Ricerca delle primitive – parte 2

Titolo nota 20/11/2012

Funzioni Razionali

In generale, quando si deve calcolare la primitiva di $\frac{1}{x^2+bx+c}$ si distinguono 3 casi

i) b^2-4c (radici reali $\neq$)

$$\int \frac{dx}{x^2+bx+c} = \int \frac{dx}{\left(x+\frac{b}{2}\right)^2 - \left(\frac{b^2}{4}-c\right)}$$

$$= \int \frac{dx}{\left(x+\frac{b}{2}-\sqrt{\frac{b^2}{4}-c}\right)\left(x+\frac{b}{2}+\sqrt{\frac{b^2}{4}-c}\right)}$$

etc

(ii) $b^2-4c=0$ (radici reali $=$)

$$\int \frac{dx}{x^2+bx+c} = \int \frac{dx}{x^2+bx+\frac{b^2}{4}} = \int \frac{dx}{\left(x+\frac{b}{2}\right)^2}$$

$$= -\frac{1}{x+\frac{b}{2}} + c \qquad c \in \mathbb{R}$$

(iii) $b^2-4c<0$

$$\int \frac{dx}{x^2+bx+c} = \int \frac{dx}{\left(x+\frac{b}{2}\right)^2+\left(c-\frac{b^2}{4}\right)} =$$

$$= \frac{1}{\left(c-\frac{b^2}{4}\right)} \int \frac{dx}{1+\left(\frac{x+\frac{b}{2}}{\sqrt{c-\frac{b^2}{4}}}\right)^2}$$

$$y = \frac{x+\frac{b}{2}}{\sqrt{c-\frac{b^2}{4}}} \qquad \frac{dy}{dx} = \frac{1}{\sqrt{c-\frac{b^2}{4}}}$$

$$= \frac{1}{\sqrt{c-\frac{b^2}{4}}} \cdot \left(\int \frac{1}{1+y^2} \cdot \frac{dy}{dx} \cdot dx\right)_{y=\frac{x+\frac{b}{2}}{\sqrt{c-\frac{b^2}{4}}}}$$

Esercizio: Calcolare

(i) $\int \frac{dx}{x^2-4x+4}$ (ii) $\int \frac{dx}{x^2-2x-6}$

(iii) $\int \frac{dx}{x^2-4x+1}$ (iv) $\int \frac{dx}{x^2+2x+7}$

Calcolare le primitive di $\frac{P(x)}{Q(x)}$

Primo passo: ci si può sempre ridurre al caso

$$\text{grado } P(x) < \text{grado } Q(x)$$

effettuando la divisione

Esempio Calcolare $\int \frac{x^3+x+2}{x^2+1} dx$

dim

$$\frac{x^3+x+2}{x^2+1} = x + \frac{2}{x^2+1}$$

$$\int \frac{x^3+x+2}{x^2+1} dx = \int x\,dx + \int \frac{2\,dx}{x^2+1}$$

$$= \frac{x^2}{2} + 2 \operatorname{arctg} x + c \qquad c \in \mathbb{R}$$

Def Dato un polinomio $P(x)$, $x=a$ si dice "radice semplice"

se $P(x) = (x-a)\,Q(x)$ con $Q(a) \neq 0$

se $P(a) = 0$
$P'(a) \neq 0$

Esempio $P(x)=x^2-1$ ha $x=1, x=-1$ radici reali semplici

Esempio $P(x)=x^2(x-1)$ ha $x=0$ radice doppia
$x=1$ " semplice

Teorema (Caso delle radici reali semplici)

Dato $f(x)=\dfrac{P(x)}{Q(x)}$ con grado $P<$ grado Q

Se $Q(x)=\prod_{i=1}^{m}(x-x_i)$

allora esistono A_i, $i=1,\dots,m$ t.c.

$$\frac{P(x)}{Q(x)}=\frac{A_1}{x-x_1}+\dots+\frac{A_m}{x-x_m}=\sum_{i=1}^{m}\frac{A_i}{(x-x_i)}$$

(non si fa la dimostrazione)

Esempio Calcolare $\displaystyle\int\frac{1}{x^2-1}\,dx$

$$\frac{1}{x^2-1}=\frac{A_1}{x-1}+\frac{A_2}{x+1}=\dots=+\frac{1}{2}\frac{1}{x-1}-\frac{1}{2}\frac{1}{x+1}$$

$$\int\frac{dx}{x^2-1}=\frac{1}{2}\int\frac{dx}{x-1}-\frac{1}{2}\int\frac{dx}{x+1}=\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+c$$

$c\in\mathbb{R}$

Teorema (radici reali coincidenti)

$f(x) = \frac{P(x)}{Q(x)}$, con P e Q polinomi,

grado P < grado Q

Se $Q(x) = (x-a)^m$

allora $\exists A_i$, $i = 1, \dots, m$ t.c.

