Lezione 22 bis - Analisi Matematica 1 - 11 novembre 2013

# Sviluppi di Taylor delle funzioni elementari in $x=0$

## (di Mc Laurin)

$\operatorname{sen} x = x + o(x)$ . per $x \to 0$ già conosciuta

$= x + o(x^2)$ " non è nota

$= x - \frac{x^3}{3!} + o(x^4)$ " " " "

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^6)$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + o(x^8) \quad (x \to 0)$

...

$$\boxed{= x - \frac{x^3}{3!} + \dots + (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})} \quad x \to 0$$

funzione pari $\longrightarrow$ potenze di $x$ pari

$$\cos x = 1 + o(x) \qquad x \to 0 \text{ è vero}$$

$$= 1 - \frac{x^2}{2} + o(x^2) \qquad x \to 0 \quad \text{noto}$$

$$= 1 - \frac{x^2}{2!} + o(x^3) \leftarrow \text{nuovo!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5)$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + o(x^7)$$

$$= \dots$$

$$= 1 - \frac{x^2}{2!} + \dots + (-1)^{n+1} \frac{x^{2n}}{(2n)!} + o(x^{2n+1}) \qquad (x \to 0)$$

Esercizio: Calcolare ordine e p.p. per $x \to 0$ di

$$f(x) = \operatorname{sen} x - x \cos x$$

dim

Devo sostituire a $\operatorname{sen} x$ e $\cos x$ i loro sviluppi

$$\operatorname{sen} x = x + o(x^2) \qquad \cos x = 1 + o(x)$$

$$x \cos x = x + o(x^2)$$

$$f(x) = \operatorname{sen} x - x\cos x = x + o(x^2) - \left(x + o(x^2)\right) \qquad (x \to 0)$$

$$= o(x^2) - o(x^2) = o(x^2)$$

$$\operatorname{sen} x = x - \frac{x^3}{6} + o(x^4)$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^3) \Rightarrow x \cos x = x - \frac{x^3}{2} + o(x^4)$$

$$f(x) = \operatorname{sen} x - x \cos x = \cancel{x} - \frac{x^3}{6} + o(x^4) - \left(\cancel{x} - \frac{x^3}{2} + o(x^4)\right)$$

$$= x^3\left(-\frac{1}{6} + \frac{1}{2}\right) + o(x^4) - o(x^4)$$

$$= \frac{x^3}{3} + o(x^4)$$

ordine $(f(x)) = 3$ $\qquad$ p.p. $(f(x)) = \frac{x^3}{3}$ $\qquad$ per $x \to 0$

Non mi basta: ho sviluppato "Troppo poco": devo prendere altri Termini dello sviluppo di $\operatorname{sen} x$

$$\operatorname{Tg}(x) = x + o(x^2)$$

$$\boxed{= x + \frac{x^3}{3} + o(x^4)} \quad (x \to 0)$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^6)$$

Questo sviluppo non è regolare ovvero

non esiste una funzione f(n) che generi i coefficienti del polinomio di Taylor di Tg x !!

dispari → potenze dispari

$$\operatorname{arctg} x = x + o(x^2)$$

$$= x - \frac{x^3}{3} + o(x^4) \quad (x \to 0)$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^6)$$

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$$= x - \frac{x^3}{3} + \dots + (-1)^{n+1} \frac{x^{2n+1}}{2n+1} + o(x^{2n+2})$$

Ricordo $\lim_{x\to 0} \frac{e^x - 1}{x} = 1 \Rightarrow \lim_{x\to 0} \frac{e^x - 1 - x}{x} = 0$

$\Rightarrow e^x - 1 - x = o(x)$ per $x \to 0$

$\Rightarrow e^x = 1 + x + o(x)$ per $x \to 0$

$$e^x = 1 + o(1)$$

$= 1 + x + o(x)$ ← lo conosco

$$= 1 + x + \frac{x^2}{2} + o(x^2)$$

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$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n)$$

$$\log(1+x) = x + o(x) \leftarrow \text{lo conosco} \qquad \frac{\log(1+x)}{x} \xrightarrow[x\to 0]{} 1$$

$$= x - \frac{x^2}{2} + o(x^2) \leftarrow \text{nuovo}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \leftarrow \text{nuovo}$$

