname{sen} x} \cdot \underset{v}{\operatorname{sen} x} \, dx = \underset{u}{(-\cos x)} \cdot \underset{v}{\operatorname{sen} x} - \int \underset{u}{(-\cos x)} \underset{v'}{\cos x} \, dx$$

$$= -\operatorname{sen} x \cos x + \int \cos^2 x \, dx$$

$$= -\operatorname{sen} x \cos x + \int (1 - \operatorname{sen}^2 x) \, dx$$

$$\int \operatorname{sen}^2 x \, dx = -\operatorname{sen} x \cos x + \int 1 \, dx - \int \operatorname{sen}^2 x \, dx$$

$$2 \int \operatorname{sen}^2 x \, dx = -\operatorname{sen} x \cos x + x + c \quad c \in \mathbb{R}$$

$$\int \operatorname{sen}^2 x \, dx = \frac{1}{2} \left(x - \operatorname{sen} x \cos x \right) + c \quad c \in \mathbb{R}$$

Sostituzione

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

$$\int (g \circ f)'(x) \, dx = \int g'(f(x)) \cdot f'(x) \, dx$$

- $$(g \circ f)(x) + c = \int g'(f(x)) \cdot f'(x) \, dx \quad c \in \mathbb{R}$$

- $$\int \operatorname{sen} x \, dx = -\cos x + c \Rightarrow \int \operatorname{sen}(\varphi(x)) \cdot \varphi'(x) \, dx = -\cos(\varphi(x)) + c \quad c \in \mathbb{R}$$

$$\frac{d}{dx} \left(\int \operatorname{sen}(\varphi(x)) \, \varphi'(x) \, dx \right) = \frac{d}{dx} \left(-\cos(\varphi(x)) + c \right)$$

$$\operatorname{sen}(\varphi(x)) \cdot \varphi'(x) \, dx = \operatorname{sen}(\varphi(x)) \cdot \varphi'(x)$$

Esempio Calcolare $\int \sqrt{1-x^2}\,dx$

dim

$$\int \sqrt{1-x^2}\,dx = \left(\int \sqrt{1-\operatorname{sen}^2 t}\,\cos t\,dt\right)_{x=\operatorname{sen} t}$$

$x = \operatorname{sen} t$

$\frac{dx}{dt} = \cos t$

$\boxed{dx = \cos t\,dt}$

$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$$= \left(\int \cos^2 t\,dt\right)_{\substack{x=\operatorname{sen} t \\ t\in[-\frac{\pi}{2},\frac{\pi}{2}]}} = \left(\frac{1}{2}\left(t+\operatorname{sen} t\cos t\right)+c\right)_{x=\operatorname{sen} t}$$

$$\left(\int \cos^2 t\,dt = \int (1-\operatorname{sen}^2 t)\,dt = t - \frac{1}{2}\left(t-\operatorname{sen} t\cos t\right)+c\right.$$

$$\left.= \frac{1}{2}\left(t+\operatorname{sen} t\cos t\right)+c \quad c\in\mathbb{R}\right)$$

$$= \frac{1}{2}\left(\operatorname{arcsen} x + x\sqrt{1-x^2}\right)+c \quad c\in\mathbb{R}$$

$\int e^{-x^2}\,dx$ $\qquad$ e^{-x^2} è una funzione continua

Oss: e^{-x^2} ha primitive che NON SONO ESPRIMIBILI in termini di funzioni elementari

Oss Tutte le primitive di e^{-x^2} sono date da

$$\int e^{-x^2}dx = \underbrace{\int_0^x e^{-t^2}dt}_{\text{Funzione integrale}} + c \quad c \in \mathbb{R}$$

$$\int \frac{\operatorname{sen} x}{x}dx = \int_0^x \frac{\operatorname{sen} t}{t}dt + c \quad c \in \mathbb{R}$$

