# Lezione 36 - Analisi Matematica 1 - dicembre 2013

## Primitive delle funzioni razionali

Abbiamo già trattato la ricerca delle primitive delle funzioni

$$f(x) = \frac{P(x)}{x^2+bx+c}$$

distinguendo tra 1) $b^2-4ac>0$ (radici reali $\neq$)
2) $b^2-4ac=0$ (radici reali $=$)
3) $b^2-4ac<0$ (radici complesse coniugate $\neq$)

**Esercizio:** Calcolare

(i) $\displaystyle\int \frac{x\,dx}{x^2-4x+4}$ (ii) $\displaystyle\int \frac{(x+1)\,dx}{x^2-2x-6}$

(iii) $\displaystyle\int \frac{(2x-3)\,dx}{x^2-4x+1}$ (iv) $\displaystyle\int \frac{dx}{x^2+2x+7}$

(i) $x^2-4x+4=(x-2)^2$ ovvero radici reali $=$
vanno determinati $A,B\in\mathbb{R}$ t.c.

$$\frac{x}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} = \frac{Ax-2A+B}{(x-2)^2} \qquad \begin{cases} A=1 \\ B=2 \end{cases}$$

$$\int \frac{x\,dx}{(x-2)^2} = \int \frac{dx}{x-2} + 2\int \frac{dx}{(x-2)^2} = \log|x-2| - \frac{1}{(x-2)} + c \quad c\in\mathbb{R}$$

(ii) $x^2-2x-6 = (x^2-2x+1)-7 = (x-1)^2-7$ ovvero radici reali $\neq$

$$\frac{x+1}{(x-1)^2-7} = \frac{A}{x-1-\sqrt{7}} + \frac{B}{x-1+\sqrt{7}} = \frac{x(A+B)-(A+B)+\sqrt{7}(A-B)}{(x-1)^2-7}$$

$$\begin{cases} A+B=1 \\ -1+\sqrt{7}(A-B)=1 \end{cases} \qquad \begin{cases} A=1-B \\ A-B=2/\sqrt{7} \end{cases} \qquad \begin{cases} \\ 1-2B=2/\sqrt{7} \end{cases}$$

$$\begin{cases} B=\frac{1}{2}\left(1-\frac{2}{\sqrt{7}}\right) \end{cases} \qquad B=\frac{\sqrt{7}-2}{2\sqrt{7}} \qquad A=\frac{2\sqrt{7}-\sqrt{7}+2}{2\sqrt{7}} = \frac{\sqrt{7}+2}{2\sqrt{7}}$$

$$\int \frac{x+1}{(x-1)^2-7}\,dx = \frac{\sqrt7+2}{2\sqrt7}\log\left(x-1-\sqrt7\right) + \frac{\sqrt7-2}{2\sqrt7}\log\left(x-1+\sqrt7\right)+c$$
$c\in\mathbb{R}$

(iii) $x^2-4x+1 = (x^2-4x+4)-3 = (x-2)^2-3$ radici reali $\neq$

$$\frac{2x-3}{(x-2)^2-3} = \frac{A}{x-2-\sqrt3} + \frac{B}{x-2+\sqrt3} = \frac{x(A+B)-2(A+B)+\sqrt3(A-B)}{(x-2)^2-3}$$

$$\begin{cases} A+B=2 \\ -4+\sqrt3(A-B)=-3 \end{cases} \quad \begin{cases} A+B=2 \\ A-B=1/\sqrt3 \end{cases} \quad \begin{cases} 2A=2+1/\sqrt3 \\ 2B=2-1/\sqrt3 \end{cases} \quad \begin{aligned} A&=1+1/2\sqrt3 \\ B&=1-1/2\sqrt3 \end{aligned}$$

$$\int \frac{2x-3}{(x-2)^2-3}\,dx = \left(1+\frac{1}{2\sqrt3}\right)\int\frac{dx}{x-2-\sqrt3} + \left(1-\frac{1}{2\sqrt3}\right)\int\frac{dx}{x-2+\sqrt3}$$
$$= \left(1+\frac{1}{2\sqrt3}\right)\log\left(x-2-\sqrt3\right) + \left(1-\frac{1}{2\sqrt3}\right)\log\left(x-2+\sqrt3\right)+c \qquad c\in\mathbb{R}$$

