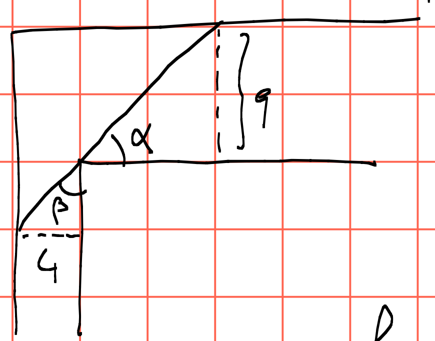
Lezione n.ro 27

Titolo nota 27/11/2011

Esercizio Calcolare lunghezza max tubo ℓ che deve essere fatto passare nell'angolo (Vedi figura)



dim

$$\ell = \frac{9}{\operatorname{sen}\alpha} + \frac{4}{\operatorname{sen}\beta} \qquad \beta+\alpha = \frac{\pi}{2}$$

$$= \frac{9}{\operatorname{sen}\alpha} + \frac{4}{\cos\alpha}$$

$$\ell \xrightarrow[\substack{\alpha \to 0 \\ \alpha \to \pi/2}]{} +\infty$$

$$\ell' = -9\frac{\cos\alpha}{\operatorname{sen}^2\alpha} + 4\frac{\operatorname{sen}\alpha}{\cos^2\alpha} = \frac{-9\cos^3\alpha + 4\operatorname{sen}^3\alpha}{\operatorname{sen}^2\alpha\cos^2\alpha} = 0$$

$$\Rightarrow \quad \operatorname{Tg}^3\theta = \frac{9}{4} \qquad \bar\theta = \operatorname{arcTg}\left(\sqrt[3]{\frac{9}{4}}\right)$$

$$\ell(\bar\theta) = \left(1+\left(\frac{9}{4}\right)^{2/3}\right)^{1/2} \cdot \left(\frac{9}{\left(\frac{9}{4}\right)^{1/3}} + 4\right)$$

$$\bar\theta = \frac{\pi}{4} \qquad \ell\left(\frac{\pi}{4}\right) = \sqrt{2}\,(9+9)$$ (se entrambi sono larghi 9)

Teorema (di Cauchy)

$f, g : [a,b] \to \mathbb{R}$.

$f, g : [a,b] \to \mathbb{R}$ continue su $[a,b]$

$f, g : [a,b] \to \mathbb{R}$ derivabile su $]a,b[$

allora $\exists z \in]a,b[$ t.c. $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(z)}{g'(z)}$

Teorema (Hopital, forme 0/0)

$f, g :]a,b[\to \mathbb{R}$ derivabili (e quindi continue)

$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$

$g'(x) \neq 0 \ \forall x \in]a,b[$

$\exists \lim_{x \to a^+} \dfrac{f'(x)}{g'(x)} = L \in \overline{\mathbb{R}}$

Allora

$\exists \lim_{x \to a^+} \dfrac{f(x)}{g(x)} = L$

Teorema (Hopital, forma ∞/∞)

$f, g :]a,b[\longrightarrow \mathbb{R}$

f, g derivabili su $]a,b[$

$\exists \lim_{x\to a^+} f(x) = \infty \quad \exists \lim_{x\to a^+} g(x) = \infty$

$g'(x) \neq 0 \quad \forall x \in]a,b[$

$\exists \lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L \in \overline{\mathbb{R}}$

Allora

$\exists \lim_{x\to a^+} \frac{f(x)}{g(x)} = L$

<u>Oss</u>

$g \xrightarrow[x\to a^+]{} 0$ e per ipotesi $g'(x) \neq 0$

$\Rightarrow g \nearrow_{x\to a^+} 0$ oppure $g(x) \searrow_{x\to a^+} 0$

$\Rightarrow g \neq 0 \quad \forall x \in]a,b[$

dim $(0/0)$ $\quad f \xrightarrow[x\to a^+]{} 0 \quad g \xrightarrow[x\to a^+]{} 0 \quad \tilde{f}(a^+)=0 \quad \tilde{g}(a^+)=0$

$$\tilde{f}(x)=\begin{cases} f(x) & x\neq a \\ 0 & x=a \end{cases} \qquad \tilde{g}(x)=\begin{cases} g(x) & x\neq a \\ 0 & x=a \end{cases}$$

