Lezione nro 33

Titolo nota — 04/12/2011

Si parla di integrale definito (proprio) di una funzione f quando

$f : [a,b] \to \mathbb{R}$ limitata.

$[a,b]$ insieme limitato.

Quando viene a mancare una di queste proprietà, si parla di integrali impropri

Esempio

$$\boxed{\int_0^1 \frac{1}{x^\alpha}\,dx = \begin{cases} +\infty & \alpha \geq 1 \\ \dfrac{1}{1-\alpha} & \alpha < 1 \end{cases}}$$

$\boxed{\alpha = 1}$

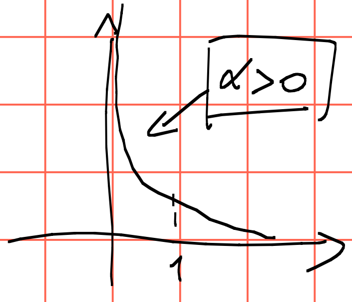
$$\lim_{\beta \to 0^+} \boxed{\int_\beta^1 \frac{dx}{x}} = \lim_{\beta \to 0^+} \Big[\log x\Big]_{x=\beta}^{x=1} = \lim_{\beta \to 0^+} \Big[-\log \beta\Big] = +\infty$$

$\alpha \neq 1$

$$\lim_{\beta \to 0^+} \int_\beta^1 \frac{dx}{x^\alpha} = \lim_{\beta \to 0^+} \int_\beta^1 x^{-\alpha}\,dx = \lim_{\beta \to 0} \left[\frac{x^{1-\alpha}}{1-\alpha}\right]_{x=\beta}^{x=1}.$$

$$= \lim_{\beta \to 0} \left[\frac{1}{1-\alpha} - \frac{\beta^{1-\alpha}}{1-\alpha}\right] = \begin{cases} \dfrac{1}{1-\alpha} & \alpha < 1 \\ +\infty & \alpha > 1 \end{cases}$$

$$\int_0^1 \frac{1}{x^\alpha}\,dx \begin{cases} \dfrac{1}{1-\alpha} & \alpha < 1 \\ +\infty & \alpha \geq 1 \end{cases}$$

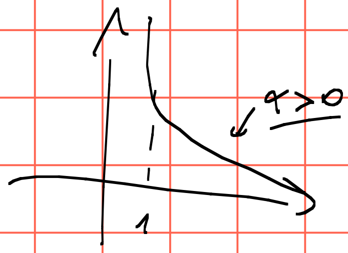


$[0,1]$ con un asintoto verticale

$[1,+\infty]$ " " " orizzontale

Esempio

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \begin{cases} +\infty & \alpha \leq 1 \\ \frac{1}{\alpha-1} & \alpha > 1 \end{cases}$$



dim

$\alpha = 1$ $$\lim_{\beta \to +\infty} \int_1^{\beta} \frac{1}{x}\,dx = \lim_{\beta \to +\infty} \Big[\log x\Big]_{x=1}^{x=\beta} = \lim_{\beta \to +\infty} \log\beta = +\infty$$

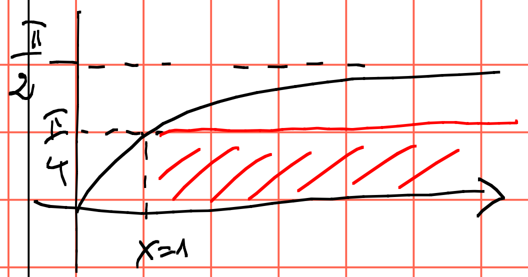
$\alpha \neq 1$ $$\lim_{\beta \to +\infty} \int_1^{\beta} \frac{1}{x^\alpha}\,dx = \lim_{\beta \to +\infty} \int_1^{\beta} x^{-\alpha}\,dx = \lim_{\beta \to +\infty} \left[\frac{x^{1-\alpha}}{1-\alpha}\right]_1^{\beta}$$

$$= \lim_{\beta \to +\infty} \left[\frac{\beta^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha}\right] = \begin{cases} +\infty & 1-\alpha > 0 \\ -\frac{1}{1-\alpha} & 1-\alpha < 0 \end{cases}$$

$$= \begin{cases} +\infty & \alpha < 1 \\ \frac{1}{\alpha-1} & 1 < \alpha \end{cases}$$

dunque $\displaystyle\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \begin{cases} \frac{1}{\alpha-1} & \alpha > 1 \\ +\infty & \alpha \le 1 \end{cases}$

