Lezione n.ro 30 del 5/12/2011

Titolo nota 02/12/2011

$$\int \frac{R(x)}{P(x)}\,dx$$

$P(x)$ e $R(x)$ sono polinomi

Di queste funzioni $f(x) = \frac{R(x)}{P(x)}$ esiste

SEMPRE

una primitiva esprimibile in termini

di f.ni elementari

(1) Data $\frac{P(x)}{Q(x)}$ con grado $(P(x)) >$ grado $(Q(x))$

allora esistono $D(x)$ e $R(x)$, con grado$(R)<$grado(Q)

tali che

$$\frac{P(x)}{Q(x)} = D(x) + \frac{R(x)}{Q(x)} \qquad \left(\begin{array}{c}\text{dove}\\ P(x) = D(x)\cdot Q(x) + R(x)\end{array}\right)$$

Esempio

$$\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$$

infatti

$$\begin{array}{r|l} x^3 & x^2+1 \\ \hline x^3+x & x \\ \hline -x & \end{array}$$

ovvero $D(x) = x$

$R(x) = -x$

e $x^3 = x\cdot(x^2+1) - x$

Da ora in avanti in $\frac{R(x)}{Q(x)}$

si avrà sempre grado$(R(x)) <$ grado$(Q(x))$

Radici Reali Semplici

① $Q(x) = (x-a_1)(x-a_2) \cdots \cdot (x-a_m)$

$Q(x)=0$ ha solo radici reali semplici

Def $x_0 = a$ è "radice semplice" (di molteplicità 1)

dell'equazione $Q(x)=0$

se $Q(a)=0$ e posto $Q(x)=(x-a)\,Q_1(x)$
(Tale fattorizzazione esiste!)

$Q_1(x) = 0$

Oss a radice semplice di $Q(x)=0$ se $\begin{cases} Q(a)=0 \\ Q'(a)\neq 0 \end{cases}$

Esempio $Q(x)=(x^2-1) = (x-1)(x+1)$

$x=\pm 1$ sono 2 radici semplici

Esempio $Q(x) = x^2\cdot(x-1)$

$x_0 = 0$ radice doppia

$x_1 = 1$ " semplice

Teorema 1 $\dfrac{R(x)}{Q(x)}$ con $\text{grado}(R) < \text{grado}(Q)$

Supponiamo $Q(x)=\prod_{i=1}^{m}(x-a_i)$ ovvero $Q(x)=0$

ha solo radici reali semplici. Allora $\exists\, A_i \cdot_{i=1,\dots,m}$

t.c. $$\frac{R(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{(x-a_2)} + \frac{A_3}{(x-a_3)} + \dots + \frac{A_m}{(x-a_m)}$$

(no dim)

Esempio $\frac{1}{x^2-1} = \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1}$

Per il Teorema precedente $\exists\, A_1\, A_2 \in \mathbb{R}$ t.c.

$$\frac{1}{x^2-1} = \frac{A_1}{x-1} + \frac{A_2}{x+1} = \frac{A_1(x+1)+A_2(x-1)}{(x-1)(x+1)}$$

identità tra polinomi

$$1 = (A_1+A_2)x + A_1 - A_2 \qquad \forall x \neq \pm 1$$

$$\begin{cases} A_1 \overset{\text{se}}{-} A_2 = 1 \\ A_1 + A_2 = 0 \end{cases} \qquad \begin{cases} -2A_2 = 1 \\ A_1 = -A_2 \end{cases} \qquad \begin{cases} A_2 = -\frac{1}{2} \\ A_1 = \frac{1}{2} \end{cases}$$

da cui

$$\frac{1}{x^2-1} = \frac{1}{2}\cdot\frac{1}{x+1} - \frac{1}{2}\,\frac{1}{x-1}$$

Se si deve calcolare $\int \frac{dx}{x^2-1}$ si procede così

$$\int \frac{dx}{x^2-1} = \int \frac{1}{2(x+1)}\,dx - \int \frac{1}{2}\cdot\frac{dx}{x-1}$$

$$= \frac{1}{2}\log|x+1| - \frac{1}{2}\log|x-1| + c$$

$$= \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| + c \qquad c \in \mathbb{R}$$