(iv) $x^2+2x+7 = (x^2+2x+1)+6 = (x+1)^2+6$ radici complesse e coniugate $\neq$
$$= 6\left[1+\left(\frac{x+1}{\sqrt6}\right)^2\right]$$

$$\int\frac{dx}{x^2+2x+7} = \frac{1}{6}\int\frac{dx}{1+\left(\frac{x+1}{\sqrt6}\right)^2} = \frac{1}{\sqrt6}\int\frac{\frac{1}{\sqrt6}\,dx}{1+\left(\frac{x+1}{\sqrt6}\right)^2}$$

$$\underset{y=\frac{x+1}{\sqrt6}}{=} \frac{1}{\sqrt6}\left(\int\frac{dy}{1+y^2}\right)_{y=\frac{x+1}{\sqrt6}} = \frac{1}{\sqrt6}\,\operatorname{arctg}\left(\frac{x+1}{\sqrt6}\right)+c$$
$c\in\mathbb{R}$

Torniamo al pb. generale:

Trovare le primitive di $\dfrac{P(x)}{Q(x)}$

La prima cosa da fare, se $(\text{grado } P) \geq (\text{grado } Q)$, è eseguire la divisione

$$P(x) = Q(x) \cdot S(x) + R(x)$$

In tal modo $\dfrac{P(x)}{Q(x)} = S(x) + \dfrac{R(x)}{Q(x)}$

dove $S(x)$ è un polinomio (la primitiva è calcolabile direttamente) mentre $(\text{grado } R(x)) < (\text{grado } Q(x))$

Quindi il problema da risolvere si riduce a

$$\boxed{\text{Trovare le primitive di } \frac{P(x)}{Q(x)} \text{ quando } (\text{gr } P(x)) < (\text{gr } Q(x))}$$

Esempio (riduzione grado numeratore)

Calcolare le primitive di $f(x) = \dfrac{x^3+x+2}{x^2+1}$

divisione

| | | | |
|---|---|---|---|
| $x^3$ | $+x$ | $+2$ | $x^2+1$ |
| $x^3$ | $+x$ | | $x$ |
| // | // | $2$ | |

ovvero

$$x^3+x+2 = x \cdot (x^2+1) + 2$$

dunque $\displaystyle\int \frac{x^3+x+2}{x^2+1}\,dx = \int x\,dx + 2\int \frac{dx}{x^2+1} =$

$$= \frac{x^2}{2} + 2\,\text{arctg}\, x + c \qquad c \in \mathbb{R}$$

# $Q(x)$ ha radici reali semplici

**Def** (radice reale semplice)

Dato un polinomio $Q(x)$, un numero $a \in \mathbb{R}$ è detto "radice reale semplice" di $Q(x)$

se $Q(a)=0$ e $Q'(a) \neq 0$

**Def** (radice reale molteplicità $m$)

Dato un polinomio $Q(x)$, un numero $a \in \mathbb{R}$ è detto "radice reale di molteplicità $m$"

se $Q(a) = Q'(a) = Q''(a) = \dots = Q^{(m-1)}(a) = 0$ e $Q^{(m)}(a) \neq 0$

Esempio:

$Q(x) = x^2-1$ ha 2 radici reali semplici $x_0 = 1$ $x_1 = -1$

(infatti $Q(1) = Q(-1)$ mentre $Q'(x) = 2x$ $Q'(1) \neq 0$ $Q'(-1) \neq 0$)

Esempio:

$Q(x) = x^2(x-1)$ ha $x_0 = 0$ radice reale doppia

$x_1 = 1$ " " semplice

infatti $Q'(x) = 2x(x-1) + x^2 = x(x+2x-2) = x(3x-2)$

$Q''(x) = 6x-2$

$x=0$ è t.c. $Q(0) = Q'(0) = 0$ e $Q''(0) = -2 \neq 0$

$x=1$ " " $Q(1) = 0$ e $Q'(1) \neq 0$

**Teorema** (radici reali semplici) (senza dim.)

Sia $f(x) = P(x)/Q(x)$ con grado $(P(x))$ < grado $(Q(x))$

Se esistono $a_i \in \mathbb{R}$ $i = 1, \dots, m$ : $Q(x) = \prod_{i=1}^{m} (x-a_i)$

allora $\exists A_i \in \mathbb{R}$ $i = 1, \dots, m$ tali che

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \dots + \frac{A_m}{x-a_m} = \sum_{i=1}^{m} \frac{A_i}{x-a_i}$$

