Lezione n.ro 9 - 13 ottobre 2011

Titolo nota 10/10/2011

$P_n = n! =$ n.ro di permutazioni (ordini) su un insieme di n elementi

$$n! \quad \begin{cases} 0! = 1 \\ (n+1)! = (n+1)\cdot n! \end{cases}$$

$$D_{n,k} = \frac{P_n}{P_{n-k}} = \frac{n!}{(n-k)!} = n\cdot(n-1)\cdot\ldots\cdot(n-k+1)$$

= disposizioni di k oggetti scelti da un insieme di n oggetti $(n \geq k)$

<u>Oss</u> $D_{n,0} \equiv P_n$

<u>Oss</u> $D_{n,k}$ tiene conto dell'ordine

$$C_{n,k} = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$ non tiene conto dell'ordine

$0 \leq k \leq n$

= Combinazioni di k oggetti presi da un insieme di n oggetti

= n.ro dei sottoinsiemi di k elementi presi da un insieme avente n elementi

Esempio

$A = \{1, 2, 3, 4, 5\}$

$$\binom{5}{0} = 1 = \binom{5}{5}$$

$$\binom{5}{1} = 5 = \binom{5}{4}$$

$$\binom{5}{2} = 10 = \binom{5}{3}$$

$A = \{1, 2, 3, 4, 5, 6\}$

$$\binom{6}{0} = 1 = \binom{6}{6}$$

$$\binom{6}{1} = 6 = \binom{6}{5}$$

$$\binom{6}{2} = 15 = \binom{6}{4}$$

$$\binom{6}{3} = 20$$

Teorema $\forall n \in \mathbb{N},\ \forall k \in \{0, \dots, n\}$

1) $\binom{n}{0} = \binom{n}{n} = 1$

2) $C_{n,k} = C_{n,(n-k)}$

dim

1) $\exists \binom{n}{0} = \binom{n}{n}$

$$\binom{n}{0} = \frac{n!}{0!\,(n-0)!} = 1 \qquad 1 = \frac{n!}{n!\,(n-n)!} = \binom{n}{n}$$

3) $\exists C_{n,k} = C_{n,(n-k)}$

$$C_{n,k} = \binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,.\,(n-(n-k))!} = \binom{n}{n-k} = C_{n,n-k}$$

$\Downarrow$

Oss $C_{n,k} = \frac{D_{n,k}}{k!} = \frac{1}{k!}\,\frac{n!}{(n-k)!} = \binom{n}{k}$

$\forall k \in \{0, 1, \dots, n\}$

$\binom{n}{k} =$ coefficiente binomiale definito $\forall k \in \{0, 1, \dots n\}$

Teorema

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad \forall n \in \mathbb{N} \quad \forall k \in \{1, \dots, n\}$$

dim

$$\binom{n+1}{k} \underset{\text{Def}}{=} \frac{(n+1)!}{k!\,(n+1-k)!}$$

$$\binom{n}{k} + \binom{n}{k-1} \underset{\text{Def}}{=} \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k-1)!\,(n+1-k)!}$$

$$= \frac{n!}{(k-1)!\,(n-k)!} \left(\frac{1}{k} + \frac{1}{n+1-k} \right)$$

$$= \frac{n!}{(k-1)!\,(n-k)!} \cdot \left(\frac{n+1}{k \cdot (n+1-k)} \right)$$

$$= \frac{(n+1)!}{k!\,(n+1-k)!} = \binom{n+1}{k}$$

Il triangolo di Pascal (Tartaglia)

$$\binom{0}{0}=1$$

$$1=\binom{1}{0} \qquad \binom{1}{1}=1 \qquad \binom{2}{1}=\binom{1}{1}+\binom{1}{0} = 2$$

$$1=\binom{2}{0} \qquad \binom{2}{1}=2 \qquad \binom{2}{2}=1 \qquad \binom{3}{1}=\underbrace{\binom{2}{0}+\binom{2}{1}}_{=3}$$

$$1=\binom{3}{0} \qquad \binom{3}{1}=3 \qquad \binom{3}{2}=3 \qquad \binom{3}{3}=1$$

$$1=\binom{4}{0} \qquad \binom{4}{1} \qquad \binom{4}{2} \qquad \binom{4}{3} \qquad \binom{4}{4}=1$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

etc

Teorema (Binomio di Newton)

