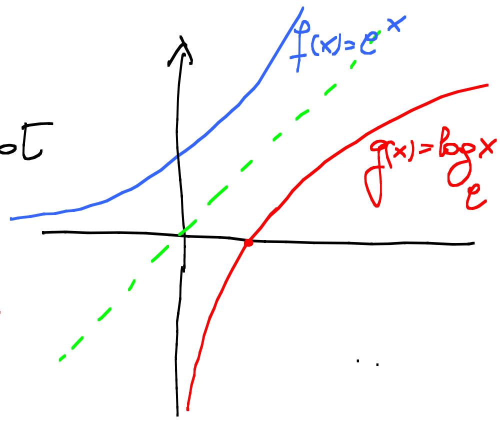
Lezione 12 - Analisi Matematica 1 - 25 ottobre 2012

$$\left.\begin{cases} f: A \to \mathbb{R} \\ f \text{ continua in } x_0 \\ x_0 \text{ p.d.a. per } A \end{cases}\right\} \Longrightarrow \lim_{x \to x_0} f(x) = f\left(\lim_{x \to x_0} x\right) = f(x_0)$$

$$\left.\begin{cases} f: A \to \mathbb{R} \text{ continua in } x_0 \\ g: B \to \mathbb{R} \quad \text{"} \quad \text{"} \quad f(x_0) \\ x_0 \text{ p.d.a. per } f^{-1}(f(A) \cap B) \\ y_0 \quad \text{"} \quad \text{"} \quad f(A) \cap B \end{cases}\right\} \Longrightarrow \lim_{x \to x_0} g(f(x)) = g\left(\lim_{x \to x_0} f(x)\right) = g(f(x_0))$$

$f(x) = e^x \qquad f: \underline{\mathbb{R}} \to]0, +\infty[$

$g(x) = \log_e x \qquad g:]0, +\infty[\to \underline{\mathbb{R}}$



$\lim_{x \to -\infty} e^x = 0^+ \qquad \left(\lim_{x \to 0^+} \log x = -\infty\right)$

$\lim_{x \to +\infty} e^x = +\infty \qquad \left(\lim_{x \to +\infty} \log x = +\infty\right)$

$e^x > 1 \ \forall x > 0 \qquad (\log x > 0 \ \forall x > 1) \qquad e^x \nearrow \qquad \log x \nearrow$

$\boxed{e^{x+y} = e^x e^y \quad \forall x, y \in \mathbb{R}} \quad \left(\log x \cdot y = \log x + \log y \quad \forall x, y > 0\right)$

$\boxed{\lim_{x \to 0} \frac{e^x - 1}{x} = 1} \qquad \left(\lim_{x \to 0} \frac{\log(1+x)}{x} = 1\right)$

$\Longrightarrow$

$x = \log(1+y) \qquad y \to 0 \Rightarrow x = \log(1+y) \to 0$

$$1 = \lim_{x \to 0} \frac{e^x - 1}{x} \underset{x = \log(1+y)}{=} \lim_{y \to 0} \frac{1 + y - 1}{\log(1+y)} = \lim_{y \to 0} \frac{y}{\log(1+y)}$$

$e^x = e^{\log(1+y)} = 1 + y$

$$\Rightarrow \lim_{y\to 0} \frac{y}{\log(1+y)} = 1 \Rightarrow \lim_{y\to 0} \frac{\log(1+y)}{y} = 1$$

$$(*)\ \underbrace{e^x \geq 1+x \quad \forall x \in \mathbb{R}}_{\text{la dimostreremo}} \qquad \left(\underline{\log x \leq x-1} \quad \forall x>0\right)$$

$\Rightarrow$

$x = \log y$

$e^x = y$

$y \geq 1 + \log y \quad \underline{y>0}$

$\underline{y-1} \geq \log y \quad \forall y > 0$

<u>Teorema</u> $f: A \to \mathbb{R}$, $B \subseteq A$, x_0 p.d.a. per B (e per A)

Se $\exists \lim_{x\to x_0} f = \ell$ allora $\exists \lim_{x\to x_0} f_{|B} = \ell$