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_m}{(x-a)^m}$$

(non viene dimostrato)

Esempio Calcolare $\int \frac{x+1}{x^2(x-1)} dx$

$$f(x) = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$= \frac{Ax(x-1) + B(x-1) + Cx^2}{x^2(x-1)} = \frac{x+1}{x^2(x-1)}$$

$\Downarrow$

$$(A+C)x^2 + (-A+B)x - B = x+1$$

$$\begin{cases} A+C=0 \\ -A+B=1 \\ -B=1 \end{cases} \quad \begin{cases} C=-A \\ -A=2 \\ B=-1 \end{cases} \quad \begin{cases} A=-2 \\ B=-1 \\ C=2 \end{cases}$$

$$f(x) = -\frac{2}{x} - \frac{1}{x^2} + \frac{2}{x-1}$$

$$\int f(x)\,dx = -\log x^2 + \frac{1}{x} + 2\log|x-1| + c \quad c\in\mathbb{R}$$

Esempio Calcolare $\int \frac{x-1}{x^3}\,dx$

$$f(x) = \frac{x-1}{x^3} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} =$$

$$f(x) = \frac{1}{x^2} - \frac{1}{x^3} \quad \Rightarrow \quad \int f(x)\,dx = -\frac{1}{x} + \frac{1}{2x^2} + c$$

$$c\in\mathbb{R}$$

Oss: Se $Q(x)$ è un polinomio a coefficienti reali t.c. $Q(z)=0$ allora $Q(\bar{z})=0$

Teorema (radici complesse-coniugate-semplici)

$$f(x)=\frac{P(x)}{Q(x)}$$ con P e Q polinomi, grado $P<$ grado Q

Se $$Q(x)=(x^2+a_1x+b_1)\cdots(x^2+a_mx+b_m)=\prod_{i=1}^{m}(x^2+a_ix+b_i)$$

allora $\exists\, A_i, B_i$ $i=1\dots m$ t.c.

$$f(x)=\frac{A_1x+B_1}{x^2+a_1x+b_1}+\frac{A_2x+B_2}{x^2+a_2x+b_2}+\dots+\frac{A_mx+B_m}{x^2+a_mx+b_m}$$

(la dimostrazione viene omessa)

Oss I termini $\frac{Ax+B}{x^2+ax+b}$ ammettono primitiva esprimibile in termini di funzioni elementari

Esempio Calcolare $\int \frac{x+1}{x^2+1}\,dx$

$$\frac{x+1}{x^2+1}=\frac{1}{2}\,\frac{2x}{x^2+1}+\frac{1}{x^2+1}$$

$$\int\frac{x+1}{x^2+1}\,dx=\frac{1}{2}\log(x^2+1)+\operatorname{arctg}x+c \qquad c\in\mathbb{R}$$

Esempio calcolare $\int \frac{x\,dx}{x^2+x+1}$

$$f(x) = \frac{x}{x^2+x+1} = \frac{1}{2}\,\frac{2x}{x^2+x+1} = \frac{1}{2}\,\frac{2x+1}{x^2+x+1} - \frac{1}{2}\,\frac{1}{x^2+x+1}$$

$$= \frac{1}{2}\,\frac{\left(x^2+x+1\right)'}{x^2+x+1} - \frac{2}{3}\,\frac{1}{1+\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2}$$

$$x^2+x+1 = \left(x^2+x+\frac{1}{4}\right)+\frac{3}{4} = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

Teorema (radici complesse e coniugate multiple)

Dato $f(x) = \frac{P(x)}{Q(x)}$, P e Q polinomi, grado $P <$ grado Q

Se $Q(x) = (x^2 + ax + b)^m$

allora esistono $A_i, B_i \in \mathbb{R}$ $i = 1, \dots, m$ t.c.

$$\frac{P(x)}{Q(x)} = \frac{A_1 x + B_1}{x^2 + ax + b} + \frac{A_2 x + B}{(x^2+ax+b)^2} + \dots + \frac{A_m x + B_m}{(x^2+ax+b)^m}$$

