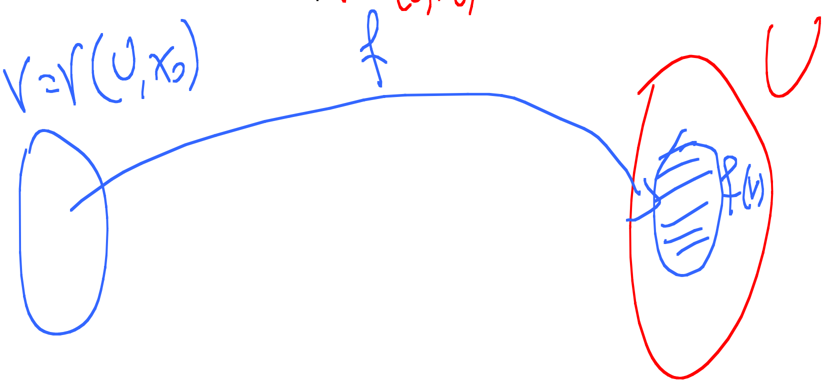
Def $f: A \to \mathbb{R}$, $x_0 \in A$, "f è continua in x_0"

se
- $\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon, x_0) > 0 : \forall x \in A \ \ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$
- $\forall U \in \mathcal{I}_{f(x_0)} \ \exists V \in \mathcal{I}_{x_0} : \forall x \in A \ \ x \in V \Rightarrow f(x) \in U$

$V = V(U, x_0)$



Def $f: A \to \mathbb{R}$ x_0 pdo per A $x_0, \ell \in \overline{\mathbb{R}}$

"f ha limite ℓ per x che tende a x_0" in simboli $\lim_{x \to x_0} f(x) = \ell$

se

$\forall U \in \mathcal{I}_\ell \ \exists V \in \mathcal{I}_{x_0} : \forall x \in A \ \ x \in V \setminus \{x_0\} \Rightarrow f(x) \in U$

*

Oss i) $f: A \to \mathbb{R}$ continua in x_0 p.d.a. per $A \Rightarrow \lim_{x \to x_0} f(x) = f(x_0)$

ii) $f: A \to \mathbb{R}$, x_0 p.d.a. per A $\lim_{x \to x_0} f(x) = f(x_0) \Rightarrow f$ continua in x_0

Teorema Se il limite esiste allora è unico.

Esercizio
$f(x) = x$
$f(x) = \sin x$
$f(x) = \cos x$
$f(x) = |x|$

sono f. ni continue $\forall x \in \mathbb{R}$

Def $\boxed{\lim_{x\to x_0} f(x) = \ell^+}$

$$\forall U \in \mathcal{J}_\ell \quad \exists V \in \mathcal{J}_{x_0} : \forall x\in A \quad x \in V\setminus\{x_0\} \Rightarrow f(x) \in U \cap]\ell,+\infty[$$

Def $\lim_{x\to x_0^-} f(x) = \ell^-$

$$\forall U \in \mathcal{J}_\ell \quad \exists V \in \mathcal{J}_{x_0} : \forall x\in A \quad x \in (V\setminus\{x_0\}) \cap]-\infty,x_0[\Rightarrow f(x) \in U \cap]-\infty,x_0[$$

Esempio verificare che $\boxed{\lim_{x\to+\infty} 1/x = 0^+}$

va provato che

$$\forall U \in \mathcal{J}_0 \quad \exists V \in \mathcal{J}_{+\infty} \quad \forall x\neq 0 \quad x\in V \Rightarrow 1/x \in U \cap]0,+\infty[$$

$$\forall]-\delta,\delta[\in \mathcal{J}_0 \quad \exists]M,+\infty[\in \mathcal{J}_{+\infty} : \forall x\neq 0 \quad x\in]M,+\infty[\Rightarrow 1/x \in]-\delta,\delta[\cap]0,+\infty[$$