INTEGRALE DEFINITO

$f : [a,b] \to \mathbb{R}$ limitata

$\mathcal{A} = \{x_0 = a < x_1 < \dots < x_n = b\}$ suddivisioni

$$S(f,\mathcal{A}) = \sum_{k=1}^{n} (x_k - x_{k-1}) \cdot \sup f([x_{k-1}, x_k])$$

$$s(f,\mathcal{A}) = \sum_{k=1}^{n} (x_k - x_{k-1}) \inf f([x_{k-1}, x_k])$$

$$S(f) = \inf\{S(f,\mathcal{A}) : \mathcal{A} \text{ suddivisione}\}$$

$$s(f) = \sup\{s(f,\mathcal{A}) : \mathcal{A} \text{ suddivisione}\}$$

Def $f : [a,b] \to \mathbb{R}$ è R-integrabile se

$$s(f) = S(f) \in \mathbb{R} \qquad s(f) = S(f) = \int_a^b f(x)dx$$

Teorema (CNS)

$f : [a,b] \to \mathbb{R}$ limitata; sono equivalenti

i) f $\mathcal{R}$-integrabile

ii) $\forall \varepsilon > 0 \ \exists \mathcal{A}_\varepsilon \ \mathcal{B}_\varepsilon$ sudd. : $S(f, \mathcal{A}_\varepsilon) - s(f, \mathcal{B}_\varepsilon) < \varepsilon$

iii) $\forall \varepsilon > 0 \ \exists \mathcal{C}_\varepsilon$ sudd. : $S(f, \mathcal{C}_\varepsilon) - s(f, \mathcal{C}_\varepsilon) < \varepsilon$

Teorema

f continua su $[a,b] \Rightarrow f$ $\mathcal{R}$-integr.

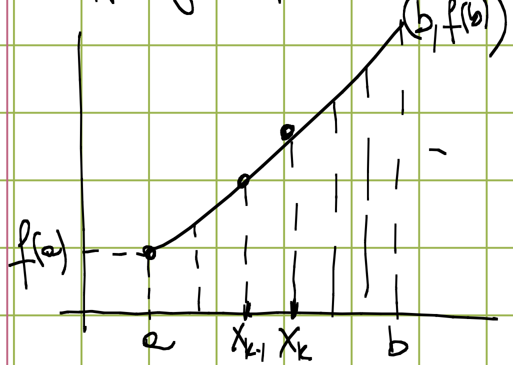
Teorema

f monotona su $[a,b] \Rightarrow f$ $\mathcal{R}$-integrabile

dim

Tesi $\forall \varepsilon > 0 \ \exists \mathcal{C}_\varepsilon$ sudd. t.c. $S(f, \mathcal{C}_\varepsilon) - s(f, \mathcal{C}_\varepsilon) < \varepsilon$

Suppongo f strettamente crescente



$$A_m = \left\{ a + k \cdot \frac{b-a}{m} \ : \ k = 0, 1, 2, \dots, m \right\}$$

$$\sup f([x_{k-1}, x_k]) = f(x_k)$$

$$\inf f([x_{k-1}, x_k]) = f(x_{k-1})$$

$$S(f, \mathcal{A}_m) = \sum_{k=1}^{m} (x_k - x_{k-1}) \sup f([x_{k-1}, x_k])$$

$$= \sum_{k=1}^{m} \frac{b-a}{m} \cdot f(x_k)$$

$$s(f, A_n) = \sum_{k=1}^{n} (x_k - x_{k-1}) \inf f([x_{k-1}, x_k])$$
$$= \sum_{k=1}^{n} \frac{b-a}{n} \cdot f(x_{k-1})$$

$$S(f, A_n) - s(f, A_n) = \frac{b-a}{n} \sum_{k=1}^{n} \left(f(x_k) - f(x_{k-1}) \right)$$
$$= \frac{b-a}{n} \left[\left(f(\cancel{x_1}) - f(x_0) \right) + \left(f(\cancel{x_2}) - f(\cancel{x_1}) \right) + \dots + \left(f(x_n) - f(\cancel{x_{n-1}}) \right) \right]$$
$$= \frac{b-a}{n} \cdot \left(f(x_n) - f(x_0) \right) = \frac{(b-a)(f(b)-f(a))}{n}$$