<u>Esempio</u> $\displaystyle\int_0^{+\infty} \operatorname{arctg} x\,dx = +\infty$



$f = \operatorname{arctg} x$

$f \nearrow$ strett. crescente

$\left(f' = \frac{1}{1+x^2} > 0 \ \forall x\right)$

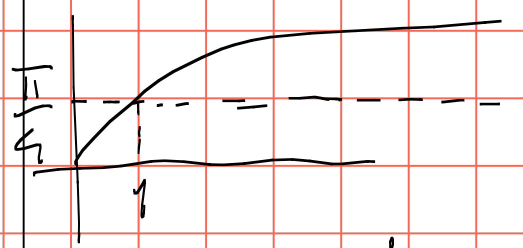
$f'' = \frac{-2x}{(1+x^2)^2} \begin{cases} <0 & x>0 \\ >0 & x<0 \end{cases}$

$\displaystyle\lim_{x\to+\infty} \operatorname{arctg} x \overset{x=\operatorname{Tg} y}{=}$

$\displaystyle= \lim_{y\to\frac{\pi}{2}} \operatorname{arctg}(\operatorname{Tg} y) = \lim_{y\to\frac{\pi}{2}} y = \frac{\pi}{2}$

dunque $f(x)$ è concava per $x>0$
convessa per $x<0$.

$\boxed{f(x) \ge \frac{\pi}{4} \quad \forall x \ge 1}$ $\quad (f \nearrow !!)$



$\forall \beta > 1$ $\displaystyle\int_0^\beta \operatorname{arctg} x\,dx = \int_0^1 \operatorname{arctg} x\,dx + \int_1^\beta \operatorname{arctg} x\,dx$

$\displaystyle\ge \int_0^1 \operatorname{arctg} x\,dx + \int_1^\beta \frac{\pi}{4}\,dx$ (Teo confronto)

$$\geq \underbrace{\left[\int_0^1 \operatorname{arctg} x \, dx\right]}_{\in \mathbb{R}} + \frac{\pi}{4}(\beta - 1)$$

$$\int_0^\beta \operatorname{arctg} x \, dx \geq \underbrace{\left[\int_0^1 \operatorname{arctg} x \, dx\right] + \frac{\pi}{4}(\beta-1)}_{\substack{\downarrow \beta \to +\infty \\ +\infty}} \qquad \forall \beta > 1$$

$$\lim_{\beta \to +\infty} \int_0^\beta \operatorname{arctg} x \, dx = +\infty$$

(Teorema del confronto tra limiti)

Oss Se $f(x)$ ha un asintoto orizzontale $y = k$

per $x \to +\infty$ e $k \neq 0$

allora $\int_a^{+\infty} f(x)\,dx$ diverge a $\pm\infty$

(a seconda che $k > 0$ o < 0) !

Oss Questo esempio suggerisce un Teorema

Se $\exists \lim_{x\to+\infty} f(x) = l \neq 0$ allora $\int_a^{+\infty} f(x)dx \notin \mathbb{R}$

(equivale alla condizione necessaria per le serie numeriche)

Def Sia data $f: [a,b[\to \mathbb{R}$ R-integrabile su $[a,\beta]$ $\forall \beta \in [a,b[$

b è il punto da studiare

"f è integrabile in senso improprio su $[a,b[$"

se $\boxed{\exists \lim_{\beta\to b^-} \int_a^\beta f(x)dx \in \mathbb{R}}$

- Integrale improprio divergente a $+\infty$ $(-\infty)$ se $\lim_{\beta\to b} \int_a^\beta f = +\infty$ $(-\infty)$
- " " non converge se $\nexists \lim_{\beta\to b} \int_a^\beta f$