Esercizio: Calcolare $\int \frac{dx}{x(x^2-1)}$

dim

$Q(x) = x(x^2-1)$ ha $a_1=0$ $a_2=1$ $a_3=-1$ radici reali semplici

$$\frac{1}{x(x^2-1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{(x^2-1)A + Bx^2+Bx + Cx^2 - Cx}{x(x^2-1)}$$

$$\begin{cases} A+B+C=0 \\ B-C=0 \\ -A=1 \end{cases} \quad \begin{cases} 2B-1=0 \\ B=C \\ A=-1 \end{cases} \quad \begin{cases} B=\frac{1}{2} \\ C=\frac{1}{2} \\ A=-1 \end{cases}$$

$$\int \frac{dx}{x(x^2-1)} = -\int \frac{dx}{x} + \frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1}$$

$$= -\log x + \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1| + c \quad c \in \mathbb{R}$$

Verifica

$$\left(-\log x + \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1|\right)' = -\frac{1}{x} + \frac{1}{2}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1}$$

$$= \frac{-2(x^2-1) + x^2 + \cancel{x} + x^2 - \cancel{x}}{2x(x^2-1)} = \frac{1}{x(x^2-1)}$$

## $Q(x)$ ha radici reali coincidenti

**Teorema** (radici reali coincidenti) (no dim)

Sia $f(x) = \frac{P(x)}{Q(x)}$ con grado$(P(x)) <$ grado$(Q(x))$

Se $Q(x) = (x-a)^m (x-b)^n$ con $m, n > 1$ $\quad a, b \in \mathbb{R}$

allora $\exists\, A_i \in \mathbb{R}$ $i = 1 \dots m$ $\quad B_j \in \mathbb{R}$ $j = 1 \dots n$ tali che

$$f(x) = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_m}{(x-a)^m} + \frac{B_1}{(x-b)} + \frac{B_2}{(x-b)^2} + \dots + \frac{B_n}{(x-b)^n}$$

$$= \sum_{i=1}^{m} \frac{A_i}{(x-a)^i} + \sum_{J=1}^{n} \frac{B_J}{(x-b)^J}$$

Nota Bene: Nel Teorema abbiamo considerato solo due radici multiple: $a$ di molteplicità $m$
$b$ " " $n$

**Esempio** Calcolare $\displaystyle\int \frac{x+1}{x^2(x-1)}\,dx$

dim

$Q(x) = x^2(x-1)$ ha $x=1$ radice reale semplice dunque
$x=0$ " " doppia

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{Ax^2 - Ax + Bx - B + Cx^2}{x^2(x-1)}$$

$$\begin{cases} A+C = 0 \\ B-A = 1 \\ -B = 1 \end{cases} \quad \begin{cases} C = 2 \\ A = -2 \\ B = -1 \end{cases} \Rightarrow \frac{x+1}{x^2(x-1)} = -\frac{2}{x} - \frac{1}{x^2} + \frac{2}{x-1}$$

$$\int \frac{x+1}{x^2(x-1)}\,dx = -2\log|x| + \frac{1}{x} + 2\log|x-1| + c \qquad c \in \mathbb{R}$$

Verifica

$$\left(-2\log|x| + \frac{1}{x} + 2\log|x-1|\right)' = -\frac{2}{x} - \frac{1}{x^2} + \frac{2}{x-1} = \frac{\cancel{-2x^2} + 2x - x + 1 + \cancel{2x^2}}{x^2(x-1)} = \frac{x+1}{x^2(x-1)}$$

Esercizio Calcolare $\displaystyle\int \frac{x^2+3}{(x^2-1)^2}\,dx$

dim Sol $\displaystyle\int f\,dx = \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{2x}{x^2-1} + c \quad c \in \mathbb{R}$