Esempio (Importante) Calcolare $\int \frac{dx}{(1+x^2)^2}$

$$\int \frac{1}{(x^2+1)^2} dx = \int \frac{x^2+1}{(x^2+1)^2} dx - \int \frac{x^2}{(x^2+1)^2} dx$$

$$= \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{(x^2+1)^2} \cdot x \, dx$$

$$= \operatorname{arctg} x - \frac{1}{2} \left\{ -\frac{1}{x^2+1} \cdot x - \int \left(-\frac{1}{x^2+1} \right) dx \right\}$$

$$= \operatorname{arctg} x + \frac{1}{2} \frac{x}{x^2+1} - \frac{1}{2} \operatorname{arctg} x + c$$

$$= \frac{1}{2} \left(\operatorname{arctg} x + \frac{x}{x^2+1} \right) + c \qquad c \in \mathbb{R}$$

verifica

$$F'(x) = \frac{1}{2} \left(\frac{1}{1+x^2} + \frac{1}{x^2+1} - \frac{2x^2}{(x^2+1)^2} \right)$$

$$= \frac{1}{x^2+1} - \frac{x^2}{(x^2+1)^2} = \frac{1+x^2-x^2}{(1+x^2)^2} = \frac{1}{(1+x^2)^2}$$

Esempio Calcolare $\int \frac{dx}{(1+x^2)^3}$

$$\int \frac{1+x^2-x^2}{(1+x^2)^3}\,dx = \underbrace{\int \frac{dx}{(1+x^2)^2}}_{=\frac{1}{2}\left(\operatorname{arctg} x + \frac{x}{x^2+1}\right)} - \int \frac{x^2\,dx}{(1+x^2)^3}$$

$$= \frac{1}{2}\left(\operatorname{arctg} x + \frac{x}{x^2+1}\right) + \frac{1}{4}\int \frac{-4x}{(1+x^2)^3}\cdot x\,dx$$

$$= \frac{1}{2}\left(\operatorname{arctg} x + \frac{x}{x^2+1}\right) + \frac{1}{4}\left\{ \frac{1}{(1+x^2)^2}\cdot x - \overbrace{\int \frac{1}{(1+x^2)^2}} \right\}$$

$$= \frac{3}{8}\left(\operatorname{arctg} x + \frac{x}{x^2+1}\right) + \frac{1}{4}\,\frac{x}{(1+x^2)^2} + c \qquad c \in \mathbb{R}$$

$$F'(x) = \frac{3}{8}\left(\frac{1}{1+x^2} + \frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2}\right) + \frac{1}{4}\,\frac{(1+x^2)^{\cancel{2}} - 4x^2\cancel{(1+x^2)}}{(1+x^2)^{\cancel{4}\,3}}$$

$$= \frac{3}{\cancel{8}\,4}\;\frac{\overset{1}{\cancel{2}}+\cancel{2x^2}-\cancel{2x^2}}{(1+x^2)^2} + \frac{1}{4}\,\frac{1-3x^2}{(1+x^2)^3}$$

$$= \frac{1}{4}\;\frac{3(1+x^2)+1-3x^2}{(1+x^2)^3} = \frac{1}{(1+x^2)^3} \qquad \checkmark$$

Esempio Calcolare $\frac{x}{(x^2+1)^2}\,dx$

$$\frac{x^2}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$= \frac{Ax(x^2+1)+B(x^2+1)+Cx+D}{(x^2+1)^2}$$

$$Ax^3+Bx^2+(A+C)x+B+D = x^2$$

$$\begin{cases} A=0 \\ B=1 \\ A+C=0 \\ B+D=0 \end{cases} \qquad \begin{cases} A=0 \\ B=1 \\ C=0 \\ D=-1 \end{cases}$$

$$f(x) = \frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}$$

$$\int \frac{1}{(x^2+1)^2}dx = \int \frac{1+x^2}{(1+x^2)^2}dx - \int \frac{x^2}{(1+x^2)^2}$$

$$= \operatorname{arctg} x - \frac{1}{2}\int \underbrace{\frac{2x}{(1+x^2)^2}}_{u'} \cdot \underbrace{x}_{v}\, dx$$

$$= \operatorname{arctg} x - \frac{1}{2}\left\{\left(-\frac{1}{1+x^2}\right)\cdot x + \int \frac{dx}{1+x^2}\right\}$$

$$= \operatorname{arctg} x + \frac{1}{2}\frac{x}{x^2+1} - \frac{1}{2}\operatorname{arctg} x + c$$

$$= \frac{1}{2}\operatorname{arctg} x + \frac{1}{2}\frac{x}{x^2+1} + c \qquad c \in \mathbb{R}$$

$$\Downarrow$$

$$\int \frac{x^2}{(1+x^2)^2}dx = \int \frac{dx}{x^2+1} - \int \frac{dx}{(x^2+1)^2}$$

$$= \operatorname{arctg} x - \frac{1}{2}\operatorname{arctg} x - \frac{1}{2}\frac{x}{x^2+1} + c$$

$$= \frac{1}{2}\operatorname{arctg} x - \frac{1}{2}\frac{x}{x^2+1} + c \qquad c \in \mathbb{R}$$