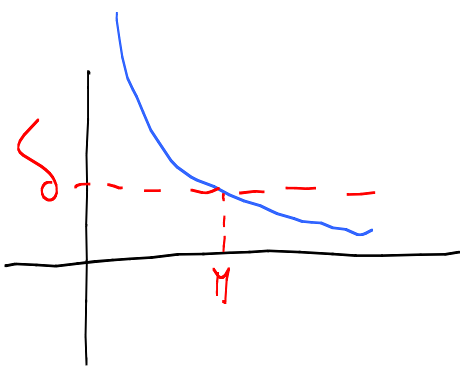
$$\forall \delta>0 \quad \exists M\in\mathbb{R} : \forall x\neq 0 \quad x>M \Rightarrow 1/x \in]0,\delta[$$

$$\forall \delta>0 \quad \exists M>0 : \quad x>M \Rightarrow 1/x \in]0,\delta[$$

$M = M(\delta)$

$$0 < 1/x < \delta \iff x > 1/\delta$$

$$\forall \delta>0 \quad \exists M = 1/\delta : \quad x>M \Rightarrow 0 < 1/x < \frac{1}{M} = \delta$$



Teorema (f continua $\Rightarrow$ f localmente limitata)

$f: A \to \mathbb{R}$ $\quad x_0 \in A$

Se f continua in x_0

allora $\exists \delta > 0$ $\exists C =$ $\quad \boxed{|f(x)| \leq C \quad \forall x \in A \cap]x_0-\delta, x_0+\delta[}$

dim

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \forall x \in A \quad x \in]x_0-\delta, x_0+\delta[\Rightarrow \underbrace{f(x_0)-\varepsilon < f(x) < f(x_0)+\varepsilon}$$

$$\Downarrow$$

$$\varepsilon = 1 \quad \exists \delta = \delta(1) : \forall x \in A \quad x \in]x_0-\delta, x_0+\delta[\Rightarrow f(x_0)-1 < f(x) < f(x_0)+1$$

$$f(x_0) - 1 < f(x) < f(x_0)+1 \leq |f(x_0)| + 1$$

$$\text{\rotatebox{90}{$\leq$}}$$

$$-|f(x_0)| - 1$$

$$|f(x)| \leq |f(x_0)| + 1 = C$$

$$\exists \delta > 0 : \quad |f(x)| \leq C \quad \forall x \in]x_0-\delta, x_0+\delta[\cap A \quad \lightning$$

Teorema (f ha limite finito allora f è limitata)

$f: A \to \mathbb{R}$ $\quad x_0$ p.d.a. per A

Se $\exists \lim_{x \to x_0} f(x) = l \in \mathbb{R}$ allora f è localmente limitata

ovvero

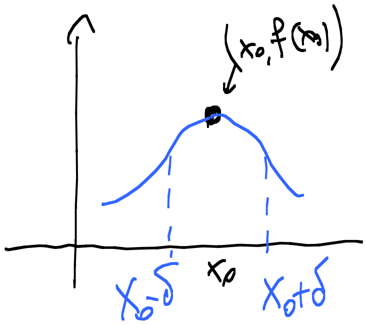
$$\exists U \in \mathcal{I}_{x_0} \quad \exists c > 0 : \quad |f(x)| \leq c \quad \forall x \in A \cap (U \setminus \{x_0\})$$

Teorema (della permanenza del segno - f.ni continue)

$f: A \to \mathbb{R}$ $\quad x_0 \in A$

f continua in x_0 e $f(x_0) > 0$

allora $\exists \delta > 0 : \forall x \in A \cap]x_0-\delta, x_0+\delta[\quad f(x) > 0$



dim

$f(x_0) > 0$

$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in A \quad x \in]x_0-\delta, x_0+\delta[\Rightarrow f(x_0)-\varepsilon < f(x) < f(x_0)+\varepsilon$

$\varepsilon = \frac{f(x_0)}{2} \quad \exists \delta > 0 : \forall x \in A \quad x \in]x_0-\delta, x_0+\delta[\Rightarrow$

$$f(x_0) - \frac{f(x_0)}{2} < f(x) < f(x_0) + \frac{f(x_0)}{2}$$

$$0 < \underbrace{\frac{f(x_0)}{2}}_{=f(x_0)-\frac{f(x_0)}{2}} \qquad \underbrace{\frac{3}{2}f(x_0)}_{=f(x_0)+\frac{f(x_0)}{2}}$$