Oss

Serie $\sum_n a_n \longleftrightarrow \int_a^b f(x)dx$ $f: [a,b[\to \mathbb{R}$

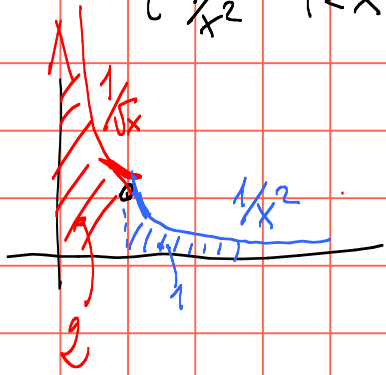
$S_n = \sum_{k=0}^n a_k$ $\forall n \in \mathbb{N}$ $\longleftrightarrow$ $\int_a^\beta f(x)dx$ $\beta \in [a,b[$

$\lim_{n\to+\infty} S_n \longleftrightarrow \lim_{\beta\to b} \int_a^\beta f(x)dx$

Problema $\exists\, f : [0,+\infty[$ continua, illimitata e t.c. $\displaystyle\int_0^{+\infty} f(x)\,dx \in \mathbb{R}$?

$$f(x) = \begin{cases} 1/\sqrt{x} & 0 < x \leq 1 \\ 1/x^2 & 1 < x \end{cases}$$

soddisfa i requisiti

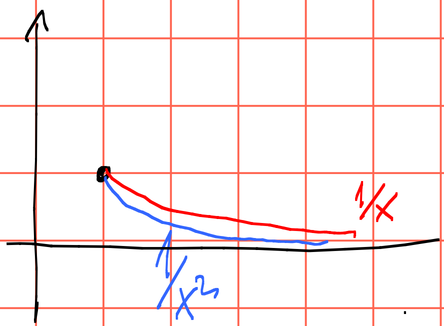


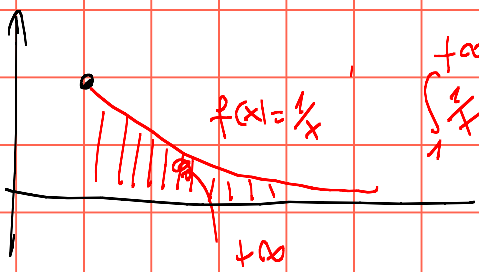
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \frac{1}{1-\frac{1}{2}} = 2$$