(limiti da dx e da sin)

<u>Def</u> $f: A \to \mathbb{R}$ $\quad B \subseteq A$ $\quad$ diciamo che

$f_{|B}(x)$ è "restrizione di f a B" se

$f_{|B}(x) = f(x) \quad \forall x \in B$

<u>Teorema</u> $f: A \to \mathbb{R}$ $\quad B \subseteq A$ $\quad x_0$ p.d.a. per B

(e quindi anche per A)

Se $\exists \lim_{x\to x_0} f$ allora $\exists \lim_{x\to x_0} f_{|B}$

Esempio $f: A \to \mathbb{R}$ $x_0 \in A$ p.d.a.

$$\exists \lim_{x \to x_0} f(x) = l \Rightarrow \exists \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0} f_{|\,]x_0, +\infty[} = l$$

$$\exists \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0} f_{|\,]-\infty, x_0[} = l$$

Teorema $f: A \to \mathbb{R}$ x_0 p.d.a. per A

Se $\exists \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = l$

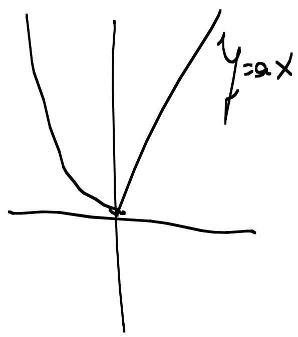
allora $\exists \lim_{x \to x_0} f(x) = l$

Esempio

Determinare $a \in \mathbb{R}$ (se esiste) t.c.

$$f(x) = \begin{cases} x^2 & x < 0 \\ ax & x \geq 0 \end{cases}$$

sia continua in $x = 0$



dim

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} ax = 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0$$

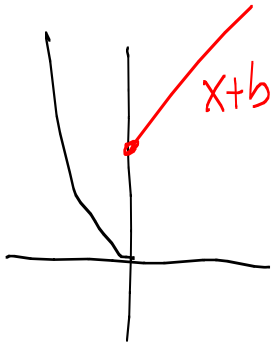
f è continua $\forall a \in \mathbb{R}$

Esercizio

$$f(x)=\begin{cases} x^2 & x<0 \\ x+b & x\geq 0\end{cases}$$

det. b in modo che f sia continua in $x=0$

dim



$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x+b = b$$

f continua in $x=0$ <u>sse</u> $b=0$ ↯

Teorema (funzioni monotone e limiti)

Sia $f: I \to \mathbb{R}$ una funzione continua $\forall x \in I$, dove I è un intervallo

$\boxed{f \text{ monotona su } I}$

Allora $\forall x_0 \in I$ $\exists \lim_{x \to x_0^-} f(x)$ e $\lim_{x \to x_0^+} f(x)$

dim

Supponiamo

f debolmente crescente $\quad \left(x < y \Rightarrow f(x) \leq f(y) \right)$

$I =]a, b[$

$x_0 = a \qquad \lim_{x \to a^+} f(x) \overset{\text{da provare}}{=} \underbrace{\inf f(]a,b[)}_{\text{esiste sempre}} \quad f(]a,b[) \subseteq \mathbb{R}$

$x_0 \in]a,b[\qquad \lim_{x \to x_0^-} f(x) = \overbrace{\sup f(]a, x_0[)}^{f(]a,x_0[) \subseteq \mathbb{R}}$

$\lim_{x \to x_0^+} f(x) = \inf f(]x_0, b[)$

$x_0 = b \qquad \lim_{x \to b^-} f(x) = \sup f(]a,b[)$

$f \nearrow = \quad I =]a,b[$

$\$ \quad \lim_{x \to a^+} f(x) = \underbrace{\inf f(]a,b[)}$

$\mathbb{R} \ni \lambda = f(]a,b[) \Leftrightarrow \begin{cases} \to \lambda \leq f(x) \quad \forall x \in]a,b[\\ \to \forall \varepsilon > 0 \ \exists \bar{x} \in]a,b[: f(\bar{x}) < \lambda + \varepsilon \end{cases}$