Thm Cauchy

$$\frac{f(x)}{g(x)}=\frac{f(x)-\overset{0}{\overset{\shortparallel}{f(a)}}}{g(x)-\underset{0}{\underset{\shortparallel}{g(a)}}}=\frac{f'(z)}{g'(z)} \qquad a<z<x$$

$x\to a$ $\quad\downarrow\quad L$ $\qquad$ $x\to a$ $\quad\downarrow\quad L$ $\qquad$ quando $x\to a$ allora $z\to a$

$$\lim_{x\to a^+}\frac{f'(x)}{g'(x)}=L \Rightarrow \lim_{x\to a^+}\frac{f(x)}{g(x)}=L$$

dim (forma $0/0$)

Le funzioni si estendono per continuità in $x=a$ ponendo $f(a)=g(a)=0$

$$\frac{f(x)}{g(x)}=\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f'(z)}{g'(z)} \qquad a<z<x$$

Quando $x\to a$, pure $z\to a$ e quindi si conclude $\quad\not{L}$

Exemplo

$$\lim_{x\to 0} \frac{\cos x - e^x}{\operatorname{sen} x^2} \overset{H}{=} \lim_{x\to 0} \frac{-\operatorname{sen} x - e^x}{2x\cos x} \quad \nexists$$

Exemplo

$$\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^3} \overset{H}{=} \lim_{x\to 0} \frac{e^x - 1 - x}{3x^2}$$

$$\overset{H}{=} \lim_{x\to 0} \frac{e^x - 1}{6x} \overset{H}{=} \lim_{x\to 0} \frac{e^x}{6} = \frac{1}{6}$$

$$\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2 - x^3/6}{x^4} = \frac{1}{4!}$$

$$\overset{H}{\parallel}$$

$$\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{4x^3}$$

$$=(H)$$

$$\lim_{x\to 0} \frac{e^x - 1 - x}{12x^2} \overset{H}{=} \lim_{x\to 0} \frac{e^x - 1}{24x} = \frac{1}{24} = \frac{1}{4!}$$

dim $\left(\infty/\infty\right)$ $\quad \frac{f'}{g'} \to L \in \mathbb{R}$

$$\forall \varepsilon>0\ \exists \delta_1>0\ \ \forall x\in]a,a+\delta_1[\quad \left|\frac{f'(x)}{g'(x)}-L\right|<\varepsilon$$

Sia $x \in]a,a+\delta_1[$: f,g soddisfano su

$[x,a+\delta_1]$ il Teorema di Cauchy:

$$\frac{f(x)}{g(x)}\,\frac{1-\frac{f(a+\delta_1)}{f(x)}}{1-\frac{g(a+\delta_1)}{g(x)}}=\frac{f(x)-f(a+\delta_1)}{g(x)-g(a+\delta_1)}=\frac{f'(t(x))}{g'(t(x))}\qquad x<t(x)<a+\delta_1$$

$$\frac{f(x)}{g(x)}=\psi(x)\,\frac{f'(t(x))}{g'(t(x))}\qquad \psi(x)=\frac{1-\frac{g(a+\delta_1)}{g(x)}}{1-\frac{f(a+\delta_1)}{f(x)}}$$

Ora $\psi(x)\xrightarrow[x\to a]{} 1$

$$\forall\varepsilon>0\ \exists\delta_2>0,\ \delta_2<\delta_1,\quad a<x<a+\delta_2\ \begin{cases}|\psi(x)-1|<\varepsilon\\ |\psi(x)|\le 2\end{cases}$$

$$\forall\varepsilon>0\ \exists\delta_2>0\quad a<x<a+\delta_2$$

$$\left|\frac{f(x)}{g(x)}-L\right|\le\left|\psi(x)\frac{f'(t(x))}{g'(t(x))}-\psi(x)L+\psi(x)L-L\right|$$

$$=|\psi(x)|\cdot\left|\frac{f'(t(x))}{g'(t(x))}-L\right|+|L|\cdot|\psi(x)-1|$$

$$\le(2+|L|)\,\varepsilon$$

Esempio (dove non usare Hopital)

$$\lim_{x\to 0}\frac{x^2\sin 1/x}{x}$$

(non è indeterminato: semplifico)

$$\left(x^2 \operatorname{sen} \tfrac{1}{x}\right)' = 2x \operatorname{sen} \tfrac{1}{x} - \cos \tfrac{1}{x} \;\cancel{\xrightarrow[x \to 0]{}}\; ?$$

$$\lim_{x \to 0} \frac{e^{x^2} - \cos x}{\operatorname{sen} x \operatorname{tg} x} \overset{H}{=} \lim_{x \to 0} \frac{2x e^{x^2} + \operatorname{sen} x}{\cos x \operatorname{tg} x - \operatorname{sen} x \frac{1}{\cos^2 x}}$$