$Q(x) = (x^2-1)^2$ ha $x=1$ radice doppia reale

$x=-1$ " " "

$$\frac{x^2+3}{(x-1)^2(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)} + \frac{D}{(x+1)^2}$$

$$= \frac{A(x-1)(x+1)^2 + B(x+1)^2 + C(x-1)^2(x+1) + D(x-1)^2}{(x^2-1)^2}$$

$$= \frac{A(x-1)(x^2+2x+1) + B(x^2+2x+1) + C(x^2-2x+1)(x+1) + D(x^2-2x+1)}{(x^2-1)^2}$$

$$= \frac{A(x^3+2x^2+x-x^2-2x-1) + B(x^2+2x+1) + C(x^3-2x^2+x+x^2-2x+1) + D(x^2-2x+1)}{(x^2-1)^2}$$

$$\begin{cases} A+C=0 \\ A+B-C+D=1 \\ -A+2B-C+2D=0 \\ -A+B+C+D=3 \end{cases}$$

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## $Q(x)$ ha radici complesse e coniugate semplici

In questo caso $Q(x)$ ha dei fattori $x^2+ax+b$ con $a^2-4b<0$, però questi fattori compaiono con potenza 1 nella fattorizzazione

**Teorema** (radici complesse - coniugate - semplici) (no dim)

Sia $f(x) = \frac{P(x)}{Q(x)}$ con $\text{grado}(P(x)) < \text{grado}(Q(x))$ $P, Q$ polinomi

Se $Q(x) = (x^2+a_1x+b_1)(x^2+a_2x+b_2) \cdot \dots \cdot (x^2+a_mx+b_m)$

$$= \prod_{i=1}^{m} (x^2+a_ix+b_i) \qquad a_i, b_i \in \mathbb{R} \quad i=1 \dots m$$
$$a_i^2-4b_i<0 \quad i=1\dots m$$

allora $\exists\, A_i, B_i \in \mathbb{R}$ $i=1\dots m$ tali che

$$f(x) = \frac{A_1x+B_1}{x^2+a_1x+b_1} + \dots + \frac{A_mx+B_m}{x^2+a_mx+b_m} = \sum_{i=1}^{m} \frac{A_ix+B_i}{x^2+a_ix+b_i}$$

Oss: I termini del tipo $\frac{Ax+B}{x^2+ax+b}$, $a^2-4b<0$, hanno primitive esprimibile in termini di funzioni elementari

Esempio Calcolare $\int \frac{x+1}{x^2+1}\,dx$

dim

$$\frac{x+1}{x^2+1} = \frac{x}{x^2+1} + \frac{1}{x^2+1} = \frac{1}{2}\,\frac{2x}{x^2+1} + \frac{1}{x^2+1}$$

$$\Downarrow$$

$$\int \frac{x+1}{x^2+1}\,dx = \frac{1}{2}\log(x^2+1) + \arctg x + c \qquad c \in \mathbb{R}$$

Esempio calcolare $\displaystyle\int \frac{x\,dx}{(x^2+1)(x^2-2x+2)}$ $\quad 1\pm i$

dim

$Q(x)$ ha come radici $x = i, -i, 1-i, 1+i$ semplici complesse e coniugate tra loro

$$\frac{x}{(x^2+1)(x^2-2x+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2-2x+2}$$

$$x = Ax^3 - 2Ax^2 + 2Ax + Bx^2 - 2Bx + 2B + Cx^3 + Cx + Dx^2 + D$$

$$\begin{cases} A+C=0 \\ -2A+B+D=0 \\ 2A-2B+C=1 \\ 2B+D=0 \end{cases} \quad \begin{cases} A=-C \\ -2C+B+D=0 \\ -2C-2B+C=1 \\ 2B+D=0 \end{cases} \quad \begin{cases} A=-C \\ -2(-1-2B)+B+D=0 \\ C=-1-2B \\ D=-2B \end{cases} \quad \begin{cases} A=-\frac{1}{3} \\ 2+5B-2B=0 \\ C=-1+\frac{4}{3}=\frac{1}{3} \\ D=\frac{4}{3} \end{cases}$$