$\to \forall x < \bar{x} \qquad f(x) \leq f(\bar{x})$

$\delta = \bar{x} - a$

$\forall \varepsilon > 0 \ \exists \delta > 0 \ : \ \forall x \in]a, a+\delta[\quad \lambda \leq f(x) \leq f(\bar{x}) < \lambda + \varepsilon$

$\forall \varepsilon > 0 \ \exists \delta > 0 \ : \ \forall x \in]a, b[\quad x \in]a, a+\delta[\Rightarrow$

$\lambda - \varepsilon < \lambda \leq f(x) < \lambda + \varepsilon$

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in]a, b[\ \ x \in]a, a+\delta[\Rightarrow |f(x) - \lambda| < \varepsilon$

$\lim_{x \to a^+} f(x) = \lambda$

Se $\lambda = -\infty = \inf f(]a, b[)$ $\quad \forall M \in \mathbb{R} \ \exists \bar{x} \in]a, b[\ \ f(\bar{x}) < M$

$f(x) \leq f(\bar{x}) \quad \forall x < \bar{x}$

fisso $\delta = \bar{x} - a$

$\Rightarrow \forall M \in \mathbb{R} \ \exists \delta > 0 \ : \ \forall x \in]a, b[\quad a < x < a + \delta$

$\Downarrow$

$f(x) \leq f(\bar{x}) < M$

$\Rightarrow \lim_{x \to a^+} f(x) = -\infty$

II parte

Supponiamo f debolmente crescente, ovvero

$\forall x, y \in I \quad x < y \Rightarrow f(x) \leq f(y)$

Sia $I =]a,b[$ e proviamo $\exists \lim_{x \to b^-} f(x)$

Sia $\Lambda = \sup f(]a,b[)$, che esiste poiché

$f(]a,b[) \subseteq \mathbb{R}$!!

Tesi $\lim_{x \to b^-} f(x) = \Lambda = \sup f(]a,b[)$

1 $\Lambda = \sup f(]a,b[) \in \mathbb{R}$

$$\begin{cases} f(x) \leq \Lambda \quad \forall x \in]a,b[\\ \forall \varepsilon > 0 \ \exists \bar{x} \in]a,b[: \quad \Lambda - \varepsilon < f(\bar{x}) \end{cases}$$

e

$\bar{x} < x \Rightarrow f(\bar{x}) \leq f(x)$

Posto $\delta = b - \bar{x}$

$$\begin{cases} f(x) \le \Lambda \quad \forall x \in]a,b[\\ \forall \varepsilon>0 \ \exists \delta>0 : b-\delta<x<b \Rightarrow \Lambda-\varepsilon<f(\bar{x}) \le f(x) \end{cases}$$

$$\Downarrow$$

$$\forall \varepsilon>0 \ \exists \delta>0 : b-\delta<x<b \Rightarrow \Lambda-\varepsilon<f(x) \le \Lambda$$

e questo significa $\lim_{x \to b^-} f(x) = \Lambda$

② $\Lambda = +\infty$

$$\begin{cases} \forall M \in \mathbb{R} \ \exists \bar{x} \in]a,b[: M<f(\bar{x}) \\ \bar{x}<x \Rightarrow f(\bar{x})<f(x) \end{cases}$$

$$\Downarrow \quad \delta = b-\bar{x}$$

$$\forall M \in \mathbb{R} \ \exists \delta>0 : b-\delta<x<b \Rightarrow M<f(\bar{x}) \le f(x)$$

$$\Downarrow$$

$$\forall M \in \mathbb{R} \ \exists \delta>0 : b-\delta<x<b \Rightarrow M<f(x)$$

e questo significa $\lim_{x \to b^-} f(x) = +\infty$

Infinitesimi - simbolo di Landau

Def (Infinitesimi - $o(1)$ per $x \to x_0$)

Dato $f: A \to \mathbb{R}$, x_0 p.d.a. per A

f si dice "infinitesima per $x \to x_0$" in simboli
"$f(x) = o(1)$ per $x \to x_0$" (o piccolo di 1 per x che tende a x_0)

se $\lim_{x \to x_0} f(x) = 0$

Esempio

$f(x) = x - 1$, "$f(x) = o(1)$ per $x \to 1$" $\Leftrightarrow$ "$\lim_{x \to 1} (x-1) = 0$"