II sviluppi

$$\lim_{x \to 0} \frac{\cancel{1} + x^2 + o(x^3) - \cancel{1} + \frac{x^2}{2} + o(x^3)}{\frac{\operatorname{sen}^2(x)}{\cos x}}$$

$$= \lim_{x \to 0} \frac{x^2 + o(x^3)}{\operatorname{sen}^2 x} \cdot \lim_{x \to 0} \cos x$$

$$= \lim_{x \to 0} \left(\frac{x}{\operatorname{sen} x}\right)^2 \cdot \frac{1 + o(x)}{1} = 1$$

Esempio Calcolare $\lim_{x\to 0} \dfrac{e^{x^2}-\cos x}{\operatorname{sen} x \operatorname{tg} x}$.

$$\frac{e^{x^2}-\cos x}{\frac{\operatorname{sen}^2 x}{\cos x}} = \frac{\cos x\, e^{x^2}-\cos^2 x}{\operatorname{sen}^2 x} =$$

$$= \cos x \frac{[e^{x^2}-\cos x]}{\operatorname{sen}^2 x} = \cos x \frac{e^{x^2}-1+1-\cos x}{\operatorname{sen}^2 x}$$

$$= \cos x \left[\frac{e^{x^2}-1}{x^2}\frac{x^2}{\operatorname{sen}^2 x} + \frac{1-\cos x}{x^2}\frac{x^2}{\operatorname{sen}^2 x}\right]$$

$$\downarrow$$

$$1\cdot\left[1\cdot 1 + \frac{1}{2}\cdot 1\right] = 3/2$$

Calcolare

$$\lim_{x\to 0}\frac{e^{-1/x}}{x^2} \overset{y=1/x}{=} \lim_{y\to+\infty}\frac{e^{-y}}{1/y^2} = \lim_{y\to+\infty}\frac{y^2}{e^y} = 0$$

Teorema

$f: U\to\mathbb{R}$ $\quad U\in I_{x_0}$

f derivabile $\forall x\neq x_0$

$\exists \lim_{x\to x_0^-} f'(x)=\alpha_- \quad \lim_{x\to x_0^+} f'(x)=\alpha_+$

Allora $f'_-(x)=\alpha_- \quad f'_+(x)=\alpha_+$

e f derivabile se $\alpha_-=\alpha_+$

$$\lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} \overset{H}{=} \lim_{x \to x_0^-} \frac{f'(x)}{1} = f'_-(x_0) = \alpha_-$$

$$\lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} \overset{H}{=} \lim_{x \to x_0} \frac{f'(x)}{1} = f'_+(x_0) = \alpha_+$$

e così via

Esercizio Data la funzione

$$f(x) = \begin{cases} (a-1)x + b - a & x > 0 \\ 3 & x = 0 \\ cx - x^2 - 3b & x < 0 \end{cases}$$

determinare a, b, c t.c. f derivabile su $\mathbb{R}$

f è continua $\forall x < 0$ e $\forall x > 0$ $\forall a, b, c$

$x = 0$ $\displaystyle \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (cx - x^2 - 3b) = -3b$

$$\overset{\shortparallel}{3} = f(0)$$

$$\shortparallel$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left((a-1)x + b - a\right) = b - a$$

$$\begin{cases} -3b = 3 \\ b - a = 3 \end{cases} \qquad \begin{cases} b = -1 \\ a = -4 \end{cases}$$

$$f(x) = \begin{cases} -5x + 3 & x > 0 \\ 3 & x = 0 \\ cx - x^2 + 3 & x < 0 \end{cases}$$

f è derivabile $\forall x \neq 0 \quad \forall c$

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} (c-2x) = c$$

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} (-5) = -5$$

$\rightarrow c = -5$

Def data $f:]x_0-\delta, x_0+\delta[\to \mathbb{R}$, un polinomio di ordine n

$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ viene detto

"Polinomio di Taylor di ordine n centrato in x_0 relativo a f"

se
$$\boxed{f(x) - P_n(x) = o(|x-x_0|^n) \quad \text{per } x \to x_0}$$

Oss $\boxed{n=1}$ $P_1(x) = f(x_0) + f'(x_0)(x-x_0)$ è la

retta tangente ovvero $f(x) - P_1(x) = o(|x-x_0|)$
per $x \to x_0$

Oss la retta tangente era unica!

