Lezione n. 12 - 20 ottobre 2011

Titolo nota 19/10/2011

$f: A \to \mathbb{R}$ continua in $x_0 \in A$, x_0 $\overbrace{\text{p.d.a.}}^{\text{punto di accumul.}}$ per A

$\Updownarrow$ Def. Cont.

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in A \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

↑ x_0 p.d.a.

$\Updownarrow$

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in A \quad 0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

(non devo testare f in x_0, in quanto è banale che mi abbia $|f(x_0) - f(x_0)| = 0 < \varepsilon$!)

$\Updownarrow$

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Concludo

f continua in x_0, x_0 p.d.a. $\iff \lim_{x \to x_0} f(x) = f(x_0)$

Osservazione: quando scriviamo

"$\lim_{x \to x_0} f(x) = \ell$" intendiamo dire che

la seguente proposizione è vera

$$\left(\forall U \in \mathcal{I}_\ell \ \exists V \in \mathcal{I}_{x_0} : \forall x \in (A \setminus \{x_0\}) \ x \in V \Rightarrow f(x) \in U \right)$$

Però questo significa "verificare" il limite

e NON CI DICE COME calcolare

$\boxed{\ell}$ quando sono dati $f(x)$ e x_0

punto di accumulazione per A

Nel seguito $\boxed{\text{P.d.a}} \equiv$ punto di accumulazione

Oss Questo problema non si pone

per la continuità poiché $f: A \to B$ e

dato $x_0 \in A$, $\exists!\, f(x_0) \in B$.

Dunque il candidato limite è

univocamente individuato

Teorema Data $f: A \to \mathbb{R}$ e x_0 p.d.a.

Se $\exists\, l = \lim_{x \to x_0} f(x)$ allora l è unico

dim

x Assurdo $\exists\, l_1, l_2$ diversi tra loro $(l_1 \neq l_2)$

Supponiamo $x_0, l_1, l_2 \in \mathbb{R}$ per semplicità

$\lim_{x \to x_0} f = l_1$ <u>se</u> $\forall \varepsilon > 0\ \exists\, \delta_1 > 0\ \forall x \in A\quad 0 < |x - x_0| < \delta_1 \Rightarrow l_1 - \varepsilon < f < l_1 + \varepsilon$

$\lim_{x \to x_0} f = l_2$ <u>se</u> " $\exists\, \delta_2 > 0$ " $0 < |x - x_0| < \delta_2 \Rightarrow l_2 - \varepsilon < f < l_2 + \varepsilon$

Sia $\delta = \min\{\delta_1, \delta_2\} = \delta_1 \wedge \delta_2$

$$\forall \varepsilon > 0\ \exists\, \delta > 0\ \forall x \in A\quad 0 < |x - x_0| < \delta \Rightarrow \begin{array}{l} l_1 - \varepsilon < f(x) < l_1 + \varepsilon \\ l_2 - \varepsilon < f(x) < l_2 + \varepsilon \end{array}$$

allora, sottraendo le disug. membro a membro

$$\forall \varepsilon > 0\ \exists\, \delta > 0\ \forall x \in A\quad 0 < |x - x_0| < \delta \Rightarrow l_1 - l_2 < 0 < l_1 - l_2$$

ASSURDO Ne segue $l_1 = l_2$ $\lightning$

Teorema (Se $f(x)$ ha limite finito allora è limitata)

$f: A \to \mathbb{R}$ $\quad x_0$ p.d.a. per A

Se $\lim_{x \to x_0} f(x) = \underline{\ell \in \mathbb{R}}$

allora $\exists c > 0 \quad \exists V \in \mathcal{I}_{x_0} : |f(x)| \leq c \quad \forall x \in (A \setminus \{x_0\}) \cap V$

<u>dim</u>

$x_0 = +\infty$

Hip $\lim_{x \to +\infty} f(x) = \ell \in \mathbb{R}$ $\qquad x \in \,]N, +\infty[$

$\forall \varepsilon > 0 \quad \exists N > 0 : \forall x \in A \quad x > N \Rightarrow \ell - \varepsilon < f(x) < \ell + \varepsilon$

$\Downarrow$

$\varepsilon = 1 \quad \exists N = N(1) > 0 : \forall x \in A \quad x > N \Rightarrow \ell - 1 < f(x) < \ell + 1$

$\Rightarrow |f(x)| < |\ell| + 1$

posto $c = |\ell| + 1$ si ha

$\exists c = |\ell| + 1 \quad \exists \,]N, +\infty[$ t.c. $\forall x \in A \cap \,]N, +\infty[\quad |f(x)| \leq c$