$$f(x) = \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+1} + \frac{1}{3}\frac{x+4}{x^2-2x+2} = -\frac{1}{3}\frac{x+2}{x^2+1} + \frac{1}{3}\frac{x+4}{x^2-2x+2}$$

$$= -\frac{1}{3}\cdot\frac{1}{2}\frac{2x}{x^2+1} - \frac{2}{3}\frac{1}{x^2+1} + \frac{1}{3}\frac{x-1}{x^2-2x+2} + \frac{5}{3}\frac{1}{(x-1)^2+1}$$

$$= -\frac{1}{6}\frac{2x}{x^2+1} - \frac{2}{3}\frac{1}{x^2+1} + \frac{1}{6}\frac{2x-2}{x^2-2x+2} + \frac{5}{3}\frac{1}{(x-1)^2+1}$$

$$\int f(x)\,dx = -\frac{1}{6}\log(x^2+1) - \frac{2}{3}\operatorname{arctg} x + \frac{1}{6}\log(x^2-2x+2) + \frac{5}{3}\operatorname{arctg}(x-1) + c$$

$c \in \mathbb{R}$

## $Q(x)$ ha radici complesse e coniugate multiple

**Teorema** (radici complesse e coniugate multiple) (no dim)

Se $f(x) = P(x)/Q(x)$ con grado $(P(x)) <$ grado $(Q(x))$ P, Q polinomi

Se $Q(x) = (x^2+ax+b)^m (x^2+cx+d)^n$ $\quad a,b,c,d \in \mathbb{R}$

$a^2-4b<0 \quad c^2-4d<0$

$n, m > 1$

allora $\exists A_i, B_i \in \mathbb{R}$ $i=1,\dots,m$ $\quad C_j, D_j \in \mathbb{R}$ $j=1,\dots,n$ tali che

$$f(x) = \frac{A_1x+B_1}{x^2+ax+b} + \frac{A_2x+B_2}{(x^2+ax+b)^2} + \dots + \frac{A_mx+B_m}{(x^2+ax+b)^m} +$$

$$+ \frac{C_1x+D_1}{x^2+cx+d} + \frac{C_2x+D_2}{(x^2+cx+d)^2} + \dots + \frac{C_nx+D_n}{(x^2+cx+d)^n}$$

$$= \sum_{i=1}^{m} \frac{A_ix+B_i}{(x^2+ax+b)^i} + \sum_{j=1}^{n} \frac{C_jx+D_j}{(x^2+cx+d)^j}$$

**Esempio** (importante) Calcolare $\int \frac{dx}{(1+x^2)^2}$

$$\int \frac{dx}{(1+x^2)^2} = \int \frac{\cancel{1+x^2}}{(1+x^2)^{\cancel{2}\,1}}dx - \int \frac{x^2}{(1+x^2)^2}dx =$$

(dim)

$$= \operatorname{arctg} x - \frac{1}{2}\int \underbrace{\frac{2x}{(1+x^2)^2}}_{u'} \cdot \underbrace{x}_{v}\, dx =$$

$$= \operatorname{arctg} x - \frac{1}{2}\left[\left(\underbrace{-\frac{1}{1+x^2}}_{u}\right)\cdot \underbrace{x}_{v} - \int\left(\underbrace{-\frac{1}{1+x^2}}_{u}\right)\cdot \underbrace{1}_{v'}\cdot dx\right]$$

$$= \operatorname{arctg} x + \frac{1}{2}\frac{x}{1+x^2} - \frac{1}{2}\operatorname{arctg} x + c \quad \text{perciò}$$

$$\int \frac{dx}{(1+x^2)^2} = \frac{1}{2}\left(\operatorname{arctg} x + \frac{x}{1+x^2}\right) + c \quad c \in \mathbb{R}$$