$\exists \delta > 0 : \forall x \in A \cap]x_0-\delta, x_0+\delta[\quad f(x) > 0$ $\quad \Downarrow$

Teorema (permanenza segno per i limiti)

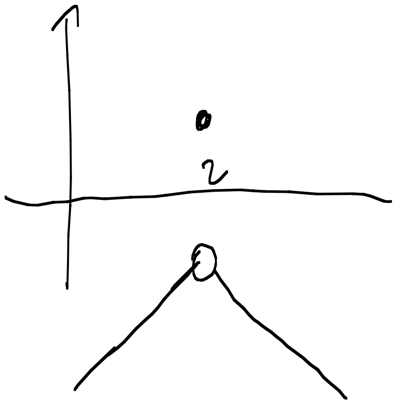
$f: A \to \mathbb{R}$, x_0 p.d.a. per A

Se $\lim_{x \to x_0} f(x) = l > 0$ allora $\exists \delta > 0 : \forall x \in (A \setminus x_0) \cap]x_0-\delta, x_0+\delta[\quad f(x) > 0$

La dimostrazione è analoga alla precedente

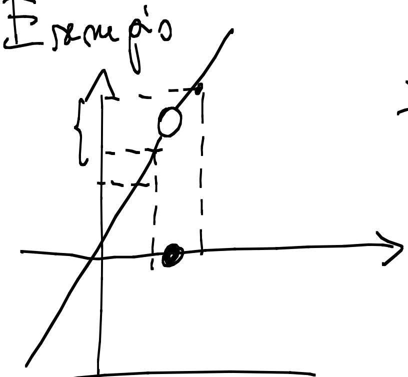
Esempio Se $f(x_0) > 0$ ma f non è continua

allora $\exists U \in \mathcal{I}_{x_0} : \quad f(x) > 0 \quad \forall x \in U \cap A$?



$$f(x)=\begin{cases} -|x-2|-1 & x\neq 2 \\ 1 & x=2 \end{cases}$$

Esempio



$$f(x)=\begin{cases} 2x & x\neq 1 \\ 0 & x=1 \end{cases}$$

$$\boxed{f(1)=0}$$

Quanto vale $\lim_{x\to 1} f(x)$? 2

$\cancel{\delta}$ $\forall\varepsilon>0\ \exists\delta>0:\quad 0<|x-1|<\delta \Rightarrow |f(x)-2|<\varepsilon$

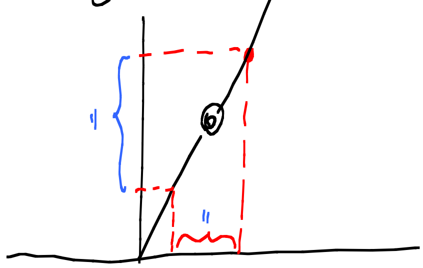
$\forall\varepsilon>0\ \exists\delta>0:\ x\neq 1\quad 1-\delta<x<1+\delta \Rightarrow |2x-2|<\varepsilon$

$\forall\varepsilon>0\ \exists\delta>0:\quad 0<|x-1|<\delta \Rightarrow 2|x-1|<\varepsilon$

$\cancel{\delta}$ $\forall\varepsilon>0\ \exists\delta>0:\quad 0<|x-1|<\delta \Rightarrow |x-1|<\varepsilon/2$

$\delta=\varepsilon/2$

$\forall\varepsilon>0\ \exists\delta=\varepsilon/2:\quad 0<|x-1|<\delta \Rightarrow |x-1|<\delta=\varepsilon/2$ ↯



$x_0 = +\infty \iff \forall M \in \mathbb{R} \quad x_0 > M \iff x_0 = \sup \mathbb{R}$

$y_0 = -\infty \iff \forall M \in \mathbb{R} \quad y_0 < M \iff y_0 = \inf \mathbb{R}$