$\forall a,b \in \mathbb{R}, \ \forall n \in \mathbb{N}$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot a^k b^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Nota $(a+b)^1 = a+b$

$$\sum_{k=0}^{1} \binom{1}{k} a^k b^{1-k}$$

$$= \binom{1}{0} b + \binom{1}{1} a = a+b$$

$(a+b)^2 = a^2+2ab+b^2$

$$\sum_{k=0}^{2} \binom{2}{k} a^k b^{2-k}$$

$$= \binom{2}{0} b^2 + \binom{2}{1} ab + \binom{2}{2} a^2$$

$$= b^2 + 2ab + a^2$$

dim

$n=0$ $(a+b)^0 = 1$

base dell'induzione

$$(a+b)^0 = \sum_{k=0}^{0} \binom{0}{k} a^k b^{0-k}$$

$$= \binom{0}{0} \cdot a^0 b^0$$

$$= 1$$

Hp induttiva $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Ts $(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}$

per ipotesi induttiva!

$$(a+b)^{n+1} = (a+b)\cdot(a+b)^{n} \overset{\text{per ipotesi induttiva!}}{=} (a+b)\sum_{k=0}^{n}\binom{n}{k}a^{k}b^{n-k}$$

$$= a\cdot\sum_{k=0}^{n}\binom{n}{k}a^{k}b^{n-k} + b\cdot\sum_{k=0}^{n}\binom{n}{k}a^{k}b^{n-k}$$

$$= \sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k} + \sum_{k=0}^{n}\binom{n}{k}a^{k}b^{n+1-k}$$

$$= \binom{n}{n}a^{n+1}\overset{1}{\cancel{b^{0}}} + \sum_{k=0}^{n-1}\binom{n}{k}\cdot a^{k+1}b^{n-k} + \sum_{k=1}^{n}\binom{n}{k}a^{k}b^{n+1-k} + \binom{n}{0}\cdot\overset{1}{\cancel{a^{0}}}\,b^{n+1}$$

$$= \binom{n+1}{n+1}a^{n+1} + \underbrace{\sum_{k=0}^{n-1}\binom{n}{k}a^{k+1}b^{n-k}}_{\text{Qui devo cambiare gli indici: voglio che vada da 1 a } n} + \underbrace{\sum_{k=1}^{n}\binom{n}{k}a^{k}b^{n+1-k}}_{\text{non lo tocco}} + \binom{n+1}{0}b^{n+1}$$

poniamo $l = k+1$: in tal modo $\sum_{k=0}^{n-1}\cdots \longrightarrow \sum_{l=1}^{n}\cdots$

dunque $\ell = k+1$ (e quindi $k = \ell - 1$)

In tal modo

$$\sum_{k=0}^{n-1} \binom{n}{k} a^{k+1} b^{n-k} = \sum_{\ell=1}^{n} \binom{n}{\ell-1} a^{\ell} b^{n+1-\ell}$$

e quindi la nostra potenza diventa

$$\binom{n+1}{n+1} a^{n+1} + \sum_{\ell=1}^{n-1} \binom{n}{\ell-1} a^{\ell} b^{n+1-\ell} + \sum_{k=1}^{n} \binom{n}{k} a^{k} b^{n+1-k} + \binom{n+1}{0} b^{n+1}$$

$$= \binom{n+1}{n+1} a^{n+1} + \sum_{r=1}^{n-1} \binom{n}{r-1} a^{r} b^{n+1-r} + \sum_{r=1}^{n} \binom{n}{r} a^{r} b^{n+1-r} + \binom{n+1}{0} b^{n+1}$$

ho posto $r=\ell$ ho posto $r=k$.

$$= \binom{n+1}{n+1} a^{n+1} + \sum_{k=1}^{n} \underline{\left[\binom{n}{k-1} + \binom{n}{k}\right]} a^{k} b^{n+1-k} + \binom{n+1}{0} b^{n+1}$$

$$= \binom{n+1}{n+1} a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^{k} b^{n+1-k} + \binom{n+1}{0} b^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{k} b^{n+1-k}$$

Esempio

$$\sum_{k=0}^{5} \binom{5}{k} a^k b^{5-k}$$

$$= \binom{5}{0} b^5 + \binom{5}{1} a^1 b^4 + \binom{5}{2} a^2 b^3 + \binom{5}{3} a^3 b^2$$

$$+ \binom{5}{4} a^4 b^1 + \binom{5}{5} a^5$$

$$= \sum_{i=0}^{5} \binom{5}{i} a^i b^{5-i} = \sum_{z=0}^{5} \binom{5}{z} a^z b^{5-z}$$

$$\sum_{k=0}^{5} \binom{5}{k} a^k b^{5-k} = \underbrace{\binom{5}{0} b^5}_{k=0} + \sum_{k=1}^{5} \binom{5}{k} a^k b^{5-k}$$

$$= \underbrace{\binom{5}{0} b^5}_{k=0} + \sum_{k=1}^{4} \binom{5}{k} a^k b^{5-k} + \underbrace{\binom{5}{5} a^5}_{k=5}$$