Il polinomio di Taylor è unico?

Teorema Il polinomio di Taylor, se esiste

allora è unico

dim

Supponiamo $P_n(x) = a_n(x-x_0)^n + a_{n-1}(x-x_0)^{n-1} + \dots + a_0$

$\overline{P_n}(x) = \overline{a_n}(x-x_0)^n + \overline{a_{n-1}}(x-x_0)^{n-1} + \dots + \overline{a_0}$

che siano 2 polinomi di Taylor

ovvero $f(x) - P_n(x) = o(|x-x_0|^n)$ $\quad x \to x_0$

$f(x) - \overline{P_n}(x) = o(|x-x_0|^n)$

$$P_m(x) - \overline{P_m}(x) = o\left(|x-x_0|^m\right) \quad (x \to x_0)$$

$$\|$$

$$c_m(x-x_0)^m + c_{m-1}(x-x_0)^{m-1} + \dots + c_1(x-x_0) + c_0$$

$$c_i = (a_i - \overline{a}_i)$$

Osserviamo che $\boxed{c_0 = 0}$

$$\lim_{x \to x_0} \left(P_m(x) - \overline{P_m}(x)\right) = c_0 = \lim_{x \to x_0} o\left(|x-x_0|^m\right) = 0$$

$$P_m(x) - \overline{P_m}(x) = c_m(x-x_0)^m + \dots + c_1(x-x_0)$$

$$= \underbrace{(x-x_0)\left[c_m(x-x_0)^{m-1} + \dots + c_2(x-x_0) + c_1\right]}$$

$$= \underbrace{o\left(|x-x_0|^m\right)}$$

$$\Downarrow$$

$$c_m(x-x_0)^{m-1} + \dots + c_2(x-x_0) + c_1 = o\left(|x-x_0|^{m-1}\right)$$

$$\uparrow$$

$c_1 = 0$ infatti

$$\lim_{x \to x_0} \left(c_m(x-x_0)^{m-1} + \dots + c_1\right) = c_1 = 0 = \lim o\left(|x-x_0|^{m-1}\right)$$

$$\dots$$

$$c_m = o(1) \Rightarrow c_m = 0 \Rightarrow P_m = \overline{P_m} \quad ⚡$$

Teorema (Formula di Taylor col resto di Peano)

$f :]a,b[\to \mathbb{R}$ $\quad x_0 \in]a,b[$

f derivabile n volte $\forall x \in]a,b[$

f " $n+1$ " in $x_0 \in]a,b[$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$$

allora $\quad f(x) - P_n(x) = o\left(|x-x_0|^n\right)$ $\quad x \to x_0$

P_n è il polinomio di Taylor ordine n centrato in x_0

$$P_n(x_0) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

- $f(x) = e^x = f'(x) = \dots = f^{(n)}(x)$

  $x_0 = 0 \quad f(0) = f'(0) = \dots = f^{(n)}(0) = 1 = e^0$

  $$\underline{P_n(x)} = \sum_{k=0}^{n} \frac{1}{k!}(x-0)^k =$$

  $$= \underline{1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}}$$

- e^x sviluppo ordine n centrato in $x=1$

  $f(1) = f'(1) = \dots = f^{(n)}(1) = e$

  $$P_n(x) = \sum_{k=0}^{n} \frac{e}{k!}(x-1)^k$$

  $$= 1 + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}\cdot(x-1)^3 +$$

  $$\dots + \frac{e}{n!}(x-1)^n$$

- $f(x) = \underbrace{3x + x^2}- x^4$

Polinomio ordine 3 centrato in $x=0$?