Algebra di $\boxed{\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}}$

- $c \in \mathbb{R}$ $\quad c + \infty = +\infty \quad c - \infty = -\infty$ **Somma**

- $c > 0$ $\quad c \cdot +\infty = +\infty \quad c \cdot (-\infty) = -\infty$ **Prodotto**

  $\frac{+\infty}{c} = +\infty \quad \frac{-\infty}{c} = -\infty$

  $\frac{1}{x} \xrightarrow[x \to +\infty]{} 0^+ \implies \frac{c}{+\infty} = 0^+$ $\quad \frac{c}{-\infty} = 0^-$ ← Esempio $\lim_{x \to -\infty} 1/x = 0^-$

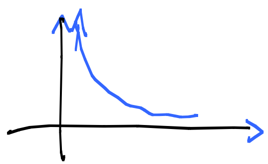
  ↑ limite da sinistra $\quad$ } limite da dx

- $c < 0$ $\quad c \cdot +\infty = -\infty \quad c \cdot (-\infty) = +\infty$ **Prodotto**

  $\frac{+\infty}{c} = -\infty \quad \frac{-\infty}{c} = +\infty$

  $\frac{c}{+\infty} = 0^- \quad \frac{c}{-\infty} = 0^+$

* $\lim_{x \to 0^+} 1/x = +\infty$



$c > 0$ $\quad$ * $\frac{c}{0^+} = +\infty$ $\quad$ $\frac{c}{0^-} = -\infty$

$c < 0$ $\quad$ $\frac{c}{0^+} = -\infty$ $\quad$ $\frac{c}{0^-} = +\infty$

$+\infty + \infty = +\infty \quad -\infty - \infty = -\infty$ Somme

$(+\infty)\cdot(+\infty) = +\infty \quad (-\infty)\cdot(-\infty) = +\infty$

$(+\infty)\cdot(-\infty) = -\infty$

$\left(1+\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{}$

Forme Indeterminate

$+\infty - \infty = ? \quad 0\cdot\infty = ? \quad \frac{\infty}{\infty} = ? \quad \frac{0}{0} = ? \quad 1^\infty = ? \quad \infty^0 = ?$

Teorema (Algebra dei limiti)

$f, g : A \to \mathbb{R}$, x_0 p. d. e. per A

1) $\underbrace{\lim_{x\to x_0} f(x) = m}$ $\underbrace{\lim_{x\to x_0} g(x) = \ell}$ $\Rightarrow \lim_{x\to x_0} (f+g)(x) = \ell + m$

quando $\ell + m$ non è una forma indeterminata

2) $\lim_{x\to x_0} f(x) = m$ $\lim_{x\to x_0} g(x) = \ell \Rightarrow \lim_{x\to x_0} (f\cdot g)(x) = \ell \cdot m$

quando $\ell \cdot m$ non è forma indeterminata

3) $\lim_{x\to x_0} f(x) = \ell$ $\lim_{x\to x_0} g(x) = m \neq 0 \Rightarrow \lim_{x\to x_0} \frac{f}{g} = \ell/m$

Non va $\frac{\infty}{\infty}$ bene

quando ℓ/m non è forma indeterminata dim.

2) dall'ipotesi, supponendo $l, m, x_0 \in \mathbb{R}$

- $\forall \varepsilon>0 \ \exists \delta_1(\varepsilon)>0 : \forall x \in A \ 0<|x-x_0|<\delta_1 \Rightarrow l-\varepsilon < f(x) < l+\varepsilon$

  $\quad " \quad \exists \delta_2(\varepsilon)>0 : \quad " \quad 0<|x-x_0|<\delta_2 \Rightarrow m-\varepsilon < g(x) < m+\varepsilon$