$f(x) = 3x^2$ "$f(x) = o(1)$ per $x \to 0$" $\Leftrightarrow$ "$\lim_{x \to 0} 3x^2 = 0$"

Esempio (non infinitesima)

$f(x) = x - 1$ "$f(x) \neq o(1)$ $x \to 0$" infatti,

$$\lim_{x \to 0} (x-1) = -1 \neq 0$$

Def $f, g: A \to \mathbb{R}$, x_0 p.d.a. per A

$f = o(1)$ $g = o(1)$ per $x \to x_0$

Diciamo che "f è un o piccolo di g per x tende x_0"

$\to$ se $\exists\, \eta(x) = o(1)$ $f(x) = \eta(x) \cdot g(x)$ $x \to x_0$

ovvero

$\to$ se $(g \neq 0)$ $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$

Esempio $x^3 = o(\operatorname{sen} x)$ per $x \to 0$

infatti $\lim_{x \to 0} \frac{x^3}{\operatorname{sen} x} = \lim_{x \to 0} x^2 \frac{x}{\operatorname{sen} x} =$

$= \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \frac{x}{\operatorname{sen} x} = 0 \cdot 1 = 0$

Osserva $x^3 = x^2 \cdot x = x^2 \cdot \frac{x}{\operatorname{sen} x} \cdot \operatorname{sen} x$

$x \in]-\pi/2, \pi/2[\setminus 0$

$$x^3 = \underbrace{\left(x^2 \cdot \frac{x}{\operatorname{sen} x}\right)} \cdot \operatorname{sen} x$$

$\eta(x) = x^2 \cdot \frac{x}{\operatorname{sen} x} = o(1) \quad x \to 0$

$\Rightarrow x^3 = \eta(x) \cdot \operatorname{sen} x \quad x \to 0$

$\eta(x) = o(1)$

Euristica

$f = o(g)$, $x \to x_0$ se $\exists\, \eta(x) = o(1)$ $x \to 0$

$f(x) = \eta(x) \cdot g(x)$

$$\left.\begin{array}{l} g \to 0 \\ \eta \to 0 \end{array}\right\} \to \eta \cdot g \to 0 \text{ più velocemente di } g$$

Oss

$$x^3 = o(x) \quad x \to 0 \qquad \left(\lim_{x\to 0} \frac{x^3}{x} = 0 \right)$$

$$x^2 = o(x) \quad x \to 0$$

$$x^{4/3} = o(x) \quad x \to 0$$

$$x^{7/3} = o(x) \quad x \to 0$$

$o(x)$ = è un insieme di funzioni

Oss $f = o(g)$ per $x \to x_0$

significa che "f tende a zero +velocemente di g"

Esempio

- $f_1(x) = x^2$ $\quad g(x) = x$ $\quad f_1 = o(g)$ per $x \to 0$
- $f_2(x) = x^3$ $\quad g(x) = x$ $\quad f_2 = o(g)$ per $x \to 0$
- $f_3 = x^4$ $\quad g(x) = x$ $\quad f_3 = o(g)$ per $x \to 0$

dunque $o(g) = o(x)$ per $x \to 0$ è

UN INSIEME DI FUNZIONI

Oss $\begin{cases} \text{Se } f = o(g) \text{ per } x \to x_0 \\ \text{allora } g \neq o(f) \text{ per } x \to x_0 \end{cases}$

$\begin{cases} \text{Se } f = o(g) \text{ e } g = o(h) \text{ per } x \to x_0 \\ \text{allora } f = o(h) \text{ per } x \to x_0 \end{cases}$

$$\lim_{x \to x_0} \frac{f}{g} = 0 \qquad \lim_{x \to x_0} \frac{g}{h} = 0$$

$$\Rightarrow \lim_{x \to x_0} \frac{f}{h} = \lim_{x \to x_0} \frac{f}{g} \cdot \frac{g}{h} = \lim_{x \to x_0} \frac{f}{g} \cdot \lim_{x \to x_0} \frac{g}{h} = 0$$