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$$\boxed{= x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^{n+1}\frac{x^n}{n} + o(x^n)}$$

$$(1+x)^\alpha \qquad \text{per } x > -1, \quad \alpha \in \mathbb{R}$$

$$\overset{\parallel}{}\; e^{\alpha \log(1+x)} \qquad \text{in } x=0$$

$$\log(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$$

$$\alpha\log(1+x) = \alpha x - \frac{\alpha}{2}x^2 + \frac{\alpha}{3}x^3 - \frac{\alpha}{4}x^4 + o(x^4)$$

$$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} + o(y^4)$$

$$= 1 + \left(\alpha x - \frac{\alpha}{2}x^2 + \frac{\alpha}{3}x^3 - \frac{\alpha}{4}x^4 + o(x^4)\right)$$

$$+ \left( \quad '' \quad \right)^2 \cdot \frac{1}{2}$$

$$+ \left( \quad '' \quad \right)^3 \cdot \frac{1}{6}$$

$$+ \left( \quad '' \quad \right)^4 \cdot \frac{1}{2} +$$

$$+ o\left[\left( \qquad \right)^4\right]$$

$$= 1+\left(\alpha x - \alpha\frac{x^2}{2} + \alpha\frac{x^3}{3} - \alpha\frac{x^4}{4}\right)$$

$$+\frac{1}{2}\left(\alpha^2x^2 + \alpha^2\frac{x^4}{4} - \alpha^2x^3 + \frac{2}{3}\alpha^2x^4\right)$$

$$+\frac{1}{6}\left(\alpha^3x^3 - \frac{3}{2}\alpha^3x^4\right) + \frac{1}{24}\alpha^4x^4 + o\left(\alpha^4x^4 + o(x^4)\right)$$

$$= 1+\alpha x + x^2\left(-\frac{\alpha}{2}+\frac{\alpha^2}{2}\right) + x^3\left(\frac{\alpha}{3}-\frac{\alpha^2}{2}+\frac{\alpha^3}{6}\right)$$

$$+ x^4\left(-\frac{\alpha}{4}+\frac{\alpha^2}{8}+\frac{\alpha^2}{3}-\frac{\alpha^3}{4}+\frac{\alpha^4}{24}\right) + o(x^4)$$

$$= 1+\alpha x + \frac{\alpha(\alpha-1)}{2}\cdot x^2 + \frac{2\alpha-3\alpha^2+\alpha^3}{6}x^3 +$$

$$+ \frac{-6\alpha+3\alpha^2+8\alpha^2-6\alpha^3+\alpha^4}{24}x^4 + o(x^4)$$

$$= 1+\alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}x^4 + o(x^4) \quad x\to 0$$

Questo è lo sviluppo di $(1+x)^\alpha$ fino all'ordine 4

In generale

$$(1+x)^\alpha = 1+\alpha x + \frac{\alpha(\alpha-1)}{2!}x + \dots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+2)(\alpha-n+1)}{n!}x^n + o(x^n) \quad \text{per } x\to 0$$

Esercizio Sviluppare $\frac{1}{1+x}$ fino al 4° ordine

$$\frac{1}{1+x} = (1+x)^{-1} = 1+(-1)\cdot x + \frac{(-1)(-1-1)}{2}x^2 + \frac{(-1)(-1-1)(-1-2)}{6}x^3 +$$

$$+ \frac{(-1)\cdot(-2)(-3)(-4)}{24}x^4 + o(x^4) \quad (x\to 0)$$

$$= 1-x+x^2-x^3+x^4+o(x^4)$$

Analogamente

$$\frac{1}{1-x} = (1-x)^{-1} = \dots = 1+x+x^2+x^3+x^4+o(x^4) \quad (x\to 0)$$

**PROBLEMA 3**

Calcolate $\lim_{x\to 0} \dfrac{e^{3x}\cos(2x)+\log(1-3x)-(1-x^2)^2}{x-\operatorname{sen} x}$.