$f(0)=0 \quad f'(x) = 3 + 2x - 4x^3 \quad f'' = 2 - 12x^2$

$$f'''(x) = -24x$$

$f(0)=0 \quad f'(0)=3 \quad f''(0)=2 \quad f'''(0)=0$

$$P_3(x) = 3x + \frac{2}{2} \cdot x^2$$

$$= 3x + x^2$$

$$f(x) = 3x + x^2 + o(x^3) \quad (x \to 0)$$

Se voglio lo sviluppo in $\boxed{x=1}$ le cose cambiano

$$f(x) = 3x + x^2 - x^4$$

$f'(x) = 3 + 2x - 4x^3 \quad f'' = 2 - 12x^2$

$f''' = -24x$

$f(1)=3 \quad f'(1)=1 \quad f''(1)=-10 \quad f'''(1)=-24$

$$P_3(x) = 3 + (x-1) + \left(\frac{-10}{2}\right)(x-1)^2 + \left(\frac{-24}{6}\right)(x-1)^3$$

$$= 3 + (x-1) - 5(x-1)^2 - 4(x-1)^3$$

$$f(x) = 3 + (x-1) - 5(x-1)^2 - 4(x-1)^3 + o((x-1)^3)$$

$$P_{n-1}(x_0) = f(x_0) \quad P'_{n-1}(x_0) = f'(x_0) \quad P''_{n-1}(x_0) = f''(x_0) \; ---$$

$$--- \; P^{(n-2)}_{n-1}(x_0) = f^{(n-2)}(x_0) \qquad P^{(n-1)}_{n-1}(x_0) = f^{(n-1)}(x_0)$$

$$\overset{\shortparallel}{P^{(n-1)}_{n-1}(x)}$$

$$\lim_{x \to x_0} \frac{f(x) - P_{n-1}(x)}{(x-x_0)^n} \overset{H}{=} \lim_{x \to x_0} \frac{f'(x) - P'_{n-1}(x)}{n(x-x_0)^{n-1}}$$

$$\overset{H}{=} \lim_{x \to x_0} \frac{f''(x) - P''_{n-1}(x)}{n(n-1)(x-x_0)^{n-2}} = --- = \lim_{x \to x_0} \frac{f^{(n-1)}(x) - P^{(n-1)}_{n-1}(x)}{n!\,(x-x_0)}$$

$$\lim_{x \to x_0} \frac{f(x) - P_n(x)}{(x-x_0)^n} = \lim_{x \to x_0} \left[\frac{f(x) - P_{n-1}(x)}{(x-x_0)^n} - \frac{f^{(n)}(x_0)}{n!} \right]$$

$$\overset{H}{=} \lim_{x \to x_0} \left[\frac{f^{(n-1)}(x) - P^{(n-1)}_{n-1}(x)}{n!\,(x-x_0)^n} - \frac{f^{(n)}(x_0)}{n!} \right]$$

$$= \frac{1}{n!} \left[\lim_{x \to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{x - x_0} - f^{(n)}(x_0) \right] = 0$$

- Con questo Teorema si possono ottenere tutti gli sviluppi visti

$$\begin{cases} f(x) = e^x \quad f'(x) = f''(x) = f'''(x) = --- = f^{(n)}(x) = e^x \\ P_n(x) = \sum_{k=0}^{n} \frac{1}{k!}(x-x_0)^k \qquad 1 = f^{(i)}(0) \quad i = 0, \dots, n \end{cases}$$

$f(x) = e^x$ se voglio lo sviluppo in $\underline{\underline{x = 1}}$

- $f(1) = e = f'(1) = \dots = f^{(n)}(1)$

$$P_{n,1}(x) = \sum_{k=0}^{n} \frac{e}{k!}(x-x_0)^k$$

- $f = \operatorname{sen} x \quad f' = \quad f'' = \quad f''' =$

- $f(x) = x^3 + 2x^5 + 3x^7$
  - Sviluppo di f centrato in $x_0 = 0$ di ordine 6

$$f(x) = x^3 + 2x^5 + o(x^6)$$

  - Sviluppo di f centrato in $x_0 = 1$ di ordine 2

$\underline{f(1) = 5} \qquad f'(x) = 3x^2 + 10x^4 + 21x^6 \quad \underline{f'(1) = 34}$

$f'' = 6x + 40x^3 + 126x^5 \quad f''(1) = 170$

$$f(x) = f(1) + f'(1)(x-5) + \frac{f''(1)}{2}(x-5)^2 + o((x-5)^2)$$

$$= 5 + 34(x-5) + 85(x-5)^2 + o(|x-5|^2)$$

<u>Problema</u>: stimare $o(|x-x_0|^6)$!!