Quindi si possono "ordinare" le funzioni

$$g \prec f \quad \text{per } x \to x_0 \quad \text{sse} \quad f = o(g) \quad x \to x_0$$

↑ va a 0 + velocemente di

Ad esempio per $x \to 0$

$$x \prec x^2 \prec \operatorname{sen}^3 x \prec (1-\cos x)^2 \quad \text{etc}$$

$$(1-\cos x)^2 = o(\operatorname{sen}^3 x) \quad \text{per } x \to 0$$

$$\Updownarrow$$

$$\lim_{x\to 0} \frac{(1-\cos x)^2}{\operatorname{sen}^3 x} = 0 \iff \lim_{x\to 0} \frac{(1-\cos x)^2}{(x^2)^2} \cdot \frac{(x^2)^2}{\operatorname{sen}^3 x} = 0$$

$$\iff \left(\lim_{x\to 0} \frac{1-\cos x}{x^2}\right)^2 \cdot \lim_{x\to 0} \frac{x^4}{\operatorname{sen}^3 x} = 0$$

$$\iff \frac{1}{4} \cdot \lim_{x\to 0} \frac{x^3}{\operatorname{sen}^3 x} \cdot \lim_{x\to 0} x = 0$$

$$\iff \frac{1}{4} \cdot \underbrace{\left(\lim_{x\to 0} \frac{x}{\operatorname{sen} x}\right)^3}_{= 1^3} \cdot \lim_{x\to 0} x = 0$$

e siamo pervenuti a una identità nota ✓

<u>Esempio</u> $\operatorname{sen} x - x = o(x), \quad x \to 0$

$$\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = 1 \iff \lim_{x\to 0} \left(\frac{\operatorname{sen} x}{x} - 1\right) = 0$$

$$\iff \lim_{x\to 0} \frac{\operatorname{sen} x - x}{x} = 0 \iff \operatorname{sen} x - x = o(x) \quad x \to 0$$

Esempio $\quad 1-\frac{x^2}{2}-\cos x = o(x^2) \quad x\to 0$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} \iff \lim_{x\to 0}\left(\frac{1-\cos x}{x^2}-\frac{1}{2}\right)=0$$

$$\iff \lim_{x\to 0} \frac{1-\cos x-\frac{x^2}{2}}{x^2} = 0$$

$$\iff 1-\cos x-\frac{x^2}{2} = o(x^2) \quad x\to 0$$

N.B. $\forall g=o(1)$ $x\to x_0$, si ha che

(la f.ne identicamente nulla) $= o(g)$ $\quad x\to x_0$

$f(x)=0 \;\; \forall x\in\mathbb{R} \implies f(x)=o(g)$ per $x\to x_0$
$\forall g=o(1)$

Algebra degli "o piccolo"

Quando $x \to 0$

(i) $o(x^\alpha) = k\, o(x^\alpha) \quad \forall k \neq 0 \quad \forall \alpha > 0$

$$f = o(x^\alpha) \quad x \to 0 \iff \exists\, \lambda = o(1) \quad x \to 0$$

$$f(x) = \lambda(x) \cdot x^\alpha$$

$$\iff f(x) = \frac{\lambda(x)}{k} \cdot k \cdot x^\alpha \quad x \to 0 \quad \lambda = o(1)$$

$\frac{\lambda}{k} = o(1)$

$$\iff \frac{f(x)}{k} = \frac{\lambda(x)}{k} \cdot x^\alpha \quad x \to 0 \quad \lambda = o(1)$$

$$\iff \frac{f(x)}{k} = o(x^\alpha) \quad x \to 0$$

$$\iff f(x) = k \cdot o(x^\alpha) \quad x \to 0$$

Oss $\quad k = 2 \qquad o(x^\alpha) = 2\, o(x^\alpha) \quad x \to 0$

$$= o(x^\alpha) + o(x^\alpha)$$

$k = -1 \qquad o(x^\alpha) = -\, o(x^\alpha)$

$$\Rightarrow o(x^\alpha) - o(x^\alpha) = o(x^\alpha) + o(x^\alpha)$$

$$\shortparallel$$

$$o(x^\alpha)$$

Oss $\quad o(x^\alpha) + o(x^\alpha) = o(x^\alpha)$

$$o(x^\alpha) - o(x^\alpha) = o(x^\alpha)$$

(ii) $x^{\alpha} \cdot o(x^{\beta}) = o(x^{\alpha+\beta}) \quad \forall \alpha, \beta > 0$