Voglio arrivare a un rapporto tra polinomi

$$e^{3x} = 1 + 3x + \frac{(3x)^2}{2} + \frac{(3x)^3}{6} + o(x^3)$$

$$\cos(2x) = 1 - \frac{(2x)^2}{2} + o(x^3) \qquad \operatorname{sen} x = x - \frac{x^3}{6} + o(x^4)$$

$$\log(1-3x) = -3x - \frac{(-3x)^2}{2} + \frac{(-3x)^3}{3} + o(x^3)$$

$$e^{3x}\cos 2x + \log(1-3x) - (1-x^2)^2 = \left(1+3x+\frac{9}{2}x^2+\frac{9}{2}x^3+o(x^3)\right)\left(1-2x^2+o(x^3)\right)$$
$$-3x-\frac{9}{2}x^2-9x^3+o(x^3)-1-x^4+2x^2$$

$$= \cancel{1}-\cancel{2x^2}+o(x^3)+\cancel{3x}-6x^3+o(x^4)+\cancel{\frac{9}{2}x^2}-9x^4+o(x^5)+\frac{9}{2}x^3-9x^5+o(x^6)$$
$$+o(x^3)+o(x^5)+o(x^6)-\cancel{3x}-\cancel{\frac{9}{2}x^2}-9x^3+o(x^3)-\cancel{1}-x^4+\cancel{2x^2}$$

$$= x^3\left(-15+\frac{9}{2}\right)\underset{\uparrow\, o(x^3)}{-10x^4}\underset{\uparrow\, o(x^3)}{-9x^5}+o(x^3)+o(x^4)+o(x^5)+o(x^6)$$

$$= -\frac{21}{2}x^3+o(x^3)$$

Denominatore $\quad x-\operatorname{sen}x = x-\left(x-\frac{x^3}{6}+o(x^4)\right) = \frac{x^3}{6}+o(x^4)$

Quindi

$$\lim_{x\to 0}\frac{\text{Numerat.}}{\text{Denomin.}} = \lim_{x\to 0}\frac{-\frac{21}{2}x^3+o(x^3)}{\frac{x^3}{6}+o(x^4)} = \lim_{x\to 0}\frac{-\frac{21}{2}x^3}{\frac{1}{6}x^3}\cdot\lim\frac{1+o(1)}{1+o(1)}$$

$$= -63$$

**PROBLEMA 4**

Calcolate lo sviluppo di Taylor di $f$ fino ai termini di ordine 4, poi scrivete l'ordine di infinitesimo per $x \to 0$ e la parte principale, dove

$$f(x) = 3e^{x^2} - 2\log(x^2+1) - x \cdot \frac{1}{1-2x} - 3\cos x + \operatorname{sen} x\ .$$

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$$3\left(1+x^2+\frac{x^4}{2}\right)-2\left(x^2-\frac{x^4}{2}\right)-x\left(1+2x+4x^2+8x^3\right)-3\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)+x-\frac{x^3}{6}+o(x^4)$$

$$\cancel{3}+3x^2+\frac{3}{2}x^4-2x^2+2x^4-\cancel{x}-2x^2-4x^3-8x^4-\cancel{3}+\frac{3}{2}x^2-\frac{x^4}{8}+\cancel{x}-\frac{x^3}{6}+o(x^4)$$

$$x^2\left(3-4+\frac{3}{2}\right)+x^3\left(-4-\frac{1}{6}\right)+x^4\left(\frac{3}{2}+2-8-\frac{1}{8}\right)+o(x^4)$$

$$\frac{x^2}{2}-\frac{25}{6}x^3+\frac{37}{8}x^4+o(x^4)$$

quindi

$f=\frac{x^2}{2}+o(x^2)$ per $x \to 0$ ordine $(f)=2$

pp $(f)=\frac{x^2}{2}$

PROBLEMA 3

Calcolate al variare di $\alpha \in \mathbb{R}$ il limite

$$\lim_{x\to 0^+} x^\alpha \left(\operatorname{sen} x \cos x + \frac{x}{x^2-1}\right).$$

