Lezione nro 29

Titolo nota 30/11/2011

Esercizio

$f(x) = \operatorname{sen}(x^3)$

Calcolare $f^{(V)}(0)$ e $f^{(VI)}(0)$

1° metodo calcolo f' f'' f''' $f^{(IV)}$ $f^{(V)}$ $f^{(VI)}$

$f^{(V)} \downarrow$ in 0, $f^{(VI)} \downarrow$ in 0

2° metodo

$f(x) = h(g(x)) \qquad h(y) = \operatorname{sen} y \quad g(x) = x^3$

$h(y) = y - \frac{y^3}{3!} + o(y^4) \qquad y \to 0$

$f(x) = h(g(x)) = x^3 - \frac{x^9}{6} + o(x^{12})$

e dunque $f^{(V)}(0) = f^{(VI)}(0) = 0$ $\quad \left(f'''(0) = 3! \right.$

$f^{(IX)}(0) = -\frac{1}{6} \cdot 9! = 4\cdot5\cdot6\cdot7\cdot8\cdot9 \left. \right)$

In quanto

$x^3 - \frac{x^9}{6} = \sum_{k=0}^{12} \frac{f^{(k)}(0)}{k!} x^k \quad !!!$

Analogamente, dato $f(x) = e^{\operatorname{sen}^2 x}$

calcolare $f'''(0)$ e $f^{(IV)}(0)$

$$x \xmapsto{g} \operatorname{sen} x \xrightarrow{h} \operatorname{sen}^2 x \xrightarrow{\ell} e^{\operatorname{sen}^2 x}$$

$$\ell(z) = e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + o(z^3)$$

$$h(y) = y^2$$

$$g(x) = \operatorname{sen} x = x - \frac{x^3}{6} + o(x^3)$$

$$e^{\operatorname{sen}^2 x} = 1 + \left(x - \frac{x^3}{6} + o(x^3)\right)^2 + \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^3)\right)^4$$
$$+ \frac{1}{6}\left(x - \frac{x^3}{6} + o(x^3)\right)^6 + o\left(x^6\right)$$

$$= 1 + x^2 - \frac{x^4}{3} + \frac{1}{2}x^4 + o(x^4)$$

$$= 1 + x^2 + \frac{x^4}{6} + o(x^4)$$

$$= f(0) + f'(0)\cdot x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(IV)}(0)}{4!}x^4 + o(x^4)$$

e dunque
$$f'(0) = 0$$
$$f''(0) = 2$$
$$f'''(0) = 0$$
$$f^{IV}(0) = 4! \cdot \frac{1}{6} = 4$$

Formula Taylor Resto Lagrange

Per provare che $f(x) = P_n(x) + \frac{f^{(n+1)}(z)}{(n+1)!}(x-x_0)^{n+1}$

dove z compreso tra x_0 e x

Si osserva che, posto

$$\boxed{g(x) = f(x) - P_n(x)}$$

si ha $g(x_0) = g'(x_0) = g''(x_0) = \dots = g^{(n)}(x_0)$

$$\boxed{h(x) = (x-x_0)^{n+1}}$$

si ha $h(x_0) = h'(x_0) = \dots$ $\qquad x_0 < x_1 < x$

$\underline{x > x_0}$ $\quad \frac{g(x)}{h(x)} = \frac{g(x) - g(x_0)}{h(x) - h(x_0)} \overset{\text{Cauchy}}{=} \frac{g'(x_1)}{h'(x_1)} = \frac{f'(x_1) - P_n'(x_1)}{(n+1)(x_1-x_0)^n}$

Posto $\boxed{g(x) = f' - P_n'}$ $\boxed{h(x) = (n+1)(x-x_0)^n}$ $\quad \underline{x_0 < x_2 < x_1}$

$$\frac{g(x)}{h(x)} = \frac{g(x_1) - g(x_0)}{h(x_1) - h(x_0)} \overset{\text{Cauchy}}{=} \frac{g'(x_2)}{h'(x_2)} = \frac{f''(x_2) - P_n''(x_2)}{(n+1)n(x_2-x_0)^{n-1}}$$