② Radici Reali Coincidenti

In questo caso $Q(x) = (x-a)^m$ (1 radice a molteplicità m)

Teorema

$Q(x) = (x-a)^m$ e grado$(R) < m =$ grado(Q)

Allora $\exists\, A_1, A_2 \dots A_m \in \mathbb{R}$ t.c.

$$\frac{R(x)}{Q(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \dots + \frac{A_m}{(x-a)^m}$$

Oss
$$\int \frac{A_i}{(x-a)^i} = \begin{cases} A_1 \log|x-a| + c & i=1 \\ A_i \dfrac{(x-a)^{1-i}}{1-i} + c & i = 2,3,\dots,m \end{cases}$$

Esempio

Calcolare $\displaystyle\int \frac{x-1}{x^3}\,dx$

dim
$$\frac{x-1}{x^3} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} = \frac{A_1 x^2 + A_2 x + A_3}{x^3}$$

$\Rightarrow$

$$A_1 x^2 + A_2 x + A_3 = x - 1$$

$\Rightarrow$

$$\begin{cases} A_1 = 0 \\ A_2 = 1 \\ A_3 = -1 \end{cases} \quad \Rightarrow \quad \frac{x-1}{x^3} = \frac{1}{x^2} - \frac{1}{x^3}$$

e quindi $\displaystyle\int \frac{x-1}{x^3}\,dx = \int \left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx = -\frac{1}{x} + \frac{1}{2x^2} + c$

fare verifica!

3° Radici complesse (e coniugate) semplici

<u>Ricordo</u> $Q(x)$ polinomio <u>coeff. reali</u> $Q(z)=0$ sse $Q(\bar{z})=0$
(se esiste una radice complessa, esiste pure la complessa coniugata)

$$z=a+ib \quad \bar{z}=a-ib \Rightarrow (x-z)(x-\bar{z})=x^2-2ax+(a^2+b^2)$$

Dunque $Q(x)$ si presenta come prodotto di trinomi di secondo grado, ovvero

$$Q(x)=(x^2+a_1x+b_1)(x^2+a_2x+b_2)\cdots(x^2+a_mx+b_m)$$

<u>Teorema</u> (no dim) Se $Q(x)=\prod_{i=1}^{m}(x^2+a_ix+b_i)$ $\quad a_i, b_i \in \mathbb{R}$

e $\text{grado}(R) < 2m = \text{grado}(Q)$

allora $\exists\, A_i, B_i \in \mathbb{R}$ $i=1,\dots,m$ t.c.

$$\frac{R(x)}{Q(x)}=\frac{A_1x+B_1}{x^2+a_1x+b_1}+\frac{A_2x+B_2}{x^2+a_2x+b_2}+\dots+\frac{A_mx+B_m}{x^2+a_mx+b_m}$$

<u>Oss</u> Ciascuno dei termini $\frac{A_ix+B_i}{x^2+a_ix+b_i}$ $\quad i=1\dots m$

è integrabile

<u>Esempio</u> $\int \frac{x+1}{x^2+1}\,dx = \frac{1}{2}\log|x^2+1|+\operatorname{arctg} x + c \quad c\in\mathbb{R}$

$$\frac{x+1}{x^2+1}=\frac{x}{x^2+1}+\frac{1}{x^2+1}=\frac{1}{2}\,\frac{2x}{x^2+1}+\frac{1}{x^2+1} \quad \text{e si ha}$$

$$\int \frac{x+1}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{dx}{x^2+1} = \frac{1}{2}\log|x^2+1| + \operatorname{arctg} x + c$$

fare verifica! $c \in \mathbb{R}$

Esempio

$$\int \frac{x}{x^2+x+1}\,dx = \frac{1}{2}\log|x^2+x+1| - \frac{1}{\sqrt{3}}\operatorname{arctg}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) + c$$