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \frac{1}{2-1} = 1$$

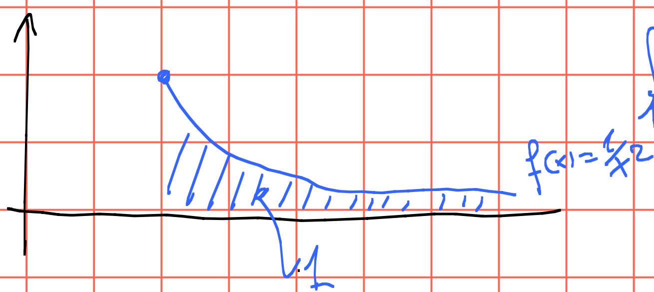
$$\int_0^{+\infty} f(x)\,dx = 1+2 = 3$$

Oss





$$\int_1^{+\infty} \frac{1}{x} = +\infty$$



$$\int_1^{+\infty} f(x)\,dx = \int_1^{+\infty} \frac{dx}{x^2} = 1$$

Oss $f(x)=\cos x$ è integrabile su $[a,b]$

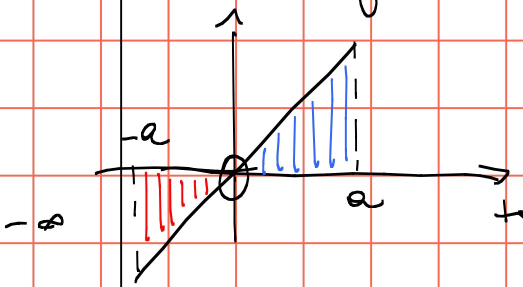
$\forall b \in [a,+\infty)$

però $\displaystyle\int_0^{+\infty} \cos x\,dx \;\not\exists$

infatti $\displaystyle\lim_{\beta\to+\infty}\int_0^{\beta} \cos x\,dx = \lim_{\beta\to+\infty}\big[\sin x\big]_0^{\beta} =$

$\displaystyle = \lim_{\beta\to+\infty} \sin\beta \;\not\exists$!!!

Problema $f(x)=x$ è integrabile su $\mathbb{R}$?



$$\int_{-a}^{a} x\,dx = \int_{-a}^{0} x\,dx + \int_{0}^{a} x\,dx$$

$$= -\int_{0}^{a} x\,dx + \int_{0}^{a} x\,dx$$

$$= 0$$

??? $\displaystyle\lim_{a\to+\infty}\int_{-a}^{a} x\,dx = 0 = \int_{-\infty}^{+\infty} x\,dx$

$$\int_{a}^{b} x\,dx = \int_{a}^{0} x\,dx + \int_{0}^{b} x\,dx$$

$$\int_{-\infty}^{+\infty} x\,dx = \lim_{a\to-\infty}\int_{a}^{0} x\,dx + \lim_{b\to+\infty}\int_{0}^{b} x\,dx = -\infty + \infty$$

è una forma indeterminata!!

$b=2a$

$$\lim_{a\to+\infty}\int_{-a}^{2a} x\,dx = \lim_{a\to+\infty}\int_{-a}^{0} x\,dx + \lim_{a\to+\infty}\int_{0}^{2a} x\,dx$$

$$= \lim_{a\to+\infty}\int_{a}^{2a} x\,dx = \lim_{a\to+\infty}\left[\frac{x^2}{2}\right]_{x=a}^{x=2a}$$

$$= +\infty$$

$$\lim_{a\to+\infty}\int_{-2a}^{a} x\,dx = \lim_{a\to+\infty}\int_{-2a}^{0} x\,dx + \lim_{a\to+\infty}\int_{0}^{a} x\,dx$$

$$= \lim_{a\to+\infty}\left[\frac{x^2}{2}\right]_{x=-2a}^{x=0} + \lim_{a\to+\infty}\left[\frac{x^2}{2}\right]_{0}^{a}$$

$$= \lim_{a\to+\infty}\left[-\frac{4a^2}{2}\right] + \lim_{a\to+\infty}\frac{a^2}{2}$$

$$= \lim_{a\to+\infty} -\frac{3}{2}a^2 = -\infty$$

Oss $\int_{-\infty}^{+\infty} x\,dx$ è una forma indeterminata, ovvero tende a $\forall$ numero reale

Esercizio Determinare la relazione tra a e b in modo che $\lim_{a\to-\infty}\int_{a}^{0} x\,dx + \lim_{b\to+\infty}\int_{0}^{b} x\,dx = 4$

dim $\lim_{a\to+\infty} -\frac{a^2}{2} + \lim_{b\to+\infty}\frac{b^2}{2} = 4$

ovvero $\frac{b^2}{2} - \frac{a^2}{2} = 4$ $\quad b^2-a^2 = 8$ $\quad b^2 = 8+a^2$

$$\boxed{b = \sqrt{a^2+8}}$$

gli integrali "parziali" devono $\exists$ finiti

Teorema

$f : [a, b[\to \mathbb{R}$, R-integrabile su $[a, \beta]$ $\forall \beta \in [a, \beta[$