Iterando n+1 volte

$$\frac{f(x) - P_n(x)}{(x-x_0)^{n+1}} = \frac{--}{--} = \dots = \boxed{\frac{f^{(n+1)}(z)}{(n+1)!}} \quad x_0 < z < \dots < x_1$$

ovvero la Tesi

$\cancel{x_0} < z < x_{n-1} < \dots < x_1$

INTEGRAZIONE PER PARTI

$$\left(f(x)g(x)\right)' = f'(x)\,g(x) + f(x)g'(x) \quad \forall x \in I$$

$$\Downarrow$$

$$f(x)g(x) + c = \int f'(x)\,g(x)\,dx + \int f(x)g'(x)\,dx \quad c \in \mathbb{R}$$

ovvero

$$\boxed{\int u' v\,dx = u v - \int u v'\,dx}$$

Calcolare $\int x e^x\,dx$

" $\int \sin^2 x\,dx$

" $\int \cos^2 x\,dx$

" $\int \ln x\,dx$

" $\int \sqrt{1-x^2}\,dx$

Calcolare $\int \sin^2 x \, dx$

$$\int \underset{u'}{\operatorname{sen} x} \, \underset{v}{\operatorname{sen} x} \, dx = \underset{u}{(-\cos x)} \cdot \underset{v}{\operatorname{sen} x} - \int \underset{u}{(-\cos x)} \, \underset{v'}{\cos x} \, dx$$

$$= -\operatorname{sen} x \cos x + \int \cos^2 x \, dx$$

$$= -\operatorname{sen} x \cos x + \int (1 - \operatorname{sen}^2 x) \, dx$$

$$= -\operatorname{sen} x \cos x + \int 1 \, dx - \int \operatorname{sen}^2 x \, dx$$

$$\int \operatorname{sen}^2 x \, dx = -\operatorname{sen} x \cos x + x - \int \operatorname{sen}^2 x \, dx$$

$$\underbrace{2 \int \operatorname{sen}^2 x \, dx}_{\infty \text{ f.ni}} = -\operatorname{sen} x \cos x + x + c \qquad c \in \mathbb{R}$$

$$\boxed{\int \operatorname{sen}^2 x \, dx = -\frac{1}{2} \operatorname{sen} x \cos x + \frac{x}{2} + c \qquad c \in \mathbb{R}}$$

Calcolare $\int \cos^2 x \, dx = \int \underset{u'}{\cos x} \, \underset{v}{\cos x} \, dx = \dots$

$$\int \cos^2 x \, dx = \int (1 - \operatorname{sen}^2 x) \, dx$$

$$= \int 1\,dx - \int \operatorname{sen}^2 x\,dx$$

$$= x - \left[-\frac{1}{2}\operatorname{sen}x\cos x + \frac{x}{2} + c \right] \qquad c \in \mathbb{R}$$

$$\boxed{\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{2}\operatorname{sen}x\cos x + c \qquad c \in \mathbb{R}}$$

$$\left(\frac{x}{2} + \frac{1}{2}\operatorname{sen}x\cos x + c\right)' = \frac{1}{2} + \frac{1}{2}\cos^2 x +$$

$$+ \frac{1}{2}\operatorname{sen}x\,(\cos x)' + 0$$

$$= \frac{1}{2} + \frac{1}{2}\cos^2 x - \frac{1}{2}\operatorname{sen}^2 x$$

$$= \frac{1}{2}\cos^2 x + \frac{1}{2}\cos^2 x = \cos^2 x$$

Calcolare $\int \sqrt{1-x^2}\,dx$

$$\left(\sqrt{1-x^2}\right)' = \left((1-x^2)^{1/2}\right)' = \frac{1}{\cancel{2}}(1-x^2)^{-1/2} \cdot (-\cancel{2}x) = -\frac{x}{\sqrt{1-x^2}}$$

$$\int \underset{v}{\sqrt{1-x^2}} \cdot \underset{u'}{1}\,dx = \underset{v}{\sqrt{1-x^2}} \cdot \underset{u}{x} - \int \underset{v'}{-\frac{x}{\sqrt{1-x^2}}} \cdot \underset{u}{x}$$

$$= x \cdot \sqrt{1-x^2} + \int \frac{x^2+1-1}{\sqrt{1-x^2}}\,dx$$

$$= x \cdot \sqrt{1-x^2} + \int \frac{x^2-1}{\sqrt{1-x^2}}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\left(\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \right) \quad = x \cdot \sqrt{1-x^2} - \int \sqrt{1-x^2}\,dx + \arcsin x$$

$$\int \sqrt{1-x^2}\,dx = x \cdot \sqrt{1-x^2} + \arcsin x - \int \sqrt{1-x^2}\,dx$$

$$\boxed{\cancel{2}\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left[x\sqrt{1-x^2} - \arcsin x \right] + c \quad c \in \mathbb{R}}$$

verificare

$$\text{x casa} \quad \left[\frac{1}{2} x \sqrt{1-x^2} + \arcsin x + c \right]' = \sqrt{1-x^2}$$

$$\int \log x\,dx = \int \underbrace{\log x}_{v} \cdot \underbrace{1}_{u'}\,dx = \underbrace{\log x}_{v} \cdot \underbrace{x}_{u} - \int \underbrace{\left(\frac{1}{\cancel{x}}\right)}_{v'} \cdot \cancel{x}\,dx$$