$c \in \mathbb{R}$

dim

$$\frac{x}{x^2+x+1} = \frac{1}{2}\,\frac{2x}{x^2+x+1} = \frac{1}{2}\,\frac{2x+1}{x^2+x+1} - \frac{1}{2}\,\frac{1}{x^2+x+1}$$

$$= \frac{1}{2}\,\frac{(x^2+x+1)'}{x^2+x+1} - \frac{1}{2}\,\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{1}{2}\,\frac{(x^2+x+1)'}{x^2+x+1} - \frac{1}{\cancel{2}}\cdot\frac{\cancel{4}^{2}}{3}\cdot\frac{1}{1+\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2}$$

$$= \frac{1}{2}\,\frac{(x^2+x+1)'}{x^2+x+1} - \frac{2}{3}\,\frac{1}{1+\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2}$$

$$\int \frac{x}{x^2+x+1}\,dx = \frac{1}{2}\int \frac{(x^2+x+1)'}{x^2+x+1}\,dx - \frac{2}{3}\int \frac{1}{1+\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2}\,dx$$

$y = \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) \quad dy = \frac{2}{\sqrt{3}}\,dx$

$$= \frac{1}{2}\log|x^2+x+1| - \frac{\cancel{2}}{3}\cdot\frac{\sqrt{3}}{\cancel{2}}\left(\int \frac{dy}{1+y^2}\right)_{y=\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)}$$

$$= \frac{1}{2}\log|x^2+x+1| - \frac{1}{\sqrt{3}}\operatorname{arctg}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) + c$$

$c \in \mathbb{R}$

fare verifica!

4° Radici complesse (e coniugate) multiple

In tal caso $Q(x) = (x^2+ax+b)^m$

Teorema (no dim)

Se $Q(x) = (x^2+ax+b)^m \quad a,b \in \mathbb{R}$

e $\text{grado}(P) < 2m = \text{grado}(Q)$

Allora $\exists\, A_i, B_i \quad i=1 \dots m$ t.c.

$$\frac{P(x)}{Q(x)} = \frac{A_1x+B_1}{x^2+ax+b} + \frac{A_2x+B_2}{(x^2+ax+b)^2} + \frac{A_3x+B_3}{(x^2+ax+b)^3} + \dots + \frac{A_mx+B_m}{(x^2+ax+b)^m}$$

Esempio $\displaystyle\int \frac{dx}{(x^2+1)^2} = \frac{1}{2}\left(\operatorname{arctg} x + \frac{x}{x^2+1}\right) + c \quad c \in \mathbb{R}$

dim

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{x^2+1-x^2}{(x^2+1)^2}\,dx = \int \frac{dx}{1+x^2} - \int \frac{x^2}{(1+x^2)^2}\,dx$$

$$= \operatorname{arctg} x - \frac{1}{2}\int \underbrace{\frac{2x}{(1+x^2)^2}}_{u'} \cdot \underbrace{x}_{v}\,dx$$

$$= \operatorname{arctg} x - \frac{1}{2}\left[-\underbrace{\frac{1}{(1+x^2)}}_{u} \cdot \underbrace{x}_{v} - \int \underbrace{\left(-\frac{1}{1+x^2}\right)}_{u} \cdot \underbrace{1}_{v'}\,dx\right]$$

$$= \operatorname{arctg} x + \frac{1}{2}\frac{x}{x^2+1} - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{1}{2}\operatorname{arctg} x + \frac{1}{2}\frac{x}{x^2+1} + c \quad c \in \mathbb{R}$$

__Esempio__ $\displaystyle\int \frac{x^2}{(x^2+1)^2}\,dx = \frac{1}{2}\left(\operatorname{arctg} x + \frac{x}{x^2+1}\right) + c$