Esercizio Determinare estremo superiore e inferiore dell' insieme $A = \{ \underbrace{\frac{1}{2^n} + \frac{1}{3^m}}_{Q(n,m)} ; n,m \in \mathbb{N} \}$

$\xrightarrow{m}$

$n \downarrow$

$$\frac{1}{2^0} + \frac{1}{3^0} ; \quad \frac{1}{2^0} + \frac{1}{3^1} ; \quad \frac{1}{2^0} + \frac{1}{3^2} ; \quad ----$$

$$\frac{1}{2^1} + \frac{1}{3^0} ; \quad \frac{1}{2^1} + \frac{1}{3^1} ; \quad ---$$

$$\frac{1}{2^2} + \frac{1}{3^0} ;$$

$$1+1 ; \quad 1 + \frac{1}{3} ; \quad 1 + \frac{1}{9} ; \quad ------> 1$$

$$1+\frac{1}{2} ; \quad \frac{1}{2} + \frac{1}{3} ; \quad \frac{1}{2} + \frac{1}{9} ; \quad ----> \frac{1}{2}$$

$$\frac{1}{4}+1 ; \quad \frac{1}{4} + \frac{1}{3} ; \quad \longrightarrow \frac{1}{4}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$1 \qquad \frac{1}{3} \qquad \frac{1}{9}$$

$$\longrightarrow 0$$

Dalle mia osservazione segue che

$$0 < \frac{1}{2^n} + \frac{1}{3^m} \leq 2 = \frac{1}{2^0} + \frac{1}{3^0} = Q(0,0)$$

(1) $2 = \max A = \sup A$

(2) $0 = \inf A$ (che non è un minimo)

(1) $2 \in A$ (ok)

$$2 \geq \frac{1}{2^n} + \frac{1}{3^m} \quad \forall n,m \in \mathbb{N}$$

$$\Uparrow$$

$$1 \geq \frac{1}{2^n} \quad e \quad 1 \geq \frac{1}{3^m} \quad \forall n,m \in \mathbb{N}$$

$$\Updownarrow$$

$$2^n \geq 1 \quad e \quad 3^m \geq 1 \quad \forall n,m \in \mathbb{N}$$

(2) (•) $0 < \frac{1}{2^n} + \frac{1}{3^m}$ ovvio $\forall n,m \in \mathbb{N}$

(••) $\forall \varepsilon > 0 \quad \exists \bar{n}, \bar{m} \in \mathbb{N} : \quad \frac{1}{2^{\bar{n}}} + \frac{1}{3^{\bar{m}}} < \varepsilon$

$$\Uparrow$$

$\forall \varepsilon > 0 \quad \exists \bar{n} : \quad \frac{1}{2^{\bar{n}}} < \varepsilon/2$

" $\quad \exists \bar{m} : \quad \frac{1}{3^{\bar{m}}} < \varepsilon/3$

$\forall \varepsilon$ $\exists \bar{n} : 2^{\bar{n}-1} > 1/\varepsilon$

$\exists \bar{m} : 3^{\bar{m}-1} > 1/\varepsilon$

$$\forall \varepsilon > 0 \begin{cases} \exists \bar{n} : \bar{n} - 1 > \log_2 1/\varepsilon \\ \exists \bar{n} : \bar{n} - 1 > \log_3 1/\varepsilon \end{cases}$$

$$\forall \varepsilon > 0 \begin{cases} \exists \bar{n} : \bar{n} > \log_2 1/\varepsilon + 1 \\ \exists \bar{n} : \bar{n} > \log_3 1/\varepsilon + 1 \end{cases}$$