Esempio Calcolare $\int \frac{dx}{(1+x^2)^3}$

dim

$$\int \frac{dx}{(1+x^2)^3} = \int \frac{1+x^2}{(1+x^2)^3}dx - \int \frac{x^2}{(1+x^2)^3}dx$$

$$= \int \frac{dx}{(1+x^2)^2} - \frac{1}{2}\int \frac{2x}{(1+x^2)^3} \cdot x\,dx$$

$$= \int \frac{dx}{(1+x^2)^2} - \frac{1}{2}\left[-\frac{1}{2(1+x^2)^2}\cdot x - \int \left(-\frac{1}{2(1+x^2)^2}\right)dx\right]$$

$$= \int \frac{dx}{(1+x^2)^2} + \frac{1}{4}\frac{x}{(1+x^2)^2} - \frac{1}{4}\int \frac{dx}{(1+x^2)^2}$$

$$= \frac{3}{4}\int \frac{dx}{(1+x^2)^2} + \frac{1}{4}\frac{x}{(1+x^2)^2} = \frac{1}{4}\left(\frac{3}{2}\operatorname{arctg} x + \frac{3}{2}\frac{x}{1+x^2} + \frac{x}{(1+x^2)^2}\right) + c$$

$$= \frac{1}{8}\left(3\operatorname{arctg} x + \frac{3x}{1+x^2} + \frac{2x}{(1+x^2)^2}\right) + c \qquad c \in \mathbb{R}$$

Esempio (radici complesse multiple)

Calcolare $\int \frac{x^2}{(x^2-2x+2)^2}dx$

dim

$$\frac{x^2}{(x^2-2x+2)^2} = \frac{Ax+B}{x^2-2x+2} + \frac{Cx+D}{(x^2-2x+2)^2}$$

$$x^2 = (Ax+B)(x^2-2x+2) + Cx+D$$

$$= Ax^3 - 2Ax^2 + 2Ax + Bx^2 - 2Bx + 2B + Cx + D$$

$$\begin{cases} A = 0 \\ -2A+B = 1 \\ 2A-2B+C = 0 \\ 2B+D = 0 \end{cases} \qquad \begin{cases} A = 0 \\ B = 1 \\ C = 2 \\ D = -2 \end{cases} \qquad \frac{x^2}{(x^2-2x+2)^2} = \frac{1}{x^2-2x+2} + \frac{2x-2}{(x^2-2x+2)^2}$$

e dunque

$$\int \frac{x^2}{(x^2-2x+2)^2}dx = \int \frac{dx}{x^2-2x+2} + \int \frac{2x-2}{(x^2-2x+2)^2}dx$$

$$= \operatorname{arctg}(x-1) - \frac{1}{x^2-2x+2} + c \qquad c \in \mathbb{R}$$

**Esempio** Calcolare $\int \frac{x^2}{(x^2+1)^2}\,dx$

dim

$$\frac{x^2}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$= \frac{Ax(x^2+1) + B(x^2+1) + Cx+D}{(x^2+1)^2}$$

$$Ax^3 + Bx^2 + (A+C)x + B + D = x^2$$

$$\begin{cases} A = 0 \\ B = 1 \\ A+C = 0 \\ B+D = 0 \end{cases} \qquad \begin{cases} A = 0 \\ B = 1 \\ C = 0 \\ D = -1 \end{cases}$$

$$f(x) = \frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}$$

$$\int f(x)\,dx = \operatorname{arctg} x - \frac{1}{2}\left(\operatorname{arctg} x + \frac{x}{x^2+1}\right) + c \quad c \in \mathbb{R}$$

$$= \frac{1}{2}\left(\operatorname{arctg} x - \frac{x}{x^2+1}\right) + c \quad c \in \mathbb{R}$$

**Oss**: Se devo calcolare $\int \frac{dx}{(1+x^2)^n} = I_n$

$$\int \frac{dx}{(1+x^2)^n} = \int \frac{dx}{(1+x^2)^{n-1}} - \frac{1}{2}\int \frac{2x}{(1+x^2)^n} \cdot x\,dx$$

$$= I_{n-1} - \frac{1}{2}\left[-\frac{1}{n-1}\cdot\frac{1}{(1+x^2)^{n-1}}\cdot x - \int\left(-\frac{1}{n-1}\cdot\frac{1}{(1+x^2)^{n-1}}\right)dx\right]$$

$$= I_{n-1} - \frac{1}{2(n-1)} I_{n-1} + \frac{1}{2(n-1)}\cdot\frac{x}{(1+x^2)^{n-1}}$$ e quindi

ci si riconduce al calcolo di $I_{n-1}$

Esercizio Calcolare $\int \frac{dx}{1+x^4}$

dim

$1+x^4=0 \qquad t^2=-1 \qquad t=i,-i \qquad x^2=i \quad x_1=\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}$

