Lezione n° 15 - 27 ottobre 2011

Titolo nota 24/10/2011

Oss Nella dimostrazione della continuità della funzione inversa si fa uso del seguente Teorema

Teorema (esistenza limiti dx e sin per f.ni monotone)

Sia $f : I \to \mathbb{R}$, dove I è un intervallo una funzione strettamente crescente in I

Allora $\forall x_0 \in I$ $\exists \lim_{x \to x_0^-} f(x)$ $\exists \lim_{x \to x_0^+} f(x)$

(per gli estremi di I ne esisterà uno solo)

dim

Sia $x_0 \in I$. Proviamo che

$$\exists \lim_{x \to x_0^-} f(x) = \sup\{f(x) : x \in I, x < x_0\}$$

$$\ell = \sup\{f(x) : x \in I, x < x_0\}$$

$$\begin{cases} f(x) \leq \ell & \forall x \in I, x < x_0 \\ \forall \varepsilon > 0 \ \exists \bar{x} \in I : \ell - \varepsilon < f(\bar{x}) \end{cases}$$

e tenendo conto che $\bar{x} < x \Rightarrow f(\bar{x}) < f(x)$

$\forall \varepsilon > 0 \ \exists \bar{x} : \forall x \in I, \bar{x} < x < x_0 \Rightarrow \ell - \varepsilon < f(x) \leq \ell < \ell + \varepsilon$

posto $x_0 - \bar{x} = \delta$

$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in I \ x_0 - \delta < x < x_0 \Rightarrow \ell - \varepsilon < f(x) < \ell + \varepsilon$

ovvero $\lim\limits_{x \to x_0^-} f(x) = \ell$

Proviamo che $\lim\limits_{x \to x_0^+} f(x) = L = \inf\limits_{\substack{x \in I \\ x_0 < x}} f(x)$

$$\begin{cases} L \leq f(x) \ \forall x \in I, x > x_0 \\ \forall \varepsilon > 0 \ \exists \bar{x} : f(\bar{x}) < L + \varepsilon \end{cases}$$

da cui, tenendo conto $f \nearrow$

$\forall \varepsilon > 0 \ \exists \bar{x} : \forall x \in I \ x_0 < x < \bar{x} \Rightarrow L - \varepsilon < L \leq f(x) < f(\bar{x}) < L + \varepsilon$

e posto $\bar{x} - x_0 = \delta$

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in I \ x_0 < x < x_0 + \delta \Rightarrow L - \varepsilon < f(x) < L + \varepsilon$

ovvero $\lim\limits_{x \to x_0^+} f(x) = L$

Algebra degli "o"

Oss In generale dovremmo lavorare con

$$o(f) \text{ per } x \to x_0$$

ma a noi interessa molto il caso $x_0 = 0$

(e comunque, a meno di una traslazione, tutti i risultati sono ancora validi)

Inoltre ci interessa il confronto non tra funzioni, ma il confronto tra

$$(f(x)) \text{ e } (\text{polinomi in } x)$$

(i) $o(x^\alpha) = k\, o(x^\alpha) \quad x \to 0, \ \forall k \neq 0 \ \forall \alpha > 0$

$f = o(x^\alpha) \quad x \to 0$ <u>sse</u> $\exists \eta(x) = o(1) \ x \to 0 \quad f(x) = \eta(x) \cdot x^\alpha$

<u>sse</u> $\exists \eta = o(1) \ x \to 0 \quad f(x) = k \cdot \frac{\eta(x)}{k} x^\alpha$

<u>sse</u> $\exists \bar{\eta} = \frac{\eta}{k} = o(1) \ x \to 0 \quad \frac{f(x)}{k} = \bar{\eta}(x) \cdot x^\alpha$

<u>sse</u> $\frac{f(x)}{k} = o(x^\alpha)$ ↯

Conseguente $o(x^\alpha) = -o(x^\alpha) \quad (k = -1) \quad x \to 0$

$$o(x^\alpha) + o(x^\alpha) = o(x^\alpha) \quad (k = 2) \quad x \to 0$$

$o(x^\alpha) - o(x^\alpha) = o(x^\alpha) + o(x^\alpha) = o(x^\alpha) \qquad x \to 0$

(ii) $\quad x^\alpha o(x^\beta) = o(x^{\alpha+\beta}) \qquad x \to 0 \qquad \forall \alpha > 0 \ \forall \beta > 0$