$\bar{\delta} = \min\{\delta_1, \delta_2\}$

$$\forall \varepsilon>0 \ \exists \bar{\delta}>0 : \forall x \in A \quad 0<|x-x_0|<\bar{\delta} \Rightarrow \begin{cases} l-\varepsilon < f(x) < l+\varepsilon \\ m-\varepsilon < g(x) < m+\varepsilon \end{cases}$$

$$|f(x)g(x) - lm| = |f(x)g(x) - f(x)\cdot m + f(x)m - lm|$$

$$\leq |f(x)(g(x)-m)| + |m(f(x)-l)|$$

$$= |f(x)|\cdot|g(x)-m| + |m|\cdot|f(x)-l|$$

$$\underset{0<|x-x_0|<\bar{\delta}}{\leq} |l+\varepsilon|\cdot\varepsilon + |m|\cdot\varepsilon \underset{\varepsilon<1}{\leq} \varepsilon\left[|l|+1+|m|\right]$$

$$\forall \varepsilon>0 \ \exists \bar{\delta}>0 : \forall x \in A \quad 0<|x-x_0|<\bar{\delta} \Rightarrow$$
$(\varepsilon<1)$

$$|f(x)g(x) - lm| \leq \underbrace{\varepsilon\cdot\left[|l|+1+|m|\right]}$$

$$\sigma = \varepsilon\left[|l|+1+|m|\right]$$

$$\forall \sigma>0 \ \exists \bar{\delta}>0 : \forall x \in A \quad 0<|x-x_0|<\bar{\delta} \Rightarrow |f\cdot g(x) - l\cdot m| < \sigma$$

Teorema (Algebra delle funzioni continue)

$f, g : A \to \mathbb{R}$ continue in $\underline{x_0 \in A}$ $\nRightarrow$ non chiedo x_0 p.d.a.

1) $f+g$ è continua in x_0 e si ha $(f+g)(x_0) = f(x_0) + g(x_0)$

2) $f \cdot g$ " " " " " " " $(f \cdot g)(x_0) = f(x_0)\, g(x_0)$

3) Se $g(x_0) \neq 0$, f/g " " " " " " " $\left(\frac{f}{g}\right)(x_0) = \frac{f(x_0)}{g(x_0)}$

$x_0, f(x_0), g(x_0) \in \mathbb{R}$

$f(x) = x$ è continua
$g(x) = \operatorname{sen} x$ " " $\forall x \in \mathbb{R}$ $\Rightarrow$
$$\begin{cases} x + \operatorname{sen} x \text{ è continua } \forall x \in \mathbb{R} \\ x \cdot \operatorname{sen} x \text{ è continua } \forall x \in \mathbb{R} \\ \dfrac{x}{\operatorname{sen} x} \text{ è continua } \forall x \in \mathbb{R} : \operatorname{sen} x \neq 0 \end{cases}$$

Teorema (I polinomi sono funzioni continue)

$$\forall a_i \in \mathbb{R} \quad i=0,\dots,n \qquad f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

è una funzione continua $\forall x \in \mathbb{R}$

dim

Va provato che x^n è continua $\forall n \in \mathbb{N}$

$n=0 \qquad x^0 = 1$ è una funzione continua $\forall x \in \mathbb{R}$

Si suppone che x^n è continua $\forall x \in \mathbb{R}$

La funzione $x^{n+1} = x \cdot x^n$ è quindi continua in quanto prodotto di funzioni continue ($f(x)=x$ è continua e $g(x)=x^n$ è continua per ipotesi induttiva)

Ma allora $a_i x^i$ è una funzione continua $\forall i$ in quanto prodotto di funzioni continue (la f.ne costante $f(x)=a_i$ e la f.ne $g(x)=x^i$)

Infine $\sum_{i=0}^{n} a_i x^i$ è continua in quanto somma di funzioni continue

OSS:

Resta da provare

- La continuità della funzione inversa

- " " di e^x e della sua inversa $\log x$

che verranno fatte più avanti

Teorema (del confronto tra limiti)

$f, g : A \to \mathbb{R}$, x_0 p.d.a. per A

1) $\exists \lim_{x \to x_0} f(x) = l$, $\exists \lim_{x \to x_0} g(x) = m$, $f(x) \le g(x)$ $\forall x \in A$

allora $l \le m$

2) $\exists \lim_{x \to x_0} f(x) = +\infty$, $f(x) \le g(x)$ $\forall x \in A \Rightarrow \exists \lim_{x \to x_0} g(x) = +\infty$