$c \in \mathbb{R}$

dim

$$\frac{x^2}{(x^2+1)^2} = \frac{A_1 x + B_1}{x^2+1} + \frac{A_2 x + B_2}{(x^2+1)^2} = \frac{(A_1 x + B_1)(x^2+1) + A_2 x + B_2}{(x^2+1)^2}$$

ne

$$x^2 = A_1 x^3 + A_1 x + B_1 x^2 + B_1 + A_2 x + B_2$$
$$= A_1 x^3 + B_1 x^2 + (A_1 + A_2) x + B_1 + B_2$$

ne

$$\begin{cases} A_1 = 0 \\ B_1 = 1 \\ A_1 + A_2 = 0 \\ B_1 + B_2 = 0 \end{cases} \qquad \begin{cases} A_1 = 0 \\ B_1 = 1 \\ A_2 = 0 \\ B_2 = -1 \end{cases}$$

e dunque $\displaystyle\frac{x^2}{(x^2+1)^2} = \frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}$

da cui segue

$$\int \frac{x^2}{(x^2+1)^2}\,dx = \int \frac{dx}{x^2+1} - \int \frac{dx}{(1+x^2)^2}$$

$$= \operatorname{arctg} x - \frac{1}{2}\left(\operatorname{arctg} x + \frac{x}{x^2+1}\right) + c$$

$$= \frac{1}{2}\left(\operatorname{arctg} x - \frac{x}{x^2+1}\right) + c$$

$c \in \mathbb{R}$

Funzioni Iperboliche

$$\operatorname{senh} x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$(\operatorname{senh} x)' = \cosh x \qquad (\cosh x)' = \operatorname{senh} x$$

$$\operatorname{senh}^2 = \frac{e^{2x} + e^{-2x} - 2}{4} \qquad (\cosh x)^2 = \frac{e^{2x} + e^{-2x} + 2}{4}$$

$$\boxed{\cosh^2 x - \operatorname{senh}^2 x = 1}$$

$y = \operatorname{senh} x$ sse $y = \dfrac{e^x - e^{-x}}{2}$

sse $2ye^x = e^{2x} - 1$

sse $e^{2x} - 2ye^x - 1 = 0$

sse $e^x = y \pm \sqrt{y^2+1}$ ma $e^x > 0$

sse $e^x = y + \sqrt{y^2+1}$

Esercizio

Calcolare $\int \sqrt{t^2+1}\, dt = I$

Si usi $t = \operatorname{senh} x \Rightarrow 1 + t^2 = 1 + \operatorname{senh}^2 x$
$dt = \cosh x \qquad = \cosh^2 x$

$$\left(\int \sqrt{1+t^2}\, dt\right) \underset{t = \operatorname{senh} x}{=} \left(\int \cosh^2 x \, dx\right)_{t = \operatorname{senh} x}$$

$= \ldots\ldots$

(si osservi che $\cosh x > 0 \ \forall x$
e $\operatorname{senh} x \nearrow$)

Esercizio Calcolare

$$\int \frac{dx}{(x^2+1)^2}$$

$$\int \frac{1}{(x^2+1)^2}\,dx = \int \frac{1+x^2}{(x^2+1)^2}\,dx - \int \frac{x^2}{(1+x^2)^2}\,dx$$

$$= \operatorname{arctg} x - \left[\frac{1}{2}\int \frac{2x}{(1+x^2)^2}\cdot x\,dx\right]$$

$$= \operatorname{arctg} x - \left[\frac{1}{2}\,\frac{(1+x^2)^{-1}}{-1}\cdot x - \frac{1}{2}\int \left(\frac{(1+x^2)^{-1}}{-1}\right)\cdot 1\,dx\right]$$

$$= \operatorname{arctg} x + \frac{\cancel{x}}{2(x^2+1)} - \frac{1}{2}\int \frac{dx}{1+x^2}$$

$$= \frac{1}{2}\operatorname{arctg} x + \frac{x}{2(1+x^2)} + c \qquad c \in \mathbb{R}$$