Infatti $f(x) = x^\alpha o(x^\beta) \quad x \to 0$

<u>se</u> $\exists\, \eta(x) = o(1) \ x \to 0 \quad f(x) = x^\alpha (\eta(x) \cdot x^\beta)$

<u>se</u> $\quad$ " $\quad$ " $\quad f(x) = \eta(x) \cdot x^{\alpha+\beta}$

<u>se</u> $\quad f(x) = o(x^{\alpha+\beta}) \qquad x \to 0$

Conseguenze: $\quad x^4 o(x^2) = x^3 o(x^3)$

$= x^5 o(x)$

$= x^6 o(1) \qquad$ per $x \to 0$

$= o(x^6)$

$= x^2 o(x^4)$

$= x\, o(x^5)$

$= \sqrt[3]{x}\, o(x^{5+\frac{2}{3}})$

etc

(iii) $\quad o(x^\alpha) + o(x^{\alpha+\beta}) = o(x^\alpha) \qquad x \to 0 \qquad \forall \alpha > 0 \ \forall \beta > 0$

$(\Rightarrow) \ f = o(x^\alpha) \qquad g = o(x^{\alpha+\beta})$

$\Rightarrow \exists\, \eta_1 = o(1) \quad \eta_2 = o(1) \quad x \to 0 \qquad f(x) = \eta_1(x) x^\alpha \quad g(x) = \eta_2(x)\, x^{\alpha+\beta}$

$\Rightarrow \quad$ " $\quad$ " $\quad$ " $\quad f(x) + g(x) = x^\alpha (\eta_1(x) + x^\beta \cdot \eta_2(x))$

$\Rightarrow \quad f + g = o(x^\alpha) \quad x \to 0 \quad$ prendendo $\eta_1(x) + x^\beta \eta_2(x) \xrightarrow[x \to 0]{} 0$

($\Leftarrow$) è immediato: $f = o(x^\alpha)$, $x \to 0$

$$\Rightarrow \exists\, \eta(x) = o(1),\ x \to 0 \quad f(x) = \eta(x) \cdot x^\alpha$$

$$\Rightarrow \exists\, \eta(x) = o(1) \quad 0 = o(1) \quad x \to 0 \quad f(x) = \eta(x) \cdot x^\alpha + 0 \cdot x^{\alpha+\beta}$$

$$\Rightarrow f = o(x^\alpha) + o(x^{\alpha+\beta})$$

Oss $f = o(x^\alpha) \Rightarrow f = o(x^\alpha) + o(x^{\alpha+\beta})$ è immediato sempre, in quanto l'insieme $o(x^\alpha) \subseteq o(x^\alpha) + o(x^{\alpha+\beta})$

(è sufficiente osservare che $0 = o(x^{\alpha+\beta})$, e

$0 + o(x^\alpha) = o(x^\alpha)$)

Conseguenze: per esempio

$$o(x^3) + o(x^5) = o(x^3) \quad x \to 0$$

$$o(x^2) + o(x) = o(x) \quad x \to 0$$

iii) $o(o(x^\alpha)) = o(x^\alpha) \quad x \to 0 \quad \forall \alpha > 0$

$\Rightarrow$ Infatti $f = o(o(x^\alpha))$ $x \to 0$

$$\Rightarrow \exists\, \eta_1(x) = o(1) \text{ per } x \to 0 \quad f(x) = \eta_1(x) \cdot o(x^\alpha)$$

$$\Rightarrow \exists\, \eta_1 = o(1)\ \ \eta_2 = o(1) \text{ per } x \to 0 \quad f(x) = \eta_1(x) \cdot \eta_2(x) \cdot x^\alpha$$

$\Rightarrow \exists\, \varphi(x) = \eta_1(x)\eta_2(x)$ per $x \to 0$ $\quad f(x) = \varphi(x) \cdot x^\alpha$.

$\Rightarrow f(x) = o(x^\alpha)$

$(\Leftarrow)$ $f = o(x^\alpha)$ $x \to 0$ $\Rightarrow \exists\, \eta(x) = o(1)$ $x \to 0$ $f(x) = \eta(x)\, x^\alpha$

$\Rightarrow \exists\, \eta(x) = o(1)$ $x \to 0$ $\quad f(x) = \eta^{1/3}(x) \cdot \eta^{2/3}(x)\, x^\alpha$

$\Rightarrow \exists\, \eta_1(x) = o(1)$ $\exists\, \eta_2(x) = o(1)$ $x \to 0$ $f(x) = \eta_1(x)\, \eta_2(x)\, x^\alpha$