$$= x \log x - \int 1\,dx$$

$$\boxed{\int \log x\,dx = x \log x - x + c \quad c \in \mathbb{R}}$$

$$\int \arctan x\,dx = \int \arctan x \cdot 1\,dx = \ldots$$

x casa

INTEGRAZIONE PER SOSTITUZIONE

$$\frac{d}{dx}(g \circ f)(x) = g'(f(x))\, f'(x)$$

$$\Downarrow$$

$$\boxed{(g \circ f)(x) + c = \int \frac{d}{dx}(g \circ f)\, dx = \int g'(f(x))\, f'(x)\, dx \quad \underline{\underline{\forall c \in \mathbb{R}}}}$$

Ovvero anche, se G è una primitiva di g

$$\boxed{\int g(\varphi(x))\, \varphi'(x)\, dx = (G \circ \varphi)(x) + c \qquad c \in \mathbb{R}}$$

$$\frac{d}{dx}\left(\int g(\varphi(x))\, \varphi'(x)\, dx \right) = g(\varphi(x)) \cdot \varphi'(x).$$

$$\frac{d}{dx}\left(G \circ \varphi \right)(x) = \frac{d}{dx}\left(G(\varphi(x)) \right) = G'(\varphi(x)) \cdot \varphi'(x)$$

$$= g(\varphi(x))\, \varphi'(x).$$

<u>OSS</u>

$$[\text{Primitiva}_1 \text{ di } f] = [\text{Primitiva}_2 \text{ di } f]$$

$$\Updownarrow$$

$$\frac{d}{dx}[\text{Primitiva}_1 \text{ di } f] = \frac{d}{dx}[\text{Primitiva}_2 \text{ di } f]$$

$$\Updownarrow$$

$$f = f$$

Calcolare $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$

$$\int \frac{f'(x)}{f(x)}\,dx \underset{y=f(x)}{=} \left(\int \frac{dy}{y}\right)_{y=f(x)} =$$

$$\frac{dy}{dx} = f'(x)$$

$$\boxed{dy = f'(x)\,dx}$$

$$= \left(\log|y| + c\right)_{y=f(x)} \quad c \in \mathbb{R}$$

$$= \log|f(x)| + c$$

$$\frac{d}{dx}\left(\log|f(x)| + c\right) = \frac{d}{dx}\log|f(x)|$$

$$= \begin{cases} \dfrac{d}{dx}\log f(x) & f(x) > 0 \\[2ex] \dfrac{d}{dx}\log(-f(x)) & f(x) < 0 \end{cases} = \begin{cases} \dfrac{f'(x)}{f(x)} & f(x) > 0 \\[2ex] \dfrac{1}{-f(x)} \cdot -f'(x) & f(x) < 0 \end{cases}$$

$$= \begin{cases} \dfrac{f'(x)}{f(x)} & f(x) > 0 \\[2ex] \dfrac{f'(x)}{f(x)} & f(x) < 0 \end{cases} = \frac{f'(x)}{f(x)} \quad \forall f \quad f(x) \neq 0$$

$$\int \operatorname{tg} x\,dx = \int \frac{\operatorname{sen} x}{\cos x}\,dx = -\int \frac{-\operatorname{sen} x}{\cos x}\,dx$$

$$= -\int \frac{(\cos x)'}{\cos x}\,dx = -\log|\cos x| + c \quad c \in \mathbb{R}$$

Calcolare $\int (\cos x^3) \cdot x^2\,dx =$

$$y = x^3$$

$$\frac{dy}{dx} = 3x^2 \qquad dy = 3x^2\,dx$$

$$\frac{dy}{3} = x^2\,dx$$

$$= \left(\frac{1}{3}\int \cos y \cdot dy\right)_{y=x^3}$$

$$= \left(\frac{\operatorname{sen} y}{3} + c\right)_{y=x^3} \qquad c \in \mathbb{R}$$

$$= \frac{1}{3}\operatorname{sen} x^3 + c \qquad c \in \mathbb{R}$$

$$\left(\frac{1}{3}\operatorname{sen} x^3 + c\right)' = \frac{1}{\cancel{3}}(\cos x^3) \cdot (\cancel{3}x^2) \quad \checkmark$$

Calcolare $\int x^2 \cos x^3\,dx =$ $\boxed{y = x^3}$

$$x = \sqrt[3]{y}$$

$$\frac{dx}{dy} = \frac{1}{3}y^{-2/3}$$

$$\boxed{dx = \frac{1}{3y^{2/3}} \cdot dy}$$

$$\left(\int \cancel{y^{2/3}} \cos y \frac{dy}{3\cancel{y^{2/3}}}\right)_{y=x^3} = \left(\operatorname{sen} y + c\right)_{y=x^3} \quad c \in \mathbb{R}$$