Calcolare $\int \frac{dx}{(x^2+a)^3}$

Come primo $\int \frac{dx}{(x^2+a)^3} = \frac{1}{a}\int \frac{x^2+a}{(x^2+a)^3}\,dx - \frac{1}{a}\int \frac{x^2}{(x^2+a)^3}$

$$= \frac{1}{a}\left(\int \frac{dx}{(x^2+a)^2}\right) - \frac{1}{2a}\int \frac{2x}{(x^2+a)^3}\cdot x\,dx$$

Vedi esercizio precedente

etc

Esempio

$$\frac{1}{(x-1)x}=\frac{A}{x-1}+\frac{B}{x}=\frac{Ax+Bx-B}{(x-1)x} \quad \text{se} \begin{cases} A+B=0 \\ -B=1 \end{cases} \quad \text{se } A=-B=1$$

per cui $\displaystyle\int\frac{1}{x(x-1)}dx=\int\frac{dx}{x-1}-\int\frac{dx}{x}=\log\left|\frac{x-1}{x}\right|+c$

$c\in\mathbb{R}$

Esempio

$$\frac{x}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}=\frac{A(x+1)^2+B(x^2-1)+C(x-1)}{(x-1)(x+1)^2}$$

$$\text{se} \begin{cases} A+B=0 \\ 2A+C=1 \\ A-B-C=0 \end{cases} \begin{cases} A=-B \\ -2B+C=1 \\ 2B=-C \end{cases} \begin{cases} A=1/4 \\ B=-1/4 \\ C=1/2 \end{cases}$$

per cui

$$\int\frac{x}{(x-1)(x+1)^2}=\frac{1}{4}\int\frac{dx}{x-1}-\frac{1}{4}\int\frac{dx}{x+1}+\frac{1}{2}\int\frac{dx}{(x+1)^2}$$

$$=\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\cdot\frac{1}{(x+1)}+c$$

$c\in\mathbb{R}$

Esercizio Calcolare $\displaystyle\int\frac{dx}{x^3(1+x^2)}$

Esercizio Calcolare $\displaystyle\int\frac{dx}{x(x^2+1)^2}$

Esempio

$$\frac{1}{x^3(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+1}$$

$$= \frac{Ax^2(x^2+1) + Bx(x^2+1) + Cx^2+C + Dx^4+Ex^3}{x^3(x^2+1)}$$

$$= \frac{Ax^4+Ax^2+Bx^3+Bx+Cx^2+C+Dx^4+Ex^3}{x^3(x^2+1)}$$

$$= \frac{x^4(A+D) + x^3(B+E)+x^2(A+C)+Bx+C}{x^3(x^2+1)}$$

me $\begin{cases} A+D=0 \\ B+E=0 \\ A+C=0 \\ B=0 \\ C=1 \end{cases}$ $\quad$ $\begin{array}{l} D=1 \\ E=0 \\ A=-1 \\ B=0 \\ C=1 \end{array}$ $\quad$ me $-\frac{1}{x}+\frac{1}{x^3}+\frac{x}{x^2+1} = f$