Funzioni Iperboliche

$$\operatorname{senh}(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\operatorname{Tgh}(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$(\cosh(x))^2 - (\operatorname{senh} x)^2 = 1$$

$$\frac{d}{dx}\operatorname{senh} x = \frac{e^x + e^{-x}}{2} = \cosh(x) > 0 \quad \forall x$$

$$\frac{d}{dx}\operatorname{senh} x = \frac{e^x - e^{-x}}{2} = \operatorname{senh}(x) \begin{cases} > 0 & x > 0 \\ < 0 & x < 0 \end{cases}$$

Quindi $y = \operatorname{senh} x$ è invertibile

$$y = \operatorname{senh}(x) \quad \underline{\text{sse}} \quad y = \frac{e^x - e^{-x}}{2}$$

$$\underline{\text{sse}} \quad e^x - e^{-x} - 2y = 0$$

$$\underline{\text{sse}} \quad e^{2x} - 2ye^x - 1 = 0$$

$$\underline{\text{sse}} \quad e^x = \begin{cases} \cancel{y - \sqrt{y^2+1}} & \text{è} < 0\,!! \\ y + \sqrt{y^2+1} \end{cases}$$

$$\underline{\text{sse}} \quad x = \log\left(y + \sqrt{y^2+1}\right)$$

ovvero $y = \operatorname{settsenh}(x) = \log\left(x + \sqrt{x^2+1}\right)$

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2+1}} \cdot \left(1 + \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}}$$

$\cosh(x) > 0 \ \forall x$

Oppure ricordando che $\cosh(x) = \sqrt{1+\operatorname{senh}^2(x)}$

$$\frac{d(\operatorname{settsenh}(x))}{dx} = \frac{1}{\cosh(\operatorname{settsenh}(x))} = \frac{1}{\sqrt{x^2+1}}$$

Analogamente

$$\frac{d(\operatorname{settcosh}(x))}{dx} = \frac{1}{\operatorname{senh}(\operatorname{settcosh} x)} = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{tgh}(x)) = \frac{(\operatorname{senh}(x))'\cosh(x) - \operatorname{senh}(x)(\cosh(x))'}{\cosh^2(x)}$$

$$= \frac{\cosh^2(x) - \operatorname{senh}^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)}$$

Esempio Calcolare $\int \sqrt{1+x^2}\,dx$

$$\int \sqrt{1+x^2}\,dx \underset{\substack{x=\operatorname{senh}(t) \\ \frac{dx}{dt}=\cosh(t)}}{=} \left(\int \cosh(t)\cdot\frac{dx}{dt}\,dt\right)_{x=\operatorname{senh}(t)}$$

$$= \left(\int \cosh^2(t)\,dt\right)_{x=\operatorname{senh}(t)}$$

$$= \left(\frac{t}{2} + \frac{1}{2}\operatorname{senh}(t)\cosh(t) + c\right)_{x=\operatorname{senh}(t)}$$

$$= \frac{1}{2}\left(x + \sqrt{x^2+1} + x\sqrt{x^2+1}\right) + c \quad c \in \mathbb{R}$$

$$\int \underset{\underset{u'}{\uparrow}}{\cosh(t)} \cdot \underset{\underset{v}{\uparrow}}{\cosh(t)}\, dt = \underset{\underset{u}{\uparrow}}{\operatorname{senh}(t)}\, \underset{\underset{v}{\uparrow}}{\cosh(t)} - \int \underset{\underset{u}{\uparrow}}{\operatorname{senh}(t)}\, \underset{\underset{v'}{\uparrow}}{\operatorname{senh}(t)}\, dt$$

$$= \operatorname{senh}(t)\cosh(t) - \int \left(\cosh^2(t) - 1\right) dt$$

$$\Downarrow$$

$$\int \cosh^2(t) = \frac{t}{2} + \frac{1}{2}\operatorname{senh}(t)\cosh(t) + c \qquad c \in \mathbb{R}$$