Calcolare $\displaystyle\int \frac{1}{x \log x}\, dx$

$$\int \frac{1}{x\log x}\, dx = \int \frac{1/x}{\log x}\, dx = \log|\log x| + c$$

$$\int \frac{1}{x\log x}\, dx =$$

$y = \log x$ | $x = e^y$

$\frac{dy}{dx} = \frac{1}{x}$ | $\frac{dx}{dy} = e^y$

$\boxed{dy = \frac{dx}{x}}$ | $\boxed{dx = e^y dy}$

$$\int \frac{1}{x\log x}\, dx \underset{\substack{x=e^y \\ dx = e^y dy}}{=} \left(\int \frac{1}{\cancel{e^y} \cdot y} \cancel{e^y} dy\right)_{y=\log x}$$

$$= \left(\log|y| + c\right)_{y=\log x} = \log|\log x| + c$$

Calcolare $\int \sqrt{1-x^2}\,dx$

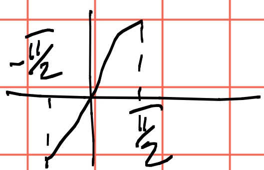
$\sin^2 t + \cos^2 t = 1 \qquad 1-\sin^2 t = \cos^2 t$

Per fare il cambiamento di variabili $x = f(t)$

la corrispondenza $t \longrightarrow f(t)=x$ dev'essere

biunivoca, e quindi si deve avere $f'(t)\neq 0$

$$\int \sqrt{1-x^2}\,dx \qquad \boxed{x=\sin t} \qquad \frac{dx}{dt}=\cos t \qquad \boxed{dx=\cos t\,dt}$$



$$= \left(\int \sqrt{1-\sin^2 t}\cdot \cos t\,dt\right)_{x=\sin t}$$

$$= \left(\int |\cos t|\,\cos t\,dt\right)_{x=\sin t}$$

in $-\frac{\pi}{2} < t < \frac{\pi}{2}$

$\cos t > 0$

$$= \left(\int \cos^2 t\,dt\right)_{x=\sin t}$$

$$= \left(\frac{t}{2} + \frac{1}{2} \operatorname{sen} t \cos t + c \right)_{\substack{x = \operatorname{sen} t \\ \Updownarrow \\ t = \operatorname{arcsen} x}} \qquad c \in \mathbb{R}$$

$$= \frac{1}{2} \operatorname{arcsen} x + \frac{1}{2} x \sqrt{1 - x^2} + c \qquad c \in \mathbb{R}$$

$$\cos t = \sqrt{1 - \operatorname{sen}^2 t}$$

Domanda

Data $f(x)$, sia $F(x)$ una sua primitiva

(suppongo che $\exists$ la primitiva)

$F(x)$ posso esplicitarla in termini di funzioni elementari ?

$$\int t^2\,dt = \frac{t^3}{3} + c$$

$$\int t e^t\,dt = te^t - e^t + c \qquad c \in \mathbb{R}$$

Però

$$\int e^{-x^2}\,dx \overset{?}{=} \text{ non so esprimerlo}$$

$f(x) = e^{-x^2}$, una primitiva esiste

(settimana prossima)

ma non so scriverla in termini

di e^x, $\operatorname{sen} x$, $\cos x$, x^n etc

$$\int \frac{dx}{x+1} = \log|x+1| + c$$

Calcolare

$$\int \frac{dx}{x^2+2x+3}$$

Si osserva che $2^2 - 4\cdot 3 < 0$
(caso dell'arcotangente!)

$$\frac{1}{x^2+2x+3} = \frac{1}{(x^2+2x+1)+2} = \frac{1}{(x+1)^2+2}$$

$$= \frac{1}{2} \frac{1}{1 + \left(\frac{x+1}{\sqrt{2}}\right)^2}$$

$$\int \frac{dx}{x^2+2x+3} = \frac{1}{2} \int \frac{1}{1+\left(\frac{x+1}{\sqrt{2}}\right)^2} dx$$