Teorema (Formula Taylor con il resto di Lagrange)

$f:]a,b[\to \mathbb{R}$ derivabile $(n+1)$ volte $\forall x \in]a,b[$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

Allora $\exists z$ compreso tra x_0 e x t.c.

$$f(x) = P_n(x) + \underbrace{R_n(z)}_{\frac{f^{(n+1)}(z)}{(n+1)!}(x-x_0)^{n+1}}$$

dim

In luogo di Hopital si usa Cauchy

$g(t) = f(t) - P_n(t)$ $\qquad h_k(t) = (t-x_0)^k$

$k = 0, \dots, n+1$

$g(x_0) = 0$ $\qquad h_k(x_0) = 0$

$$x > x_0 \quad \frac{g(x)}{h_{n+1}(x)} = \frac{g(x) - g(x_0)}{h_{n+1}(x) - h_{n+1}(x_0)} \overset{\text{Cauchy}}{=} \frac{g'(x_1)}{h'_{n+1}(x_1)} = \frac{g'(x_1)}{(n+1)\cdot h_n(x_1)} \qquad x_0 < x_1 < x$$

$$= \frac{g'(x_1) - \overset{=0}{g'(x_0)}}{(n+1)\, h_n(x_1) - (n+1)\underbrace{h_n(x_0)}_{0}} \overset{\text{Cauchy}}{=} \frac{g''(x_2)}{(n+1)\cdot n \cdot h_{n-1}(x_2)} \qquad x_0 < x_2 < x_1$$

$$\frac{g''(x_2) - g''(x_0)}{(n+1)\cdot n\left[h_{n-1}(x_2) - h_{n-1}(x_0)\right]} = \frac{g'''(x_3)}{(n+1)\, n\, h'_{n-1}(x_3)} = \frac{g'''(x_3)}{(n+1)\cdot n(n-1)\cdot h_{n-2}(x_3)} \qquad 0 < x_3 < x_2$$

etc

$$\frac{g^{(n)}(x_n)}{n!\, h_1(x_n)} = \frac{g^{(n)}(x_n) - g^{(n)}(x_0)}{--} = \frac{g^{(n+1)}(x_{n+1})}{(n+1)!}$$

e quindi si arriva a dire $\left(P_n^{(n+1)}(x) = 0 \, !\right)$

$$\frac{f(x) - P_n(x)}{(x-x_0)^{n+1}} = \frac{g(x)}{h_{n+1}(x)} = \frac{g^{(n+1)}(x_{n+1})}{(n+1)!} = \frac{f^{(n+1)}(x_{n+1})}{(n+1)!}$$

posto $z = x_{n+1}$ $\quad f(x) - P_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}(x-x_0)^{n+1}$

Questo Teorema mi permette di approssimare $\sqrt{65}$

$$\sqrt{1+64} = 8\sqrt{1+1/64}$$

$$= 8\left\{1 + \frac{1}{2}\cdot\frac{1}{64} - \frac{3}{8}\left[\frac{1}{64}\right]^2 + \text{Resto}\right\}$$

dove $|\text{Resto}| \leq \dfrac{1}{128\,(63)^{5/2}}$

$f(x) = (1+x)^{1/2}$ $\quad f'(x) = \frac{1}{2}(1+x)^{-1/2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$ $\quad f'''(x) = +\frac{3}{8}(1+x)^{-5/2}$

$$f(x) = \sum_{k=0}^{2} \frac{f^{(k)}(0)}{k!}x^k + \frac{f'''(\xi)}{3!}x^3$$

$$= 1 + \frac{x}{2} - \frac{1}{8}x^2 + \frac{3}{48}\,\frac{1}{(1+\xi)^{5/2}}\cdot x^3 \ldots$$

Dovendo approssimare $f(1/64)$,

$$\sup_{-\frac{1}{64}\le x\le \frac{1}{64}} \left|R_3(\xi)\right| = \sup_{x\in\left[-\frac{1}{64},\frac{1}{64}\right]} \frac{1}{16}\,\frac{x^3}{(1+\xi)^{5/2}}$$

$$\le \frac{1}{16}\cdot\left(\frac{1}{64}\right)^3\cdot\frac{1}{\left(1-\frac{1}{64}\right)^{5/2}} = \frac{1}{2}\cdot\frac{1}{8^{\cancel{7}2}}\cdot\frac{\cancel{8^5}}{(63)^{5/2}}$$

$$= \frac{1}{128\cdot(63)^{5/2}}$$