3) $\exists \lim_{x \to x_0} g(x) = -\infty$, $f(x) \le g(x)$ $\forall x \in A \Rightarrow \exists \lim_{x \to x_0} f(x) = -\infty$

<u>dim</u>

1) $l, m, x_0 \in \mathbb{R}$

$\forall \varepsilon > 0 \; \exists \delta_1 > 0 : \forall x \in A \;\; 0 < |x - x_0| < \delta_1 \Rightarrow l - \varepsilon < f(x) < l + \varepsilon$

$\forall \varepsilon > 0 \; \exists \delta_2 > 0 : \;\; " \;\; 0 < |x - x_0| < \delta_2 \Rightarrow m - \varepsilon < g(x) < m + \varepsilon$

$\bar{\delta} = \min\{\delta_1, \delta_2\}$

$$\begin{cases} \forall \varepsilon > 0 \; \exists \bar{\delta} > 0 \;\; \forall x \in A \;\; 0 < |x - x_0| < \bar{\delta} \Rightarrow \begin{cases} \underbrace{l - \varepsilon < f(x)} < l + \varepsilon \\ m - \varepsilon < \underbrace{g(x) < m + \varepsilon} \end{cases} \\ \underbrace{f(x) \le g(x)} \;\; \forall x \in A \text{ ipotesi che ho fatto} \end{cases}$$

$\forall \varepsilon > 0 \; \exists \delta > 0 : \forall x \in A \;\; 0 < |x - x_0| < \delta \Rightarrow \underbrace{l - \varepsilon} < f(x) \le g(x) < \underbrace{m + \varepsilon}$

$\to \forall \varepsilon > 0 \quad l - \varepsilon < m + \varepsilon$

$\forall \varepsilon > 0 \quad l - m < 2\varepsilon$

$l - m \le 0$

$l \le m$

$\to x < 1/M \;\; \forall M \Rightarrow x \le 0$

$l - m < 2\varepsilon \;\; \forall \varepsilon > 0 \Rightarrow l - m \le 0$

2) $\lim_{x\to x_0} f = +\infty$ e $f(x) \le g(x)\ \forall x\in A$ $\quad x_0$ p.d.a. per A $x_0 \in \mathbb{R}$

$$\begin{cases} \forall M>0\ \exists \delta>0 : \forall x\in A\ \ 0<|x-x_0|<\delta \Rightarrow \underbrace{M<f(x)} \\ \underbrace{f(x) \le g(x)}\ \forall x\in A \end{cases}$$

$\Downarrow$

$\forall M>0\ \exists \delta>0 : \forall x\in A\ \ 0<|x-x_0|<\delta \Rightarrow M<g(x)$

che è poi

$\lim_{x\to x_0} g(x) = +\infty$

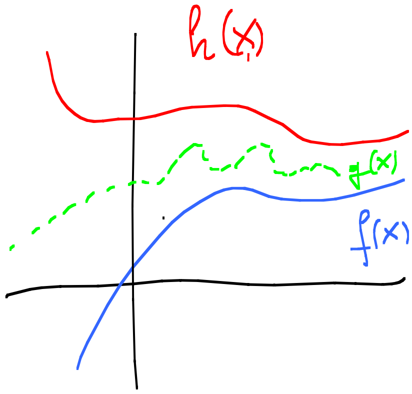
3) analogo a 2) $\leftarrow$ Esercizio per casa!

Teorema (dei 2 carabinieri per i limiti)

$f, g, h : A \to \mathbb{R}$ $\quad x_0$ p.d.a. per A

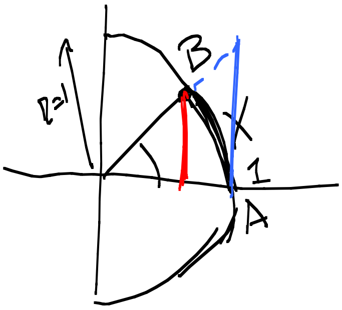
- $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = l \in \overline{\mathbb{R}}$
- $f(x) \le g(x) \le h(x)\ \ \forall x\in A$