Si opera la sostituzione $\boxed{y = \frac{x+1}{\sqrt{2}}}$ $\boxed{dy = \frac{dx}{\sqrt{2}}}$ $\boxed{dx = \sqrt{2}\, dx}$

$$= \frac{1}{2} \left(\int \frac{1}{1+y^2} \cdot \sqrt{2}\, dy \right)_{y = \frac{x+1}{\sqrt{2}}}$$

$$= \frac{1}{\sqrt{2}} \left(\int \frac{dy}{1+y^2} \right)_{y = \frac{x+1}{\sqrt{2}}}$$

$$= \frac{1}{\sqrt{2}} \left(\operatorname{arctg} y + c \right)_{y = \frac{x+1}{\sqrt{2}}} \quad c \in \mathbb{R}$$

ovvero

$$\int \frac{dx}{x^2+2x+3} = \frac{1}{\sqrt{2}} \operatorname{arctg}\left(\frac{x+1}{\sqrt{2}}\right) + c \qquad c \in \mathbb{R}$$

$$a^2 - 4b > 0$$

Calcolare $\displaystyle\int \frac{dx}{x^2+4x+1}$

$$\frac{1}{x^2+4x+1} = \frac{1}{x^2+4x+4-4+1} = \frac{1}{(x+2)^2-3}$$

Calcolare $\displaystyle\int \frac{dx}{x^2+4x+1}$

$$\frac{1}{x^2+4x+1} = \frac{1}{x^2+4x+4-3} = \frac{1}{(x+2)^2-3} = \frac{1}{(x+2-\sqrt{3})(x+2+\sqrt{3})}$$

$$= \frac{A}{x+2-\sqrt{3}} + \frac{B}{x+2+\sqrt{3}} = \frac{1}{2\sqrt{3}}\left(\frac{1}{x+2-\sqrt{3}} - \frac{1}{x+2+\sqrt{3}}\right)$$

Infatti, $1 = A(x+2+\sqrt{3}) + B(x+2-\sqrt{3})$

ne $\begin{cases} A+B=0 \\ (2+\sqrt{3})A + (2-\sqrt{3})B = 1 \end{cases}$ da cui $A = -B = \dfrac{1}{2\sqrt{3}}$

$$\int \frac{1}{x^2+4x+1}\,dx = \int \frac{dx}{(x+2-\sqrt{3})(x+2+\sqrt{3})} =$$

$$= \frac{1}{2\sqrt{3}} \int \left(\frac{1}{x+2-\sqrt{3}} - \frac{1}{x+2+\sqrt{3}}\right) dx$$

$$= \frac{1}{2\sqrt{3}} \int \frac{dx}{x+(2+\sqrt{3})} - \frac{1}{2\sqrt{3}} \int \frac{dx}{x+(2-\sqrt{3})}$$

$$= \frac{1}{2\sqrt{3}} \left\{ \log|x+2-\sqrt{3}| - \log|x+2+\sqrt{3}| \right\} + c$$

$$= \frac{1}{2\sqrt{3}} \log\left|\frac{x+2-\sqrt{3}}{x+2+\sqrt{3}}\right| + c \qquad c \in \mathbb{R}$$

Calcolare $\int \frac{dx}{x^2-6x+9}$ $\quad (-6)^2-4\cdot 9=0$

$$\int \frac{dx}{x^2-6x+9} = \int \frac{dx}{(x-3)^2} = \quad \begin{array}{l} y=x-3 \\ dy=dx \end{array}$$

$$= \left(\int \frac{dy}{y^2} \right)_{y=x-3} = \left(\frac{y^{-2+1}}{-2+1} + c \right)_{y=x-3} \quad c\in\mathbb{R}$$

$$= -\frac{1}{(x-3)} + c \qquad c\in\mathbb{R}$$

In generale, dovendo calcolare

$$\int \frac{dx}{x^2+2bx+c} = \int \frac{dx}{(x+b)^2+c-b^2}$$

① $c-b^2>0$

$$\int \frac{dx}{x^2+2bx+c} = \frac{1}{c-b^2} \cdot \int \frac{dx}{1+\left(\frac{x+b}{\sqrt{c-b^2}}\right)^2}$$

e con la trasf. $y=\frac{x+b}{\sqrt{c-b^2}}$ si arriva a

$$\int \frac{dx}{x^2+2bx+c} = \frac{1}{\sqrt{c-b^2}} \operatorname{arctg}\left(\frac{x+b}{\sqrt{c-b^2}}\right) + c \qquad c\in\mathbb{R}$$

② $c-b^2=0$

$$\int \frac{dx}{x^2+2bx+c} = \int \frac{dx}{(x+b)^2} = \qquad \begin{matrix} y = x+b \\ dy = dx \end{matrix}$$

$$= \left(\int \frac{dy}{y^2} \right)_{y=x+b} = -\frac{1}{x+b} + c$$

$c \in \mathbb{R}$

③ $c-b^2<0$

$$\int \frac{dx}{x^2+2bx+c} = \int \frac{dx}{(x+b)^2-(b^2-c)}$$

$$= \int \frac{dx}{(x+b-\sqrt{b^2-c})(x+b+\sqrt{b^2-c})}$$

$$= \int \left(\frac{\frac{1}{2\sqrt{b^2-c}}}{x+b-\sqrt{b^2-c}} + \frac{-\frac{1}{2\sqrt{b^2-c}}}{x+b+\sqrt{b^2-c}} \right) dx$$

$$= \frac{1}{2\sqrt{b^2-c}} \log\left|x+b-\sqrt{b^2-c}\right| - \frac{1}{2\sqrt{b^2-c}} \log\left|x+b+\sqrt{b^2-c}\right|$$