$\Rightarrow \exists\, \eta_1(x) = o(1)$ $x \to 0$ $\quad f(x) = \eta_1(x)\, o(x^\alpha)$

$\Rightarrow \quad f(x) = o(o(x^\alpha))$ $\quad x \to 0$ $\quad \not\downarrow$

(IV) $o(x^\alpha + o(x^\alpha)) = o(x^\alpha)$

$(\Rightarrow)$ $f = o(x^\alpha + o(x^\alpha))$ $x \to 0$ $\Rightarrow$

$\Rightarrow \exists\, \eta_1(x) = o(1)$ $x \to 0$: $f(x) = \eta_1(x) \cdot (x^\alpha + o(x^\alpha))$

$\Rightarrow \exists\, \eta_1 = o(1)$ $\eta_2 = o(1)$ $x \to 0$: $f(x) = \eta_1(x)\, (x^\alpha + \eta_2(x)\, x^\alpha)$

$\Rightarrow$ " " " $f(x) = (\eta_1(x) + \eta_1(x)\eta_2(x))\, x^\alpha$

$\Rightarrow \exists\, \bar{\eta} = o(1)$ $x \to 0$: $f(x) = \bar{\eta}(x) \cdot x^\alpha$

$\Rightarrow f(x) = o(x^\alpha)$

$(\Leftarrow)$ $f(x) = o(x^\alpha)$ $x \to 0$ $\Rightarrow$

$\Rightarrow \exists\, \eta(x) = o(1)$ $x \to 0$ $\quad f(x) = \eta(x) \cdot x^\alpha$

$\Rightarrow \exists\, \eta(x) = o(1) \quad x \to 0 \quad f(x) = \left(\eta(x) + o \cdot \eta(x)\right) x^{\alpha}$

$\Rightarrow \exists\, \eta_1 = o(1) \ \exists\, \eta_2(x) = o(1) \quad x \to 0 \quad f(x) = \eta_1 \left(x^{\alpha} + \eta_2 x^{\alpha}\right)$

$\Rightarrow f(x) = o\left(x^{\alpha} + o(x^{\alpha})\right) \quad x \to 0 \quad \lightning$

Conseguenze

$$o\left(x^{\alpha} + x^{\alpha+\beta}\right) = o(x^{\alpha}) \quad x \to 0 \quad \forall \alpha > 0 \ \forall \beta > 0$$

(V) $o(x^{\alpha})\, o(x^{\beta}) = o(x^{\alpha+\beta}) \quad x \to 0 \quad \forall \alpha > 0 \ \forall \beta > 0$

$(\Rightarrow)$ $f = o(x^{\alpha}) \quad g = o(x^{\beta}) \quad x \to 0$

$\Rightarrow \exists\, \eta_1 = o(1) \ \exists\, \eta_2 = o(1) \quad x \to 0 \quad h = \eta_1(x) x^{\alpha} \cdot \eta_2(x) x^{\beta} = f \cdot g$

$\Rightarrow \exists\, \bar{\eta} = o(1) \quad x \to 0 \quad h(x) = \bar{\eta}(x) \cdot x^{\alpha+\beta}$

$\Rightarrow h = o(x^{\alpha+\beta})$

$(\Leftarrow)$ $h = o(x^{\alpha+\beta}) \quad x \to 0 \quad \Rightarrow$

$\Rightarrow \exists\, \tilde{\eta}(x) = o(1) \ x \to 0 \ : \ h(x) = \eta(x)\, x^{\alpha+\beta}$

$\Rightarrow \exists\, \eta_1(x) = \eta^{1/3} = o(1) \quad \eta_2 = \eta^{2/3} = o(1) \quad h(x) = \left(\eta_1 x^{\alpha}\right) \cdot \left(\eta_2 x^{\beta}\right)$

$\Rightarrow h(x) = f(x) \cdot g(x) \qquad f = o(x^{\alpha}) \quad g = o(x^{\beta}) \quad \lightning$

(VI) $$\frac{o\left(x^{\alpha+\beta}\right)}{x^{\beta}} = o(x^{\alpha}) \qquad \forall \alpha, \beta > 0 \quad x \to 0$$