<u>Nota</u>

$\ell - 1 < f(x) < \ell + 1$ <u>ne</u> $-\max\{|\ell - 1|, |\ell + 1|\} < |f(x)| < \max\{|\ell - 1|, |\ell + 1|\}$

cioè $|f(x)| \leq \max\{|l-1|, |l+1|\}$

$$\leq \max\{|l|+1, |l|+1\} = |l|+1$$

Supponiamo, $x_0 \in \mathbb{R}$

$f \to l \quad \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in A \; 0 < |x - x_0| < \delta \quad l - \varepsilon < f(x) < l + \varepsilon$

Fissiamo $\underline{\underline{\varepsilon = 1}}$ $\exists \delta_1 > 0 \; \forall x \in A \; 0 < |x - x_0| < \delta \quad l - 1 < f(x) < l + 1$

Dunque, posto $c = |l| + 1$

$$\forall x \in (A \setminus \{x_0\}) \cap \,]x_0 - \delta, x_0 + \delta[\quad |f(x)| \leq c \qquad ↯$$

- $c \in \mathbb{R}$ $\quad c+\infty=+\infty \quad c-\infty=-\infty$

$c>0$
$$\begin{cases} c\cdot(+\infty)=+\infty & c\cdot(-\infty)=-\infty \\ \dfrac{+\infty}{c}=+\infty & \dfrac{-\infty}{c}=-\infty \\ \dfrac{c}{+\infty}=0^{+} & \dfrac{c}{-\infty}=0^{-} \end{cases} \qquad \frac{c}{\infty}=0$$

limite da dx $\quad$ lim da sin

$c<0$
$$\begin{cases} c\cdot(+\infty)=-\infty & c\cdot(-\infty)=-\infty \\ \dfrac{+\infty}{c}=-\infty & \dfrac{-\infty}{c}=+\infty \\ \dfrac{c}{+\infty}=0^{-} & \dfrac{c}{-\infty}=0^{+} \end{cases}$$

$$\frac{c}{0}=\infty$$

$c>0$ $\quad \dfrac{c}{0^+}=+\infty \quad \dfrac{c}{0^-}=-\infty$ $\quad$ (in generale $c/0=\infty$ se non distinguo)

$c<0$ $\quad \dfrac{c}{0^+}=-\infty \quad \dfrac{c}{0^-}=+\infty$ $\quad$ (//)

$$(+\infty)+(+\infty)=+\infty \qquad (+\infty)\cdot(+\infty)=+\infty$$

$$(-\infty)+(-\infty)=-\infty \qquad (-\infty)\,(-\infty)=+\infty$$

$$(-\infty)\cdot(+\infty)=-\infty$$

Forme indeterminate:

$$+\infty-\infty\ ;\quad \infty\cdot 0\ ;\quad \frac{\infty}{\infty}\ ;\quad \frac{0}{0}\ ;\quad 1^{\infty}\ ;\quad \infty^{0}$$

$\dfrac{0}{0}=\pi$ infatti $\pi\cdot 0=0$ $\qquad \dfrac{0}{0}=-\sqrt{2}$ infatti $-\sqrt{2}\cdot 0=0$

Teorema (Teorema algebrico)

$f, g : A \to \mathbb{R}$, x_0 p.d.a. per A

1) $\lim_{x \to x_0} f = l$ $\quad \lim_{x \to x_0} g(x) = m \Rightarrow \lim_{x \to x_0} (f+g)(x) = l+m$

2) " $\quad$ " $\Rightarrow \lim_{x \to x_0} (f \cdot g)(x) = l \cdot m$

3) " $\quad$ " $\quad m \neq 0 \Rightarrow \lim_{x \to x_0} \left(\frac{f}{g}\right)(x) = \frac{l}{m}$

Oss 1) Il Teorema non tiene conto dei casi $+\infty - \infty$, $0 \cdot \infty$ e $\frac{0}{0}$ o $\frac{\infty}{\infty}$

$x \to 0$

$f(x) = x$ $\quad \lim_{x \to +\infty} f = +\infty$ $\quad$ e $\quad \lim_{x \to +\infty} f+g = 0$

$g(x) = -x$ $\quad \lim_{x \to +\infty} g = -\infty$

$f(x) = x$ $\quad \lim_{x \to +\infty} f = +\infty$ $\quad \lim_{x \to +\infty} f+g = \lim_{x \to +\infty} (-x) = -\infty$