$+ c \quad c \in \mathbb{R}$

$$= \frac{1}{2\sqrt{b^2-c}} \cdot \log \left| \frac{x+b-\sqrt{b^2-c}}{x+b+\sqrt{b^2-c}} \right| + c \quad c \in \mathbb{R}$$

Si è fatto uso del fatto che

$$\frac{1}{\left(x+b-\sqrt{b^2-c}\right)\left(x+b+\sqrt{b^2-c}\right)}=\frac{A}{x+b-\sqrt{b^2-c}}+\frac{B}{x+b+\sqrt{b^2-c}}$$

Calcoliamo A e B

$$A\left(x+b+\sqrt{b^2-c}\right)+B\left(x+b-\sqrt{b^2-c}\right)=1$$

$$\underset{\rightarrow}{\text{ne}}\begin{cases} A+B=0 \\ \left(b+\sqrt{b^2-c}\right)A+\left(b-\sqrt{b^2-c}\right)B=1 \end{cases}$$

$$\underset{\rightarrow}{\text{ne}}\begin{cases} A=-B \\ B\left(\cancel{b}-\sqrt{b^2-c}-\cancel{b}-\sqrt{b^2-c}\right)=1 \end{cases}$$

$$A=-B=\frac{1}{2\sqrt{b^2-c}}$$

Esercizi per casa

Calcolare $\displaystyle\int\frac{dx}{x^2-4x+4}$; $\displaystyle\int\frac{dx}{x^2-2x-6}$

$\displaystyle\int\frac{dx}{x^2-4x-1}$; $\displaystyle\int\frac{dx}{x^2+2x+7}$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad n \neq -1$$

$$\int \frac{dx}{(x+1)^2} = \int (x+1)^{-2} \, dx = \frac{(x+1)^{-2+1}}{-2+1} + c$$

$$\int \frac{1}{x+1} \, dx = \log|x+1| + c \qquad (n=-1)$$

Integrazione delle funzioni razionali

$$\int \frac{P(x)}{Q(x)} \, dx$$

<u>1° passo</u> se grado $(P(x)) \geq$ grado $(Q(x))$

allora $\dfrac{P_1(x) \cdot Q(x) + R_1(x)}{Q(x)} = P_1(x) + \dfrac{R_1(x)}{Q(x)}$

con grado $(R_1(x)) <$ grado $(Q(x))$

<u>2° passo</u> $Q(x)$ formato da radici reali semplici grado $(P(x)) <$ grado $(Q(x))$

$$\frac{P(x)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_m)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \cdots + \frac{A_m}{x-\alpha_m}$$

Esempio $\displaystyle\int \frac{1}{x(x+1)} = \log|x| - \log|x+1| + c$
$c \in \mathbb{R}$

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)}$$

$$\begin{cases} A+B=0 \\ A=1 \end{cases} \qquad = \frac{1}{x} - \frac{1}{x+1}$$

2) $Q(x)$ ha radici complesse e coniugate semplici

$$\frac{P_1(x)}{Q(x)} = \frac{A_1x+B_1}{x^2+p_1x+q_1} + \dots + \frac{A_mx+B_m}{x^2+p_mx+q_m}$$

Esempio

$$\int \frac{1}{(x+1)(x^2+1)}\,dx$$

In questo caso $x=-1$ $x=\pm i$

$$\int \frac{dx}{1+x^4}$$

$1+x^4=0$ ha 4 radici $\frac{1+i}{\sqrt2}$ $\frac{1-i}{\sqrt2}$ $\frac{-1-i}{\sqrt2}$ $\frac{-1+i}{\sqrt2}$