Esempio $\quad \frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

$$= \frac{(x^2+1)^2 \cdot A + (x^3+x)(Bx+C) + Dx^2+Ex}{x(x^2+1)^2}$$

$$= \frac{Ax^4+2Ax^2+A+Bx^4+Cx^3+Bx^2+Cx+Dx^2+Ex}{x(x^2+1)^2}$$

$$= \frac{(A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A}{x(x^2+1)^2}$$

me $\begin{cases} A+B=0 \\ C=0 \\ 2A+B+D=0 \\ C+E=0 \\ A=1 \end{cases}$ $\quad$ $\begin{array}{l} B=-1 \\ C=0 \\ D=-1 \\ E=0 \\ A=1 \end{array}$ $\quad$ $f(x) = \frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}$

$$\int \sqrt{1+x^2}\,dx \underset{x=\operatorname{senh} y}{=} \left(\int \cosh y \cdot \cosh y \; dy \right)_{x=\operatorname{senh} y}$$

$$\frac{dx}{dy} = \cosh y$$

$$dx = \cosh y \cdot dy$$

$$(\cosh y)^2 = \left(\frac{e^y+e^{-y}}{2}\right)^2 = \frac{1}{4}\left(e^{2y}+e^{-2y}+2\right)$$

$$= \left(\int \frac{e^{2y}+e^{-2y}+2}{4}\, dy \right)_{y=\log(x+\sqrt{1+x^2})} =$$

$$= \left(\frac{e^{2y}}{8} - \frac{e^{-2y}}{8} + \frac{y}{2} + c \right)_{y=\log(x+\sqrt{1+x^2})}$$

$$= \frac{\left(x+\sqrt{1+x^2}\right)^2}{8} - \frac{1}{8\left(x+\sqrt{1+x^2}\right)^2} + \frac{1}{2}\log\left(x+\sqrt{1+x^2}\right) + c$$

$c \in \mathbb{R}$

$$= \frac{1}{2} x \sqrt{1+x^2} + \frac{1}{2}\log\left(x+\sqrt{1+x^2}\right) + c$$

$c \in \mathbb{R}$

ovvero $\displaystyle\int \sqrt{1+x^2}\,dx \underset{\substack{x=\operatorname{senh} y \\ dx=\cosh y\,dy}}{=} \int \cosh y \cosh y\,dy$

$\cosh^2 y - \operatorname{senh}^2 y = 1$

$$\int \cosh y \cosh y\,dy = \cosh y \operatorname{senh} y - \int \operatorname{senh}^2 y\,dy$$

$$= \operatorname{senh} y \cosh y + y - \int \cosh^2 y\,dy$$

da cui si deduce

$$\left(\int \cosh^2 y\,dy\right)_{y=\dots} = \left(\frac{y}{2} + \frac{1}{2}\operatorname{senh} y \cosh y + c\right)_{y=\dots} \quad c \in \mathbb{R}$$

$y = \operatorname{settsh} x$

$$= \frac{\operatorname{settsh} x}{2} + \frac{1}{2} x\sqrt{x^2+1} + c \quad c \in \mathbb{R}.$$

$\cosh^2 y - \operatorname{senh}^2 y = 1$

$\cosh y = \sqrt{1+\operatorname{senh}^2 y}$

$$= \frac{1}{2}\log\left(x+\sqrt{1+x^2}\right) + \frac{1}{2} x\sqrt{1+x^2} + c$$

$$\int \frac{dx}{1+x^4}$$

$1+x^4=0$ ha 4 radici $\quad \frac{1+i}{\sqrt{2}} \quad \frac{1-i}{\sqrt{2}} \quad \frac{-1-i}{\sqrt{2}} \quad \frac{-1+i}{\sqrt{2}}$

$$1+x^4 = (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$$

$$\frac{1}{1+x^4} = \frac{Ax+B}{x^2-\sqrt{2}x+1} + \frac{Cx+D}{x^2+\sqrt{2}x+1}$$

$$= \frac{Ax^3+A\sqrt{2}x^2+Ax+Bx^2+B\sqrt{2}x+B}{//} + \frac{Cx^3-C\sqrt{2}x^2+Cx+Dx^2-D\sqrt{2}x+D}{//}$$