$f(x) = x^{\alpha} \cdot o(x^{\beta}) \ x \to 0 \iff \exists \lambda(x) \ \lambda = o(1)$

$$f(x) = x^{\alpha} \cdot (\lambda(x) \cdot x^{\beta})$$

$$= \lambda(x) \cdot x^{\alpha+\beta}$$

$x \to 0$

$$\iff f(x) = o(x^{\alpha+\beta}) \quad x \to 0$$

<u>Oss</u> $x^4 o(x^2) = x^3 \cdot o(x^3)$

$= x^5 \cdot o(x)$

$= o(x^6)$

$= x^2 \cdot o(x^4)$ etc

- $o(o(x^{\alpha})) = o(x^{\alpha}) \quad x \to 0$

$f = o(o(x^{\alpha})) \ x \to 0 \iff \exists \lambda(x) \ \lambda = o(1) : f(x) = \lambda(x) \cdot o(x^{\alpha})$

$x \to 0$

$\iff \exists \lambda(x) \ \exists \eta(x) \ \lambda = o(1) \ \eta = o(1) \quad x \to 0$

$$f(x) = [\lambda(x) \cdot \eta(x)] \cdot x^{\alpha}$$

$\lambda \cdot \eta = o(1)$

$\iff f(x) = o(x^{\alpha}) \quad x \to 0$

(iii) $o(x^\alpha) + o(x^{\alpha+\beta}) = o(x^\alpha)$ $x \to 0$ $\forall \alpha, \beta > 0$

è + veloce di x^α

$\Leftarrow$ è banale poiché $0 = o(x^{\alpha+\beta})$

$f = o(x^\alpha)$ per $x \to 0$

$\Rightarrow \exists \zeta = o(1) : f(x) = \zeta(x) \cdot x^\alpha \quad x \to 0$

$\Rightarrow \quad f(x) = \zeta(x) \cdot x^\alpha + 0 \cdot x^{\alpha+\beta}$

$\Rightarrow \quad f(x) = o(x^\alpha) + o(x^{\alpha+\beta}) \quad x \to 0$

$(\Rightarrow)$ $f = o(x^\alpha) + o(x^{\alpha+\beta}) \quad x \to 0$

$\Leftrightarrow \exists \lambda = o(1) \ \exists \zeta = o(1) : f = \lambda(x) \cdot x^\alpha + \zeta(x) \cdot x^{\alpha+\beta} \quad x \to 0$

$\Rightarrow \exists \lambda = o(1) \ \exists \zeta = o(1) : f(x) = \left(\lambda(x) + \zeta(x) \cdot x^\beta\right) x^\alpha \quad x \to 0$

$\lambda + \zeta x^\beta = o(1)$

$\Rightarrow f = o(x^\alpha) \quad x \to 0$

Oss $A, B \subseteq \mathbb{R}$ $\quad 0 \in B$

Tesi $\quad A \subseteq A + B$

$A + B = \{a + b : a \in A, b \in B\}$

$$A = \{a \in A\} = \{a + 0 \; ; \; a \in A\} \subseteq A + B$$

Oss $\mathbb{R} \subseteq \mathbb{C}$

Oss $o(x^\alpha) \subseteq o(x^\alpha) + o(x^{\alpha+\beta})$ in quanto $0 \in o(x^{\alpha+\beta})$