$x_2=-\frac{\sqrt2}{2}-i\frac{\sqrt2}{2}$

$x^2=-i \quad x_3=-\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}$

$x_4=\frac{\sqrt2}{2}-i\frac{\sqrt2}{2}$

$$x^4+1=\left(x-\left(\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}\right)\right)\left(x-\left(\frac{\sqrt2}{2}-i\frac{\sqrt2}{2}\right)\right)\cdot\left(x-\left(-\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}\right)\right)\left(x-\left(-\frac{\sqrt2}{2}-i\frac{\sqrt2}{2}\right)\right)$$

$$=\left(x^2-1\ -\sqrt2\cdot x\right)\left(x^2-1\ +\sqrt2\,x\right)$$

$$\frac{1}{x^4+1}=\frac{Ax+B}{x^2-\sqrt2x+1}+\frac{Cx+D}{x^2+\sqrt2x+1}$$

$$1=Ax(x^2+\sqrt2x+1)+B(x^2+\sqrt2x+1)+Cx(x^2-\sqrt2x+1)+D(x^2-\sqrt2x+1)$$

$$\begin{cases} A+C=0\\ \sqrt2A+B-\sqrt2C+D=0\\ \cancel{A}+\sqrt2B+\cancel{C}-\sqrt2D=0\\ B+D=1\end{cases}\qquad \begin{cases} A=-C\\ -\cancel{2}\sqrt2C+\cancel{2}B=0\\ B=D\\ B=\frac12\end{cases}\qquad \begin{matrix} A=-\frac{1}{2\sqrt2}\\ C=\frac{1}{2\sqrt2}\\ D=\frac12\\ B=\frac12\end{matrix}$$

$$\int\frac{1}{x^4+1}dx=-\frac{1}{2\sqrt2}\int\frac{x-\sqrt2}{x^2-\sqrt2x+1}dx+\frac{1}{2\sqrt2}\int\frac{x+\sqrt2}{x^2+\sqrt2x+1}dx$$

$$\int\frac{x+\sqrt2}{x^2+\sqrt2x+1}dx=\frac12\int\frac{2x+\sqrt2}{x^2+\sqrt2x+1}dx+\frac{1}{\sqrt2}\int\frac{dx}{\left(x^2+2\cdot\frac{\sqrt2}{2}\cdot x+\frac12\right)+\frac12}$$

$$=\frac12\log(x^2+\sqrt2x+1)+\sqrt2\cdot\int\frac{dx}{1+\left[\left(x+\frac{\sqrt2}{2}\right)\cdot\sqrt2\right]^2}$$

$$=\frac12\log(x^2+\sqrt2x+1)+\operatorname{arctg}(\sqrt2x+1)+c \qquad c\in\mathbb{R}$$

$$\int \frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1} = \frac{1}{2}\int \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}dx - \frac{1}{\sqrt{2}}\int \frac{1}{x^2-\sqrt{2}x+\frac{1}{2}+\frac{1}{2}}dx$$

$$= \frac{1}{2}\log\left(x^2-\sqrt{2}x+1\right) - \frac{1}{\sqrt{2}}\int \frac{dx}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}$$

$$= \frac{1}{2}\log\left(x^2-\sqrt{2}x+1\right) - \int \frac{\sqrt{2}\,dx}{1+(\sqrt{2}x-1)^2}$$

$$= \frac{1}{2}\log\left(x^2-\sqrt{2}x+1\right) - \operatorname{arctg}\left(\sqrt{2}x-1\right) + c \quad c\in\mathbb{R}$$

$$= \frac{1}{2}\log\left(x^2-\sqrt{2}x+1\right) - 3\operatorname{arctg}\left(\sqrt{2}x-1\right) + c$$