Principio di sostituzione dell'infinitesimo

Teorema

$f_1, f_2, g_1, g_2 : A \longrightarrow \mathbb{R}$ $\quad x_0$ p.d.a. per A

- f_1, f_2, g_1, g_2 infinitesime per $x \to x_0$
- $f_1 = o(f_2)$ $\quad g_1 = o(g_2)$ $\quad x \to x_0$

allora sono equivalenti le seguenti proposizioni

1) $\displaystyle \lim_{x \to x_0} \frac{f_1(x) + f_2(x)}{g_1(x) + g_2(x)}$ — esiste finito / diverge a $\pm\infty$ / non esiste

2) $\displaystyle \lim_{x \to x_0} \frac{f_1(x)}{g_1(x)}$ — esiste finito / diverge a $\pm\infty$ / non esiste

La dimostrazione segue da

per $x \to x_0$
$$\frac{f_1(x) + f_2(x)}{g_1(x) + g_2(x)} = \frac{f_1(x) + o(f_1)}{g_1(x) + o(g_1)} = \frac{f_1(x)}{g_1(x)} \cdot \frac{1 + o(1)}{1 + o(1)}$$

osservando che $\displaystyle \lim_{x \to x_0} \frac{1 + o(1)}{1 + o(1)} = 1$ $\checkmark$

Esempio Calcolare $\lim_{x\to 0} \dfrac{1-\frac{x^2}{2}-\cos x - \operatorname{sen} x}{\operatorname{sen}^3 x + x}$

dim

$$\lim_{x\to 0} \frac{-\operatorname{sen}^2 x + \left[1-\frac{x^2}{2}-\cos x\right]}{x^2+\left[\operatorname{sen}^3 x\right]} =$$

$1-\frac{x^2}{2}-\cos x = o(x^2)$

ma

$\lim_{x\to 0} \frac{\operatorname{sen}^2 x}{x^2} = 1$

$\Downarrow$

$1-\frac{x^2}{2}-\cos x = o(\operatorname{sen}^2 x)$

$$= \lim_{x\to 0} \frac{-\operatorname{sen}^2 x + o(\operatorname{sen} x)}{x^2 + o(x)}$$

$$= \lim_{x\to 0} -\frac{\operatorname{sen}^2 x}{x^2} = -1 \quad \checkmark$$

Esercizio $\dfrac{1}{1-x} = 1 + x + x^2 + o(x^2), \quad x\to 0$

$$1 = 1 - x^2 + x^2 - x^3 + x^3$$

$$= 1 - x^2 + x^2 - x^3 + o(x^2)$$

$$= (1-x)(1+x) + x^2(1-x) + o(x^2)$$

$$\frac{1}{1-x} = 1 + x + x^2 + \frac{o(x^2)}{1-x}$$

$$\frac{1}{1-x} = 1 + x + x^2 + o(x^2) \qquad x\to 0$$

In generale $\dfrac{1}{1-x} = 1 + x + x^2 + \dots + x^n + o(x^n)$

(dim per induzione)

Sviluppi delle funzioni elementari

$$\operatorname{sen} x = o(1) \qquad x \to 0$$

$$= x + o(x^2) \qquad x \to 0$$

$$= x - \frac{x^3}{3!} + o(x^5) \qquad x \to 0$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)$$

$$\vdots$$

$$= x - \frac{x^3}{3!} + \dots + (-1)^{n+1} \cdot \frac{x^{2n+1}}{(2n+1)!} + o\left(x^{2n+2}\right)$$

<u>Oss</u> Naturalmente

$$\operatorname{sen} x = x + o(x^2) \quad x \to 0 \quad \Rightarrow \quad \operatorname{sen} x = x + o(x) \quad x \to 0$$

$$\cos x = 1 + o(x) \qquad x \to 0$$

$$= 1 - \frac{x^2}{2!} + o(x^3) \qquad x \to 0$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5) \qquad x \to 0$$

$$\vdots$$

$$= 1 - \frac{x^2}{2!} + \dots + (-1)^{n+1} \frac{x^{2n}}{(2n)!} + o\left(x^{2n+1}\right)$$

$$\operatorname{tg}(x) = x + o(x^2)$$

$$= x + \frac{x^3}{3} + o(x^4)$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^6)$$

$$\frac{1}{1-x} = 1+x+$$

$$\frac{1}{1+x} = 1-x+x^2-x^3$$

$$\frac{1}{1+x^2} = 1-x^2+x^4-x^6$$

$$\arctan = x - \frac{x^3}{3} + \frac{x^5}{5}$$

$$\arctan x = x + o(x^2)$$

$$= x - \frac{x^3}{3} + o(x^4)$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^6)$$

$$\vdots$$

$$= x - \frac{x^3}{3} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + o(x^{2n+1})$$