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$f(x) = \operatorname{sen} x \cos x + \dfrac{x}{x^2-1}$ per $x \to 0$

$$= \left(x - \frac{x^3}{6} + o(x^4)\right)\left(1 - \frac{x^2}{2} + o(x^3)\right) + \frac{x}{x^2-1}$$

$$= x - \frac{x^3}{2} + o(x^4) - \frac{x^3}{6} + \frac{x}{x^2-1}$$

$$= x - \frac{2}{3}x^3 + o(x^4) + \frac{x}{x^2-1}$$

$$= \frac{x^3 - \cancel{x} + \frac{2}{3}x^3 + o(x^4) + \cancel{x}}{x^2-1} = \frac{\frac{5}{3}x^3 + o(x^4)}{x^2-1}$$

$$\lim_{x\to 0^+} x^\alpha f(x) = \lim_{x\to 0^+} \frac{\frac{5}{3}x^{3+\alpha} + o(x^{4+\alpha})}{x^2-1}$$

$$= -\lim_{x\to 0^+}\left(\frac{5}{3}x^{3+\alpha} + o(x^{4+\alpha})\right)$$

$\alpha > -3$
$3+\alpha > 0 \Rightarrow 4+\alpha > 0 \Rightarrow -\lim_{x\to 0^+}\left(\frac{5}{3}x^{3+\alpha} + o(x^{4+\alpha})\right) = 0$

$\alpha = -3 \Rightarrow 3+\alpha = 0 \quad 4+\alpha > 0 \Rightarrow -\lim_{x\to 0}\left(\frac{5}{3} + o(x^{4+\alpha})\right) = -\frac{5}{3}$

$\alpha < -3 \Rightarrow -\lim_{x\to 0^+} \frac{5}{3}x^{3+\alpha}\left(1 + o(x)\right) = -\infty$

**PROBLEMA 3**

Calcolate il limite

$$\lim_{x\to 0^+} \frac{3x - \operatorname{sen}(3x)}{e^{2x} + \log(1-2x) - 1}\,.$$

Calcolate poi, al variare dell'esponente $\alpha > 0$, il limite

$$\lim_{x\to 0^+} \frac{3x^\alpha - \operatorname{sen}(3x)}{e^{2x} + \log(1-2x) - 1}\,.$$

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$$\text{Numeratore} = 3x^\alpha - \left(3x - \frac{9}{2}x^3\right) + o(x^4)$$
$$\overset{!}{=} 3(x^\alpha - x) + \frac{9}{2}x^3 + o(x^3)$$

$$\text{Denominatore} = \cancel{1} + \cancel{2x} + \cancel{2x^2} + \frac{8}{6}x^3 + o(x^3)$$
$$+ \left(-\cancel{2x} - \cancel{2x^2} - \frac{8}{3}x^3 - o(x^3)\right) - \cancel{1}$$
$$= -\frac{8}{6}x^3 + o(x^3)$$

Se $\alpha = 1$ allora $f(x) = \dfrac{\frac{9}{2}x^3 + o(x^3)}{-\frac{4}{3}x^3 + o(x^3)} \xrightarrow[x\to 0^+]{} \frac{9}{2}\cdot\frac{3}{4} = -\frac{27}{8}$

Se $\alpha > 1$ allora $f(x) = \dfrac{-3x + o(x^2)}{-\frac{4}{3}x^3 + o(x^3)} \xrightarrow[x\to 0^+]{} +\infty$

Se $0 < \alpha < 1$ allora $f(x) = \dfrac{3x^\alpha + o(x^\alpha)}{-\frac{4}{3}x^3 + o(x^3)} \xrightarrow[x\to 0^+]{} -\infty$