Esempio $]0,1[+]2,3[=]2,4[$

$\{x+y : x \in]0,1[\; y \in]2,3[\}$

Esempio $]-3,2[+]1,3[=]-2,5[$

Oss $o(x^3) + o(x^5) = o(x^3) \quad x \to 0$

$o(x) + o(x^2) = o(x) \quad x \to 0$

(iv) $o(o(x^\alpha)) = o(x^\alpha) \quad x \to 0 \quad \forall \alpha > 0$

$(\Rightarrow)$ $f = o(o(x^\alpha)) \quad x \to 0$

$\Leftrightarrow \exists \zeta = o(1) \ : \ f = \zeta \cdot o(x^\alpha) \quad x \to 0$

$\Leftrightarrow \exists \zeta = o(1) \ \exists \lambda = o(1) \quad f = \zeta(x) \cdot \lambda(x) \cdot x^\alpha \quad x \to 0$

$\Rightarrow \exists \zeta \cdot \lambda = o(1) \quad f = (\zeta(x) \cdot \lambda(x)) x^\alpha \quad x \to 0$

$\Rightarrow f(x) = o(x^\alpha) \quad x \to 0$

$(\Leftarrow)$ $f = o(x^\alpha) \quad x \to 0$

$\Leftrightarrow \exists \zeta = o(1) \quad f(x) = \zeta(x) \cdot x^\alpha = \sqrt[3]{\zeta(x)} \sqrt[3]{\zeta^2(x)} \cdot x^\alpha \quad x \to 0$

$\Leftrightarrow \exists \sqrt[3]{\zeta} = o(1), \sqrt[3]{\zeta^2} = o(1) \ : \ f(x) = \sqrt[3]{\zeta(x)} \sqrt[3]{\zeta^2(x)}\, x^\alpha \quad x \to 0$

$\Leftrightarrow \exists \sqrt[3]{\zeta} = o(1) \ : \ f(x) = \sqrt[3]{\zeta(x)} \cdot o(x^\alpha) \quad x \to 0$

$\Leftrightarrow f(x) = o(o(x^\alpha)) \quad x \to 0$ $\quad \lightning$

(V) $\boxed{o\left(x^\alpha + o(x^\alpha)\right) = o(x^\alpha)}$ $x \to 0$

($\Rightarrow$) $f = o\left(x^\alpha + o(x^\alpha)\right)$ $x \to 0$

$\Leftrightarrow \exists\, \varepsilon_1 = o(1)\ \ \varepsilon_2 = o(1) \ : \ f(x) = \varepsilon_1 \left(x^\alpha + \varepsilon_2 \cdot x^\alpha\right)$ $x \to 0$

$\Leftrightarrow \exists \left(\varepsilon_1 + \varepsilon_1 \varepsilon_2\right) = o(1) \ : \ f(x) = \left(\varepsilon_1 + \varepsilon_1 \varepsilon_2\right) \cdot x^\alpha$ $x \to 0$

$\Leftrightarrow f = o(x^\alpha)$ $x \to 0$

($\Leftarrow$) $f = o(x^\alpha)$ $x \to 0$

$\Leftrightarrow \exists\, \varepsilon_1 = o(1) \ : \ f(x) = \varepsilon_1 \cdot x^\alpha$ $x \to 0$

$\{0 = o(1) \quad x \to 0\}$

$\Leftrightarrow \exists\, \varepsilon_1 = o(1) \ : \ f(x) = \left(\varepsilon_1 + \varepsilon_1 \cdot 0\right) \cdot x^\alpha$ $x \to 0$

$\Leftrightarrow \exists\, \varepsilon_1 = o(1)\ \ \varepsilon_2 = 0 = o(1) \quad f(x) = \varepsilon_1 \left(x^\alpha + 0 \cdot x^\alpha\right)$ $x \to 0$

$\Leftrightarrow f(x) = o\left(x^\alpha + o(x^\alpha)\right)$ $x \to 0$ ✓

(vi) $\quad o(x^{\alpha})\, o(x^{\beta}) = o(x^{\alpha+\beta}) \quad x \to 0$

(si veda $\quad x^{\alpha} o(x^{\beta}) = o(x^{\alpha+\beta}) \quad !$)

(vii) $\quad \dfrac{o(x^{\alpha+\beta})}{x^{\beta}} = o(x^{\alpha}) \qquad \forall \alpha, \beta > 0 \quad x \to 0$