$$\begin{cases} A+C=0 \\ B+D+A\sqrt{2}-C\sqrt{2}=0 \\ A+C+B\sqrt{2}-D\sqrt{2}=0 \\ B+D=1 \end{cases} \quad \begin{cases} A=-C \\ 2\sqrt{2}A=-1 \\ B=D=\frac{1}{2} \\ B+D=1 \end{cases} \quad \begin{cases} C=\frac{1}{2\sqrt{2}} \\ A=-\frac{1}{2\sqrt{2}} \\ B=D=\frac{1}{2} \end{cases}$$

$$\frac{-\frac{x}{2\sqrt{2}}+\frac{1}{2}}{x^2-\sqrt{2}x+1} + \frac{\frac{x}{2\sqrt{2}}+\frac{1}{2}}{x^2+\sqrt{2}x+1}$$

$$-\frac{1}{4\sqrt{2}}\,\frac{2x-2\sqrt{2}}{x^2-\sqrt{2}x+1} + \frac{1}{4\sqrt{2}}\,\frac{2x+2\sqrt{2}}{x^2+\sqrt{2}x+1}$$

$$= -\frac{1}{4\sqrt{2}}\underbrace{\frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}}_{\to \log|x^2-\sqrt{2}x+1|} + \frac{1}{4}\underbrace{\frac{1}{x^2-\sqrt{2}x+1}}_{\to \sqrt{2}\,\mathrm{arctg}(\sqrt{2}x-1)} +$$

$$+ \frac{1}{4\sqrt{2}}\underbrace{\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}}_{\to \log|x^2+\sqrt{2}x+1|} + \frac{1}{4}\underbrace{\frac{1}{x^2+\sqrt{2}x+1}}_{\to \sqrt{2}\,\mathrm{arctg}(\sqrt{2}x+1)} = \int$$

$$\int f\,dx = -\frac{1}{4\sqrt{2}}\log|x^2-\sqrt{2}x+1| + \frac{1}{4\sqrt{2}}\log|x^2+\sqrt{2}x+1|$$

$$+\frac{1}{2\sqrt{2}}\arctg(\sqrt{2}x+1) + \frac{1}{2\sqrt{2}}\arctg(\sqrt{2}x-1) + c$$

$$x^2-\sqrt{2}x+1 = \left(x^2-\sqrt{2}x+\frac{1}{2}\right)+\frac{1}{2} = \left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}$$

$$(x^2+\sqrt{2}x+1) = \left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}$$

$$\int\frac{1}{x^2+\sqrt{2}x+1} = 2\int\frac{1}{1+\left[\sqrt{2}\left(x+\frac{1}{\sqrt{2}}\right)\right]^2}$$

$$= 2\left(\int\frac{\frac{1}{\sqrt{2}}\,dy}{1+y^2}\right)_{y=\sqrt{2}\left(x+\frac{1}{\sqrt{2}}\right)} = \sqrt{2}\arctg(\sqrt{2}x+1)+c$$

$$\int\frac{1}{x^2-\sqrt{2}x+1}\,dx = \text{----} = \sqrt{2}\arctg(\sqrt{2}x-1)+c$$

Esempio

$$\int \frac{dx}{(x^2+1)^2\, x}$$

$$\frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$\frac{A(x^4+2x^2+1) + (Bx^2+Cx)(x^2+1) + Dx^2+Ex}{}$$

$$\parallel$$

$$\frac{(A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A}{}$$

$$\parallel$$

$$\begin{cases} A=1 \\ C+E=0 \\ 2+B+D=0 \\ C=0 \\ B=-1 \end{cases} \qquad \begin{cases} A=1 \\ E=0 \\ C=0 \\ B=-1 \\ D=-1 \end{cases}$$

$$f(x) = \frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}$$

$$\int f(x)\,dx = \log|x| - \frac{1}{2}\log(x^2+1) - \frac{1}{2}\,\frac{(x^2+1)^{-1}}{-1} + c$$

$$= \log|x| - \log(x^2+1)^{1/2} + \frac{1}{2(x^2+1)} + c \qquad c \in \mathbb{R}$$