Vero x il principio di Archimede

$\Downarrow$

Esercizio

$\# A = n$ allora $\# \mathbb{P}(A) = 2^n$ (senza binomio Newton)

dim

Vediamo gli esempi

$n = 0$ $\quad A = \emptyset$ $\qquad \mathbb{P}(A) = \mathbb{P}(\emptyset) = \{\emptyset\}$

non ci sono elementi

e dunque

$$\# \mathbb{P}(A) = \# \mathbb{P}(\emptyset) = 1$$

$$2^0 = 1 \checkmark$$

$n = 1$ $\quad \# A = 1$ $\qquad \mathbb{P}(A) =$ insieme dei sottoinsiemi di A

$A = \{a\}$

$$= \{\emptyset, \{a\}\}$$

$$\# \mathbb{P}(A) = 2 = 2^1 \checkmark$$

$n = 2$ $\quad \# A = 2$ $\qquad \mathbb{P}(A) = \{\emptyset, A, \{a\}, \{b\}\}$

$A = \{a, b\}$

$$\# \mathbb{P}(A) = 2^2 = 4 \checkmark$$

$\underline{n=3}$ $\quad \# A = 3 \quad \mathbb{P}(A) = \{\emptyset, A, \{a\}, \{b\}, \{c\}$

$A = \{a, b, c\} \qquad \ldots \}$

$$\# \mathbb{P}(A) = 8$$

$$2^3 = 8 \quad \checkmark$$

$\underline{\text{Hp}}$ induttiva $\quad \# A = n$ allora $\# \mathbb{P}(A) = 2^n$

$\underline{\text{Ts}}$ $\quad \# A = n+1$ allora $\# \mathbb{P}(A) = 2^{n+1}$

$$A = \{x_1, x_2, \ldots, x_n, x_{n+1}\}$$

$$= \{x_1, \ldots, x_n\} \cup \{x_{n+1}\}$$

$$= B \cup \{x_{n+1}\} \qquad \# B = n$$

ovvero posso applicare l'Hp ind. a B!!!

$C \subseteq A$

- $x_{n+1} \in C \Rightarrow C = D \cup \{x_{n+1}\}$ con $D \in \mathbb{P}(B)$
- $x_{n+1} \notin C \Rightarrow C \in \mathbb{P}(B)$

Quanti sono i sottoinsiemi $C \ni x_{n+1}$?

Essendo $C = D \cup \{x_{n+1}\}$ con $D \in \mathcal{P}(B)$

sono tanti quanti i sottoinsiemi di B

$\Rightarrow$ sono 2^n

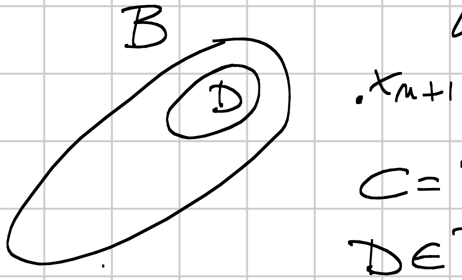
Quanti sono i sottoinsiemi $C \not\ni x_{n+1}$?

Essendo $C \in \mathcal{P}(B)$ (non contiene x_{n+1} !),

questi sono 2^n

Ma allora $\# \mathcal{P}(A) = 2^n + 2^n = 2^{n+1}$

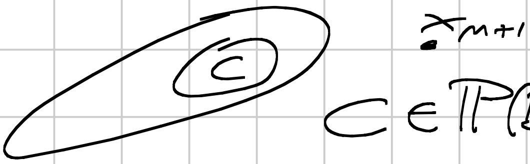
Quando $x_{n+1} \in C$



$C = D \cup \{x_{n+1}\}$

$D \in \mathcal{P}(B)$

Quando $x_{n+1} \notin C$



$C \in \mathcal{P}(B)$

Oss $\quad 2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$

$\underbrace{2^n}_{\text{n.ro di}}$

Tutti i sottoinsiemi di A $\quad \#A = n$

$$= \underbrace{\binom{n}{0}}_{\text{sottoinsiemi con 0 el.ti}} + \underbrace{\binom{n}{1}}_{\text{sottoinsieme con 1 elemento}} + \dots + \underbrace{\binom{n}{n}}_{\text{sottoins. con n el.ti}}$$

Oss

1) $\#A = n \qquad \#\mathbb{P}(A) = 2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$

$$= \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$$

2) Essendo $(1-1)^n = 0 \quad \forall n \geq 1$

Essendo $(1-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$

$$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} +$$

si deduce

$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \qquad \forall n \geq 1$$

Esempio

$n=3 \quad \sum_{k=0}^{3} \binom{3}{k}(-1)^k$

$$= \binom{3}{0} - \binom{3}{1} + \binom{3}{2} - \binom{3}{3}$$

$$= 1 - 3 + 3 - 1$$

etc