$\boxed{f \geq 0}$ (≤ 0)

allora $\int_a^b f(x)\,dx$ $\boxed{\text{esiste finito}}$ oppure $\boxed{\text{diverge a } +\infty \ (-\infty)}$

dim

Ricordare le serie a termini positivi

per dimostrare il Teorema è sufficiente provare

che la funzione integrale

$$F(\beta) = \int_a^\beta f(x)\,dx \quad \text{è crescente}$$

ovvero

$$\forall x, y \in [a, b[\quad a < x < y \Rightarrow F(x) \leq F(y)$$

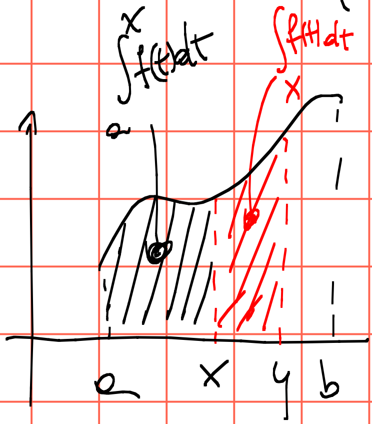
$$F(y) = \int_a^y f(t)\,dt \underset{\text{Teorema spezzamento}}{=} \int_a^x f(t)\,dt + \underbrace{\boxed{\int_x^y f(t)\,dt}}_{\geq 0}$$

$f(t) \geq 0$ per ipotesi
e $y > x$
$\Downarrow$ Teorema confronto
$\int_x^y f(t)\,dt \geq 0$

$$\geq \int_a^x f(t)\,dt = F(x)$$

Allora, essendo $F(\beta)$ crescente, esiste

$$\lim_{\beta \to b} F(\beta) = \sup_{\beta \in [a,b[} F(\beta) \begin{cases} +\infty \\ \ell \in \mathbb{R} \end{cases}$$



<u>Oss</u> Se f continua su $[a,b[$

allora f è u.c. su $[a,\beta]$ $\forall \beta \in [a,b[$

allora f è R-Integrabile su $[a,\beta]$ $\forall \beta \in [a,b[$

<u>Teorema</u>

int. illimitato f continua

$$\left.\begin{array}{l} f:[a,+\infty[\longrightarrow \mathbb{R} \text{ continua} \\ \bullet)\ f(x) \geq 0 \quad \forall x \geq a \quad \text{f positiva} \\ \bullet)\ \int_a^{+\infty} f(x)\,dx \in \mathbb{R} \quad \text{integrale finito} \\ \bullet)\ \exists \lim_{x \to +\infty} f(x) = l \quad \exists \text{ asintoto orizzontale} \end{array}\right\} \Longrightarrow \boxed{l = 0}$$

dim

Per assurdo $l > 0$, $l \in \mathbb{R}$

essendo $y = l$ asintoto orizzontale

$$\forall \varepsilon > 0 \quad \exists M > 0 : x > M \Rightarrow l - \varepsilon < f(x) < l + \varepsilon$$

$$\Rightarrow \varepsilon = \frac{l}{2} \quad \exists \overline{M} : x > \overline{M} \Rightarrow \frac{l}{2} < f(x)$$

$$\Rightarrow \quad \forall \beta > \overline{M} \quad \int_{\overline{M}}^{\beta} f(t)\,dt \geq \int_{\overline{M}}^{\beta} \frac{l}{2}\,dt = \frac{l}{2}(\beta - \overline{M})$$

Applico il Teorema Confronto Integ. a $f(t)$ e $\frac{l}{2}$ sull'intervallo $[\overline{M}, \beta]$

$\Rightarrow$ per il Teorema del confronto tra limiti

essendo $\lim_{\beta \to +\infty} \frac{l}{2}(\beta - \overline{M}) = +\infty$

dovrà essere $\lim_{\beta \to +\infty} \int_{\overline{M}}^{\beta} f(t)\,dt = +\infty$

$$\Rightarrow \int_{a}^{+\infty} f(t)\,dt = \underbrace{\int_{a}^{\overline{M}} f(t)\,dt}_{\in \mathbb{R}} + \int_{\overline{M}}^{+\infty} f(t)\,dt = +\infty$$

dunque $\int_{a}^{+\infty} f(t)\,dt = +\infty$ Assurdo.