$g(x) = -2x$ $\quad \lim_{x \to +\infty} g = -\infty$

$f(x) = x+1$ $\quad \lim_{x \to +\infty} f = +\infty$ $\quad \lim_{x \to +\infty} f+g = 1$

$g(x) = -x$ $\quad \lim_{x \to +\infty} g = -\infty$

dim $x_0, l, m \in \mathbb{R}$ $\lim_{x \to x_0} f = l \iff f \xrightarrow[x \to x_0]{} l$

1) $f(x) \xrightarrow[x \to x_0]{} l \iff \forall \varepsilon > 0 \; \exists \delta_1 > 0 \; \forall x \in A \;\; 0 < |x - x_0| < \delta_1 \Rightarrow l - \varepsilon < f(x) < l + \varepsilon$

$g(x) \xrightarrow[x \to x_0]{} m \iff \forall \varepsilon > 0 \; \exists \delta_2 > 0 \; \forall x \in A \;\; 0 < |x - x_0| < \delta_2 \Rightarrow m - \varepsilon < g(x) < m + \varepsilon$

prendo $\delta = \delta_1 \wedge \delta_2 = \min\{\delta_1, \delta_2\}$

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in A \;\; 0 < |x - x_0| < \delta \Rightarrow \begin{matrix} l - \varepsilon < f(x) < l + \varepsilon \\ m - \varepsilon < g(x) < m + \varepsilon \end{matrix}$$

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in A \;\; 0 < |x - x_0| < \delta \Rightarrow l + m - 2\varepsilon < f + g < l + m + 2\varepsilon$$

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in A \;\; 0 < |x - x_0| < \delta \Rightarrow |(f+g) - (l+m)| < 2\varepsilon$$

$$\lim_{x \to x_0} (f+g)(x) = l + m$$

$2\varepsilon = \sigma$

$$\forall \sigma > 0 \; \exists \delta > 0 \; \forall x \in A \;\; 0 < |x - x_0| < \delta \Rightarrow |(f+g)(x) - (l+m)| < \sigma$$

2) $f(x) \to l \to \forall \varepsilon>0 \ \exists \delta_1>0 \ \forall x \in A \ 0<|x-x_0|<\delta_1 \Rightarrow l-\varepsilon<f(x)<l+\varepsilon$

$g(x) \to m \to \ '' \ \exists \delta_2>0 \ '' \ 0<|x-x_0|<\delta_2 \Rightarrow m-\varepsilon<g(x)<m+\varepsilon$

prendo $\delta = \delta_1 \wedge \delta_2$

$$\forall \varepsilon>0 \ \exists \delta>0 \ \forall x \in A \ 0<|x-x_0|<\delta \Rightarrow \begin{cases} l-\varepsilon<f(x)<l+\varepsilon \\ m-\varepsilon<g(x)<m+\varepsilon \end{cases}$$

obbiettivo $|f(x)g(x) - l\cdot m| < \varepsilon$

$$\forall \varepsilon>0 \ \exists \delta>0 : \forall x \in A \ 0<|x-x_0|<\delta \Rightarrow \begin{cases} |f(x)-l|<\varepsilon \\ |g(x)-m|<\varepsilon \end{cases}$$

$$|f(x)g(x)-lm| = |f(x)g(x) - f(x)m + f(x)m - lm|$$

$$\leq |f(x)g(x) - f(x)m| + |f(x)\cdot m - l\cdot m|$$

$$= |f(x)|\cdot|g(x)-m| + |m|\cdot|f(x)-l|$$

$f(x) \to l \in \mathbb{R}$
$\Rightarrow f(x)$ è limitata $\leq C \cdot |g(x)-m| + |m|\cdot|f(x)-l|$

$|g(x)-m|<\varepsilon$ $\quad < C\cdot\varepsilon + |m|\cdot\varepsilon = (C+|m|)\cdot\varepsilon$

$|f(x)-m|<\varepsilon$

$\forall \varepsilon > 0 \; \exists \delta > 0 : \forall x \in A \; 0 < |x - x_0| < \delta \Rightarrow |f(x)g(x) - l \cdot m|$

$$\wedge$$

$$(c + |m|) \cdot \varepsilon$$

$$\lim_{x \to x_0} f(x)g(x) = l \cdot m$$

3) Il rapporto non lo dimostriamo $\cancel{}$

Un teorema che è già contenuto nel precedente ma conviene enunciare lo stesso

Teorema

Siano date $f, g : A \to \mathbb{R}$ x_0 p.d.a. per A

1) $\lim_{x \to x_0} g(x) = +\infty$ $\Longrightarrow \exists \lim_{x \to x_0} (f+g)(x) = +\infty$