$\forall \varepsilon > 0 \quad \exists\, n > \dfrac{(b-a)(f(b)-f(a))}{\varepsilon} \quad \exists\, A_n$ suddivisione

t.c. $\quad S(f, A_n) - s(f, A_n) < \varepsilon$

Teorema (della Media)

$f : [a,b] \to \mathbb{R}$

1) f R integrabile $\implies \inf f([a,b]) \le \overbrace{\frac{1}{b-a} \int_a^b f(x)\,dx} \le \sup f([a,b])$

2) f continua $\implies \exists\, z \in [a,b] : f(z) = \dfrac{1}{b-a} \displaystyle\int_a^b f(x)\,dx$

dim

1) $\quad \inf f([a,b]) \le f(x) \le \sup f([a,b]) \quad \forall x \in [a,b]$

$\Downarrow$ Teorema Confronto

$$\int_a^b \inf f([a,b])\,dx \le \int_a^b f(x)\,dx \le \int_a^b \sup f([a,b])\,dx$$

$$(b-a)\cdot \inf f([a,b]) \le \int_a^b f(x)\,dx \le (b-a)\sup f([a,b])$$

2) f continua su $[a,b]$

$\Rightarrow$ (Thm Weierstrass) $\min f([a,b]) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \max f([a,b])$

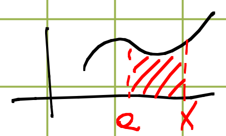
$\Rightarrow$ (Thm Valori Intermedi) $\exists z \in [a,b] : f(z) = \frac{1}{b-a}\int_a^b f(x)\,dx$

Intervalli orientati $\quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dX$

Teorema (Fondamentale Calcolo)

$f: I \to \mathbb{R}$, I intervallo, f continua. Fissato $a \in I$

$F(x) = \int_a^x f(t)\,dt$



Allora $F'(x) = f(x) \quad \forall x \in I$

dim

Fisso $x_0 \in I$ e provo $F'(x_0) = f(x_0)$

$$\frac{F(x)-F(x_0)}{x-x_0} = \frac{1}{x-x_0}\left(\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt\right)$$

$$= \frac{1}{x-x_0}\left(\int_a^x f(t)\,dt + \int_{x_0}^a f(t)\,dt\right)$$

$$= \frac{1}{x-x_0} \cdot \int_{x_0}^{x} f(t)\,dt = f(z) \qquad z \text{ compreso tra } x \text{ e } x_0$$

Passando al limite

$$F'(x_0) = \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = \lim_{x \to x_0} f(z) = \lim_{z \to x_0} f(z) = f(x_0)$$

Pb Calcolare $\frac{d}{dx}\left(\int_{x \operatorname{sen} x}^{2} e^{-t^2}\,dt \right)$

Pb calcolare $\frac{d}{dx}\left(\int_{x}^{3} e^{-t^2}\,dt \right) =$

$$= \frac{d}{dx}\left(- \int_{3}^{x} e^{-t^2}\,dt \right) = -\frac{d}{dx}\left(\int_{3}^{x} e^{-t^2}\,dt \right)$$

teor fond.

$$= -e^{-x^2}$$

Calcolare

$$\frac{d}{dx}\left(\int_{x \operatorname{sen} x}^{3} e^{-t^2}\,dt \right) = -\frac{d}{dx}\left(\int_{3}^{x \operatorname{sen} x} e^{-t^2}\,dt \right)$$

$$= -\frac{d}{dx}\left(F(x \operatorname{sen} x) \right) \qquad x \mapsto x \operatorname{sen} x \mapsto F(x \operatorname{sen} x)$$

$$= -F'(x \operatorname{sen} x) \cdot (x \operatorname{sen} x)' = -e^{-(x \operatorname{sen} x)^2} \cdot (\operatorname{sen} x + x \cos x)$$