$\Rightarrow \exists \lim_{x\to x_0} g(x) = l$



Esempio (MOLTO IMPORTANTE)

$$\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = 1 \qquad \frac{\operatorname{sen}(0)}{0} = \frac{0}{0} \text{ è forma indeterminata}$$



$x =$ lunghezza $(\overset{\frown}{AB})$

$0 < x < \pi/2 \qquad \operatorname{sen} x < x < \operatorname{Tg} x$

$\Vert$

$$1 < \frac{x}{\operatorname{sen} x} < \frac{1}{\cos x}$$

$$0 < x < \pi/2 \qquad 1 > \frac{\operatorname{sen} x}{x} > \cos x \qquad \Vert\Vert$$

$\Downarrow$

$$0 < -x < \pi/2 \qquad 1 > \frac{\operatorname{sen}(-x)}{(-x)} > \cos(-x)$$

$$0 > x > -\pi/2 \qquad 1 > \frac{+\operatorname{sen} x}{+x} > \cos x$$

$$0 > x > \pi/2 \qquad 1 > \frac{\operatorname{sen} x}{x} > \cos x \qquad \Vert\Vert$$

$\Downarrow$

$$\boxed{0 < |x| < \pi/2 \qquad 1 > \frac{\operatorname{sen} x}{x} > \cos x}$$

$f(x) = \cos x \quad g(x) = \frac{\operatorname{sen} x}{x} \quad h(x) = 1 \qquad f, g, h : \overbrace{]-\tfrac{\pi}{2}, \tfrac{\pi}{2}[\setminus \{0\}}^{A} \to \mathbb{R}$

0 p.d.a. per A

$f(x) \le g(x) \le h(x) \quad \forall x \in A$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 1$

$\Rightarrow$ (per il teorema dei 2 carabinieri)

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{\operatorname{sen} x}{x} = 1$$

Esempio $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

$$\cos x = \cos 2\cdot\frac{x}{2} = \cos^2\frac{x}{2} - \operatorname{sen}^2\frac{x}{2}$$

$$\frac{1-\cos x}{x^2} = \frac{1-\cos^2\frac{x}{2}+\operatorname{sen}^2\frac{x}{2}}{x^2} =$$

$$= \frac{2\operatorname{sen}^2\frac{x}{2}}{x^2} = \frac{\operatorname{sen}^2\frac{x}{2}}{\frac{x^2}{2}} = \frac{1}{2}\,\frac{\operatorname{sen}^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}$$

$$= \frac{1}{2}\left(\frac{\operatorname{sen}\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}\right)^2$$

$$\lim_{x\to 0}\frac{1}{2}\left(\frac{\operatorname{sen}\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}\right)^2 = \frac{1}{2}\left[\lim_{x\to 0}\frac{\operatorname{sen}\frac{x}{2}}{\frac{x}{2}}\right]^2$$

$$= \frac{1}{2}\left[\lim_{y\to 0}\frac{\operatorname{sen} y}{y}\right]^2 = \frac{1}{2}\cdot 1^2 = \frac{1}{2}$$

Cambio variabili nei limiti

Esempio (MOLTO IMPORTANTE)

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

dim

$$1-\cos x = 1 - \cos 2 \cdot \frac{x}{2} = 1 - \cos^2 \frac{x}{2} + \operatorname{sen}^2 \frac{x}{2}$$

$$= 2 \operatorname{sen}^2 \frac{x}{2}$$

$$\frac{1-\cos x}{x^2} = \frac{2 \operatorname{sen}^2 \frac{x}{2}}{x^2} = \frac{1}{2} \frac{4 \operatorname{sen}^2 \frac{x}{2}}{x^2} = \frac{1}{2} \frac{\operatorname{sen}^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2} \xrightarrow[x \to 0]{} \frac{1}{2}$$

poichè $\displaystyle \lim_{x \to 0} \frac{\operatorname{sen}^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2} = \left[\lim_{x \to 0} \frac{\operatorname{sen} \frac{x}{2}}{\frac{x}{2}}\right]^2 = 1^2 = 1$ ✓