($f(x)$ limitata inferiormente)

2) $\lim_{x \to x_0} g(x) = 0$ $\Longrightarrow \underbrace{\exists \lim_{x \to x_0} (f \cdot g)(x) = 0}_{\text{limitato per infinitesimo è infinitesimo}}$

($f(x)$ limitata)

dim

1) $f(x) \xrightarrow[x \to x_0]{} +\infty$ ma $x_0 \in \mathbb{R}$

$|g(x)| \le c \quad \forall x \in A$

$\rightarrow \forall M > 0 \ \exists \delta > 0 \ \forall x \in A \ 0 < |x - x_0| < \delta \Rightarrow f(x) > M$

$\rightarrow \forall x \in A \quad -c \le g(x) \le c$

$$\Downarrow$$

$$\forall M>0\ \exists \delta>0 : \forall x\in A\quad 0<|x-x_0|<\delta \Rightarrow$$

$$f(x)+g(x) > \underline{\underline{M-c}}$$

$$\forall \bar{M} = M-c \overset{>0}{} \ \exists \delta>0 : \forall x\in A\quad 0<|x-x_0|<\delta \Rightarrow$$

$$f(x)+g(x) > \bar{M}$$

$$\lim_{x\to x_0} (f+g) = +\infty$$

2) $\lim_{x\to x_0} f(x) = 0$

g limitata

$$\begin{cases} \forall \varepsilon>0\ \exists \delta>0\ \forall x\in A\quad 0<|x-x_0|<\delta \Rightarrow -\varepsilon<f(x)<\varepsilon \\ \exists c>0 : \quad -c\le g(x)\le c\quad \forall x\in A \end{cases}$$

$$\forall \varepsilon>0\ \exists \delta>0 : \forall x\in A\quad 0<|x-x_0|<\delta$$

$$|f(x)\cdot g(x)| \le |f(x)|\cdot|g(x)| \le \underline{\varepsilon\cdot c}$$

$$\forall \sigma = c\varepsilon>0\ \exists \delta>0\ \forall x\in A\quad 0<|x-x_0|<\delta \Rightarrow |(f\cdot g)(x)|<\sigma \ \Downarrow$$

Il teorema algebrico sui limiti si traduce in un teorema algebrico sulle funzioni continue

<u>Teorema</u>

$f, g : A \to \mathbb{R}$ continue in $x_0 \in A$

allora

1) $(f+g)(x)$ è continua in x_0

2) $(f \cdot g)(x)$ " " " "

3) $\left(f/g\right)(x)$ " " " " , se $g(x_0) \neq 0$

La dimostrazione di questo teorema non viene fatta, in quanto ricalca l'analogo teorema sui limiti quando

x_0, $f(x_0) = l$, $g(x_0) = m \in \mathbb{R}$

Teorema (del confronto per i limiti)

$f, g: A \to \mathbb{R}$ $\quad x_0$ p.d.a. per A

1) $\bullet \lim_{x \to x_0} f(x) = l \quad \bullet \lim_{x \to x_0} g(x) = m$

$\bullet f(x) < g(x) \quad \forall x \in A \qquad \Longrightarrow l \leq m$

2) $\bullet \lim_{x \to x_0} f(x) = +\infty$

$\bullet f(x) < g(x) \ \forall x \in A \qquad \Longrightarrow \exists \lim_{x \to x_0} g(x) = +\infty$

(2 risultati)

3) $\bullet \lim_{x \to x_0} g(x) = -\infty$

$\bullet f(x) < g(x) \ \forall x \in A \qquad \Longrightarrow \exists \lim_{x \to x_0} f(x) = -\infty$

2 risultati

dim

1) Supponiamo $x_0, l, m \in \mathbb{R}$

$\forall \varepsilon > 0 \ \exists \delta_1 > 0 \ \forall x \in A \quad 0 < |x - x_0| < \delta_1 \Rightarrow l - \varepsilon < f(x) < l + \varepsilon$

$\forall \varepsilon > 0 \ \exists \delta_2 > 0 \ \forall x \in A \quad 0 < |x - x_0| < \delta_2 \Rightarrow m - \varepsilon < g(x) < m + \varepsilon$

prendiamo $\delta = \delta_1 \wedge \delta_2 = \min\{\delta_1, \delta_2\}$

tenendo conto che $f(x) < g(x) \ \forall x \in A$

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in A \quad 0 < |x - x_0| < \delta$