(x ipotesi $\int_{a}^{+\infty} f\,dx < +\infty$) $\Rightarrow l = 0$ ↯

Teorema (1° criterio integrabilità)

Se $f : [a,b[\to \mathbb{R}$ $\quad$ ($f :]a,b] \to \mathbb{R}$)

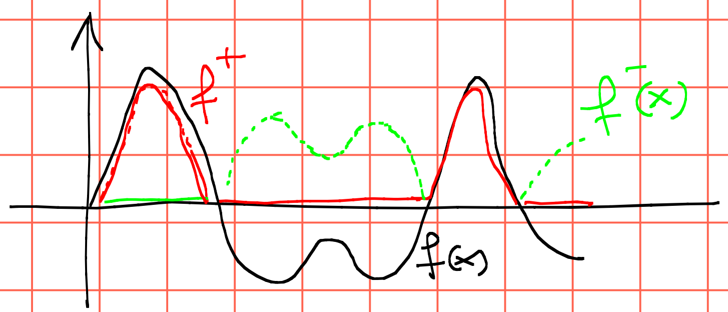
- f continua su $[a,b[$ ← mi serve per avere $\int_a^\beta f \in \mathbb{R}$
- $\boxed{|f(x)| \le \varphi(x) \ \forall x \in [a,b[}$ ← ipotesi di dominazione
- $\boxed{\varphi \text{ integrabile impropriamente}}$ su $[a,b[$ ←

allora

f integrabile impropriamente su $[a,b[$

dim

$|f(x)| = f^+(x) + f^-(x)$, $\quad f^+(x) = \max\{f(x), 0\}$ $\quad f^-(x) = \max\{-f, 0\}$



- $f(x) = f^+(x) - f^-(x)$

$f(x)$ è continua $\Rightarrow$ f^+ ed f^- sono continue

$\Rightarrow |f|$ è continua

$\Rightarrow \int_a^\beta f^+ \quad \int_a^\beta f^- \quad \int_a^\beta |f| \quad \int_a^\beta f$

esistono finiti $\forall \beta \in [a,b[$

Per ipotesi

$$|f| \leq \varphi(x) \implies \begin{cases} f^+(x) \leq \varphi(x) \\ f^-(x) \leq \varphi(x) \end{cases}$$

① $f^+ \geq 0 \implies \int_a^b f^+ dx$ esiste $\begin{cases} \text{finito} \\ +\infty \end{cases}$

Ma $\int_a^b \varphi dx < +\infty$, $f^+ \leq \varphi \ \forall x \in [a,b[$ $\implies \int_a^\beta f^+ dx < \int_a^\beta \varphi dx \quad \forall \beta \in [a,b[$

$$\implies \int_a^b f^+ dx < \int_a^b \varphi dx < +\infty \implies \int_a^b f^+ dx < +\infty$$

① $f^- \geq 0 \implies \int_a^b f^- dx$ esiste $\begin{cases} \text{finito} \\ +\infty \end{cases}$

Ma $\int_a^b \varphi dx < +\infty$, $f^- \leq \varphi \ \forall x \in [a,b[$ $\implies \int_a^\beta f^- dx < \int_a^\beta \varphi dx \quad \forall \beta \in [a,b[$

$$\implies \int_a^b f^- dx < \int_a^b \varphi dx < +\infty \implies \int_a^b f^- dx < +\infty$$

Ne segue che $\int_a^b f\, dx = \int_a^b (f^+ - f^-)(x)\, dx$

$$= \underset{\substack{\uparrow \\ +\infty}}{\int_a^b f^+ dx} - \underset{\substack{\uparrow \\ +\infty}}{\int_a^b f^- dx}$$

$$\implies \int_a^b f\, dx < +\infty \quad !!$$