$$1+x^4=(x^2-\sqrt2 x+1)(x^2+\sqrt2 x+1)$$

$$\frac{1}{1+x^4}=\frac{Ax+B}{x^2-\sqrt2x+1}+\frac{Cx+D}{x^2+\sqrt2x+1}$$

$$=\frac{Ax^3+A\sqrt2x^2+Ax+Bx^2+B\sqrt2x+B}{//}$$

$$+\frac{Cx^3-C\sqrt2x^2+Cx+Dx^2-D\sqrt2x+D}{//}$$

$$\begin{cases}A+C=0\\ B+D+A\sqrt2-C\sqrt2=0\\ A+C+B\sqrt2-D\sqrt2=0\\ B+D=1\end{cases}\quad\begin{cases}A=-C\\ 2\sqrt2A=-1\\ B=D=\frac12\\ B+D=1\end{cases}\quad\begin{cases}C=\frac{1}{2\sqrt2}\\ A=-\frac{1}{2\sqrt2}\\ B=D=\frac12\end{cases}$$

$$\frac{-\frac{x}{2\sqrt2}+\frac12}{x^2-\sqrt2x+1}+\frac{\frac{x}{2\sqrt2}+\frac12}{x^2+\sqrt2x+1}$$

$$-\frac{1}{4\sqrt2}\frac{2x-2\sqrt2}{x^2-\sqrt2x+1}+\frac{1}{4\sqrt2}\frac{2x+2\sqrt2}{x^2+\sqrt2x+1}$$

$$=-\frac{1}{4\sqrt2}\frac{2x-\sqrt2}{x^2-\sqrt2x+1}+\frac14\frac{1}{x^2-\sqrt2x+1}+$$

$\to \log|x^2-\sqrt2x+1|$ $\to \sqrt2\arctg(\sqrt2x-1)$

$$+\frac{1}{4\sqrt2}\frac{2x+\sqrt2}{x^2+\sqrt2x+1}+\frac14\frac{1}{x^2+\sqrt2x+1}=\int$$

$\to \sqrt2\arctg(\sqrt2x+1)$

$\to \log|x^2+\sqrt2x+1|$

$$\int f\,dx = -\frac{1}{4\sqrt{2}}\log|x^2-\sqrt{2}x+1| + \frac{1}{4\sqrt{2}}\log|x^2+\sqrt{2}x+1|$$

$$+\frac{1}{2\sqrt{2}}\arctg(\sqrt{2}x+1) + \frac{1}{2\sqrt{2}}\arctg(\sqrt{2}x-1) + c$$

$$x^2-\sqrt{2}x+1 = \left(x^2-\sqrt{2}x+\frac{1}{2}\right)+\frac{1}{2} = \left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}$$

$$(x^2+\sqrt{2}x+1) = \left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}$$

$$\int\frac{1}{x^2+\sqrt{2}x+1} = 2\int\frac{1}{1+\left[\sqrt{2}\left(x+\frac{1}{\sqrt{2}}\right)\right]^2}$$

$$= 2\left(\int\frac{\frac{1}{\sqrt{2}}\,dy}{1+y^2}\right)_{y=\sqrt{2}\left(x+\frac{1}{\sqrt{2}}\right)} = \sqrt{2}\arctg(\sqrt{2}x+1)+c$$

$$\int\frac{1}{x^2-\sqrt{2}x+1}\,dx = \text{----} = \sqrt{2}\arctg(\sqrt{2}x-1)+c$$

3) Radici reali multiple

$$\frac{1}{(x-\alpha)^3} = \frac{A}{x-\alpha} + \frac{B}{(x-\alpha)^2} + \frac{C}{(x-\alpha)^3}$$

Esempio

$$\int \frac{1}{x^3(x+1)^2}\,dx$$

$$\frac{1}{x^3(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1} + \frac{E}{(x+1)^2}$$

$$= \frac{Ax^2(1+x)^2 + Bx(1+x)^2 + C(1+x)^2 + Dx^3(1+x) + Ex^3}{}$$

- - - - -

$A = 3$

$B = -2$

$C = 1$

$D = -3$

$E = 1$

$$f(x) = \frac{3}{x} - \frac{2}{x^2} + \frac{1}{x^3} - \frac{3}{x+1} + \frac{1}{(x+1)^2}$$

$$\int f(x)\,dx = 3\log|x| + \frac{2}{x} - \frac{1}{2x^2} - 3\log|1+x| - \frac{1}{(x+1)} + c$$

4) Radici complesse multiple

$$\frac{1}{(x^2+2\alpha+\beta)^3} = \frac{Ax+B}{x^2+2\alpha+\beta} + \frac{Cx+D}{(x^2+2\alpha+\beta)^2} + \frac{Ex+F}{(x^2+2\alpha+\beta)^3}$$

con $\alpha^2-\beta<0$

Esempio

$$\int \frac{dx}{(x^2+1)^2\, x}$$

$$\frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$\frac{A(x^4+2x^2+1) + (Bx^2+Cx)(x^2+1) + Dx^2+Ex}{}$$