$\Downarrow$

$l - \varepsilon < f(x) < g(x) < m + \varepsilon$

e quindi $\forall \varepsilon > 0 \quad \ell - \varepsilon < m + \varepsilon$

ovvero $\forall \varepsilon > 0 \quad \ell - m < 2\varepsilon$

ovvero $\ell - m \leq 0$ ovvero $\ell \leq m$

2) $\lim_{x \to x_0} f(x) = +\infty \qquad f(x) < g(x) \ \forall x \in A$

$\Leftrightarrow \forall M > 0 \ \exists \delta > 0 \ \forall x \in A \ 0 < |x - x_0| < \delta \Rightarrow M < f(x)$

$\Rightarrow$ " " " " $\Rightarrow M < g(x)$

$\Rightarrow \lim_{x \to +\infty} g(x) = +\infty$

etc

__Oss__ le disuguaglianze, passando al limite, si "rilassano" ovvero

"<" $\xrightarrow[\text{al limite}]{\text{passando}}$ "≤"

__Esempio__ $f(x) = \frac{1}{x^2} > 0 \ \forall x \in \mathbb{R}$, però

$\lim_{x \to +\infty} f(x) = 0$!!

Teorema (dei due Carabinieri)

$f, g, h : A \to \mathbb{R}$ $\quad x_0$ p.d.a. per A

1) $f(x) \leq g(x) \leq h(x) \quad \forall x \in A$

2) $\lim_{x \to x_0} f(x) = l = \lim_{x \to x_0} h(x)$

allora

$\exists \lim_{x \to x_0} g(x) = l$

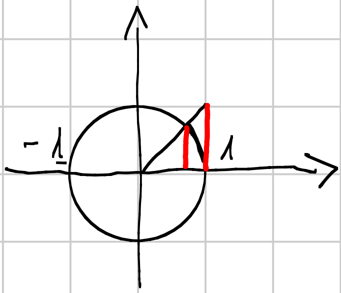
dim

Esempio (Importante)

$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$$

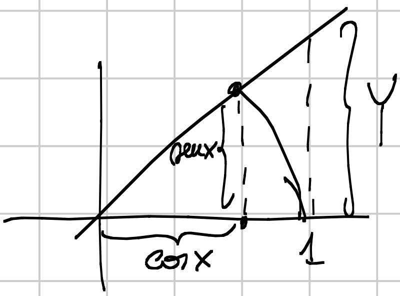
Note: è della forma $\frac{0}{0}$!

dim



Preso $0 < x < \frac{\pi}{2}$

$\operatorname{sen} x < x < \operatorname{Tg} x$ da cui



$\cos x : 1 = \operatorname{sen} x : Y$

$Y = \operatorname{Tg} x$

$\operatorname{sen} x < x < \operatorname{Tg} x \qquad x \in]0, \frac{\pi}{2}[$

$1 < \frac{x}{\operatorname{sen} x} < \frac{1}{\cos x} \qquad$ "

$1 > \frac{\operatorname{sen} x}{x} > \cos x \qquad$ "

poi

$1 > \frac{\operatorname{sen}(-x)}{-x} > \cos(-x) \qquad x \in]-\frac{\pi}{2}, 0[$

$1 > \frac{+\operatorname{sen} x}{+x} > \cos x \qquad$ "

$$\cos x < \frac{\operatorname{sen} x}{x} < 1 \qquad \forall x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\setminus \{0\}$$

$f(x)$ $\qquad$ $h(x)$

$$\left.\begin{array}{l} \lim_{x \to 0} f(x) = 1 \\ \lim_{x \to 0} h(x) = 1 \end{array}\right\} \xRightarrow{\text{th. carab.}} \lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$$

Esempio

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2} \qquad \left(\frac{0}{0}\right)$$

$$\frac{1 - \cos x}{x^2} = \frac{1 - \left(\cos^2 \frac{x}{2} - \operatorname{sen}^2 \frac{x}{2}\right)}{x^2}$$

$$= \frac{2 \operatorname{sen}^2 \frac{x}{2}}{x^2} = \frac{1}{2} \frac{\operatorname{sen}^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2}$$

$$= \frac{1}{2} \left[\frac{\operatorname{sen} \frac{x}{2}}{\frac{x}{2}}\right]^2 \xrightarrow[x \to 0]{} \frac{1}{2}$$

$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = \lim_{\frac{x}{2} \to 0} \frac{\operatorname{sen} \frac{x}{2}}{\frac{x}{2}} = 1$$