Principio di sostituzione dell'infinitesimo

Teorema

$f_1, f_2, g_1, g_2 : A \to \mathbb{R}$ $\quad x_0$ p.d.a. per A

- f_1, f_2, g_1, g_2 infinitesime per $x \to x_0$
- $f_1 = o(f_2)$, $g_1 = o(g_2)$ $\quad x \to x_0$

allora

$$\frac{f_1+f_2}{g_1+g_2}$$ ha lo stesso carattere di $$\frac{f_2}{g_2}$$ quando $x \to x_0$

ovvero $\exists \lim_{x\to x_0} \frac{f_1+f_2}{g_1+g_2} = \lim_{x\to x_0} \frac{f_2}{g_2}$

oppure $\nexists$ entrambi i limiti

$x \to x_0$

$$\frac{f_1+f_2}{g_1+g_2} = \frac{o(f_2)+f_2}{o(g_2)+g_2} = \frac{f_2}{g_2} \cdot \frac{1+o(1)}{1+o(1)}$$

$$\boxed{\lim_{x\to x_0} \frac{1+o(1)}{1+o(1)} = \frac{1}{1}}$$ e dunque

$$\frac{f_1+f_2}{g_1+g_2}$$ si comporta come $$\frac{f_2}{g_2}$$ quando $x \to x_0$

Esempio Calcolare $\lim_{x\to 0} \frac{\sin^2 x + x^3}{1-\cos x + x^4}$

dim

Si osserva che $x^3 = o(\sin^2 x) \quad x \to 0$

$x^4 = o(1-\cos x) \quad x \to 0$

$\lim_{x\to 0} \frac{x^3}{\sin^2 x} = 0 \qquad \lim_{x\to 0} \frac{x^4}{1-\cos x} =$

$= \lim_{x\to 0} x^2 \cdot \frac{x^2}{1-\cos x} = 0$

$$\lim_{x\to 0} \frac{\sin^2 x + x^3}{1-\cos x + x^4} = \lim_{x\to 0} \frac{\sin^2 x + o(\sin^2 x)}{1-\cos x + o(1-\cos x)}$$

$$\overset{\Downarrow}{=} \lim_{x\to 0} \frac{\sin^2 x}{1-\cos x} = \lim_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \frac{x^2}{1-\cos x} =$$

$= 2$

Sviluppi f.ni elementari

$$\operatorname{sen} x = x + o(x)$$

$$= x + o(x^2)$$

$$= x - \frac{x^3}{3!} + o(x^4)$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^6)$$

$$\cdots$$

$$= x - \frac{x^3}{3!} + \cdots + (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

$$\cos x = 1 + o(x)$$

$$= 1 - \frac{x^2}{2!} + o(x^3)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5)$$

$$\dots$$

$$= 1 - \frac{x^2}{2!} + \dots + (-1)^{m+1} \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$$

$$Tg(x) = x + o(x^2)$$

$$= x + \frac{x^3}{3} + o(x^4)$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^6) \qquad x \to 0$$

$$\operatorname{arctg} x = x + o(x^2)$$

$$= x - \frac{x^3}{3} + o(x^4)$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^6)$$

$$\ldots$$

$$= x - \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^{2n+1}}{2n+1} + o(x^{2n+2})$$

$$e^x = 1 + o(1)$$

$$= 1 + x + o(x)$$

$$= 1 + x + \frac{x^2}{2} + o(x)$$

$$\ldots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n)$$

$$\log(1+x) = x + o(x)$$

$$= x - \frac{x^2}{2} + o(x^2)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

$$---$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + o(x^n)$$

$$(1+x)^\alpha = 1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\cdots$$

$$\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n+o(x^n)$$

$$(1+x)^\alpha = e^{\alpha\log(1+x)}$$

$$\log(1+x) = \cdots$$

$$\alpha\log(1+x) = \cdots$$

$$e^{\alpha y} = \cdots$$

$$e^{\alpha\log(1+x)} = \cdots$$

<u>Esercizio</u> Calcolare lo sviluppo di $\frac{1}{1-x}$

(ovvero $(1-x)^\alpha$ con $\alpha=-1$!)