$$\parallel$$

$$\frac{(A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A}{}$$

$$\parallel$$

$$\begin{cases} A=1 \\ C+E=0 \\ 2+B+D=0 \\ C=0 \\ B=-1 \end{cases} \qquad \begin{cases} A=1 \\ E=0 \\ C=0 \\ B=-1 \\ D=-1 \end{cases}$$

$$f(x) = \frac{1}{x} - \frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}$$

$$F(x) = \log|x| - \operatorname{arctg} x - \frac{1}{2}\operatorname{arctg} x - \frac{x}{2(x^2+1)} + C$$

$$\int \frac{dx}{(1+x^2)^2} = \int \frac{1+x^2}{(1+x^2)^2} - \int \frac{x^2}{(1+x^2)^2}$$

$$= \operatorname{arctg} x + \frac{1}{2}\int \frac{-2x}{(1+x^2)^2}\cdot x$$

$$= \operatorname{arctg} x + \frac{1}{2}\left[\frac{1}{1+x^2}\cdot x - \int \frac{1}{1+x^2}\cdot 1\, dx\right]$$

$$= \frac{1}{2}\operatorname{arctg} x + \frac{x}{2(x^2+1)} + C$$

$$\frac{1}{x^3(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1} + \frac{E}{(x+1)^2}$$

$$= \frac{Ax^2(x+1)^2 + Bx(x+1)^2 + C(x+1)^2 + Dx^3(x+1) + Ex^3}{}$$

$$= \frac{Ax^4 + 2Ax^3 + Ax^2 + Bx^3 + 2Bx^2 + Bx + Cx^2 + 2Cx + C + Dx^4 + Dx^3 + Ex^3}{}$$

$$\frac{x^4(A+D) + x^3(2A+B+D+E) + x^2(A+2B+C) + x(B+2C) + C}{}$$

$$\begin{cases} A+D=0 & D=-3 & A=3 \\ 2A+B+D+E=0 & E=-2A-B-D=-6+2+3=1 & B=-2 \\ A+2B+C=0 & A=-2B-C=3 & C=1 \\ B+2C=0 & B=-2 & D=-3 \\ C=1 & C=1 & E=1 \end{cases}$$

$$\frac{3}{x} - \frac{2}{x^2} + \frac{1}{x^3} - \frac{3}{(x+1)} + \frac{1}{(x+1)^2}$$

$$3\log|x| + \frac{2}{x} - \frac{1}{2x^2} - 3\log|x+1| - \frac{1}{(x+1)}$$

Funzioni Iperboliche

$$\operatorname{senh} x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$(\cosh x)^2 - (\operatorname{senh} x)^2 = 1$$

$$y = \frac{e^x - e^{-x}}{2} \qquad e^{2x} - 1 - 2ye^x = 0$$

$$e^{2x} - 2y\,e^x - 1 = 0$$

$$e^x = y \pm \sqrt{y^2 + 1} = y + \sqrt{y^2+1}$$

$$y = \operatorname{settsenh}(x) = \log\left(x + \sqrt{x^2+1}\right)$$

analog

$$y = \frac{e^x + e^{-x}}{2} \qquad e^{2x} + 1 - 2ye^x = 0 \qquad e^x = y \pm \sqrt{y^2 - 1}$$

e qui trovo l'inversa solo quando $x > 0$ $(o\ x < 0)$
infatti $\cosh x$ è pari !!!

$$\int \sqrt{1+x^2}\,dx \underset{x = \operatorname{senh} y}{=} \left(\int \cosh y \cdot \cosh y \; dy \right)_{x = \operatorname{senh} y}$$

$$\frac{dx}{dy} = \cosh y$$

$$dx = \cosh y \cdot dy$$

$$= \left(\int \frac{e^{2y} + e^{-2y} + 2}{2} \, dy \right)_{y = \log\left(x + \sqrt{1+x^2}\right)} =$$

$$= \left(\frac{e^{2y}}{4} - \frac{e^{-2y}}{4} + y + c \right)_{y =} \_\_$$

$$= \frac{1}{4}\left[\left(x+\sqrt{1+x^2}\right)^2 - \left(\frac{1}{x+\sqrt{1+x^2}}\right)^2\right] + \log\left(x+\sqrt{1+x^2}\right) + c$$

$\cos^2 = \operatorname{sen} + 1$

$$\int \cosh x \cosh x \, dx = \cosh x \operatorname{senh} x - \int \operatorname{senh}^2 x$$

$$= \cosh x \operatorname{senh} x + x - \int \cosh x \cosh x$$

$$\int \cosh^2 x \, dx = \frac{x}{2} + \frac{1}{2} \cosh x \operatorname{senh} x + c$$

$$= \frac{\operatorname{settsenh} y}{2} + \frac{1}{2} \qquad - \qquad + c$$