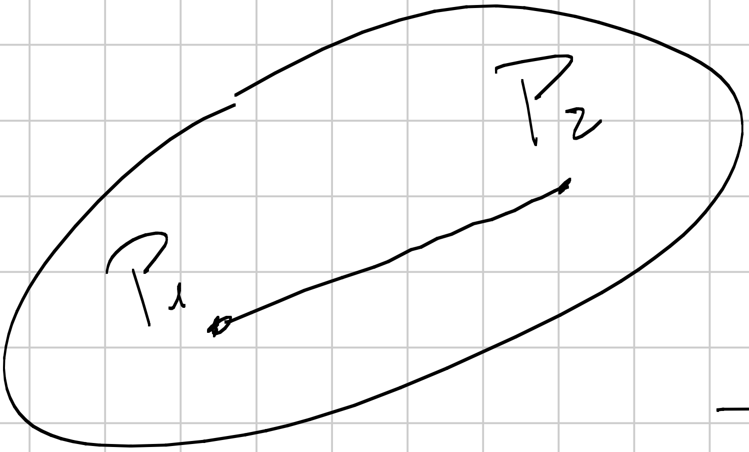
Convessità in breve



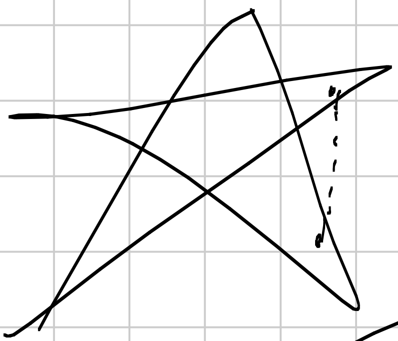
Ω si dice convesso

se

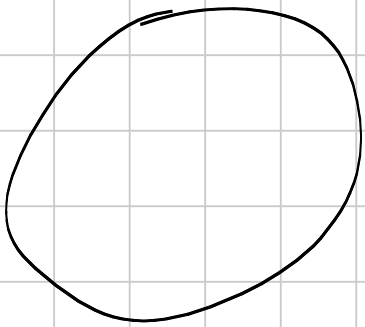
presi due punti P_1 e $P_2 \in \Omega$

il segmento che li congiunge

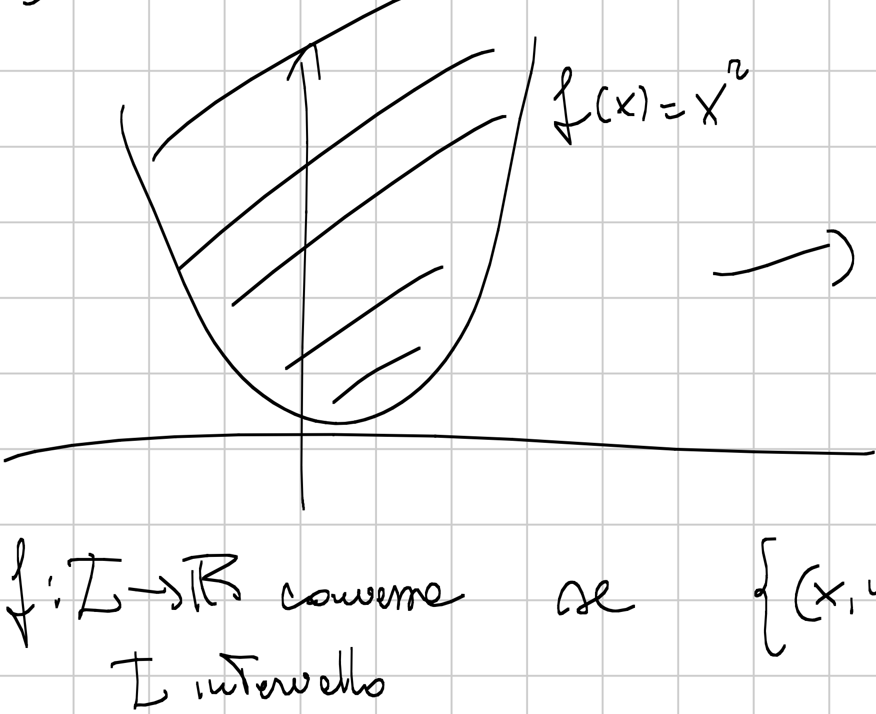
è contenuto in Ω



NON è convesso



convesso



$f(x) = x^2$

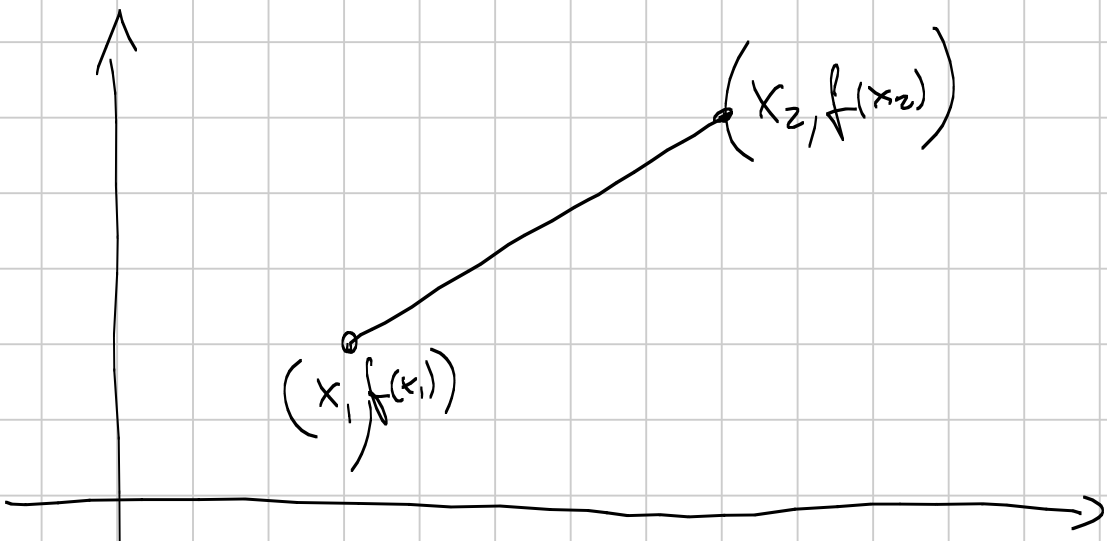
$\longrightarrow \{(x,y) : f(x) \le y,\ x \in \mathbb{R}\}$

$f : I \to \mathbb{R}$ convessa se $\{(x,y) : f(x) \le y,\ x \in I\}$ è convesso

I intervallo

Def. Dato $f: I \to \mathbb{R}$, dove I è un intervallo, questa si dice "convessa" se $\forall x_1, x_2 \in I$

$$f(x_1 \cdot t + x_2(1-t)) \le t f(x_1) + (1-t) f(x_2)$$



$$P(t) = (t x_1 + (1-t) x_2, \ t f(x_1) + (1-t) f(x_2))$$

$$= t (x_1, f(x_1)) + (1-t)(x_2, f(x_2))$$

$t = 0 \quad P(0) = (x_2, f(x_2)) \qquad t = 1 \quad P(1) = (x_1, f(x_1))$

$$\begin{cases} x = t x_1 + (1-t) x_2 \\ y = t f(x_1) + (1-t) f(x_2) \end{cases} \qquad \begin{cases} x - x_2 = t(x_1 - x_2) \\ y - f(x_2) = t(f(x_1) - f(x_2)) \end{cases}$$

$$\frac{y - f(x_2)}{f(x_1) - f(x_2)} = \frac{x - x_2}{x_1 - x_2}$$

Interpretazione Geometrica

I punti $\left(tx+(1-t)y,\ t f(x)+(1-t) f(y)\right)$ $t\in[0,1]$

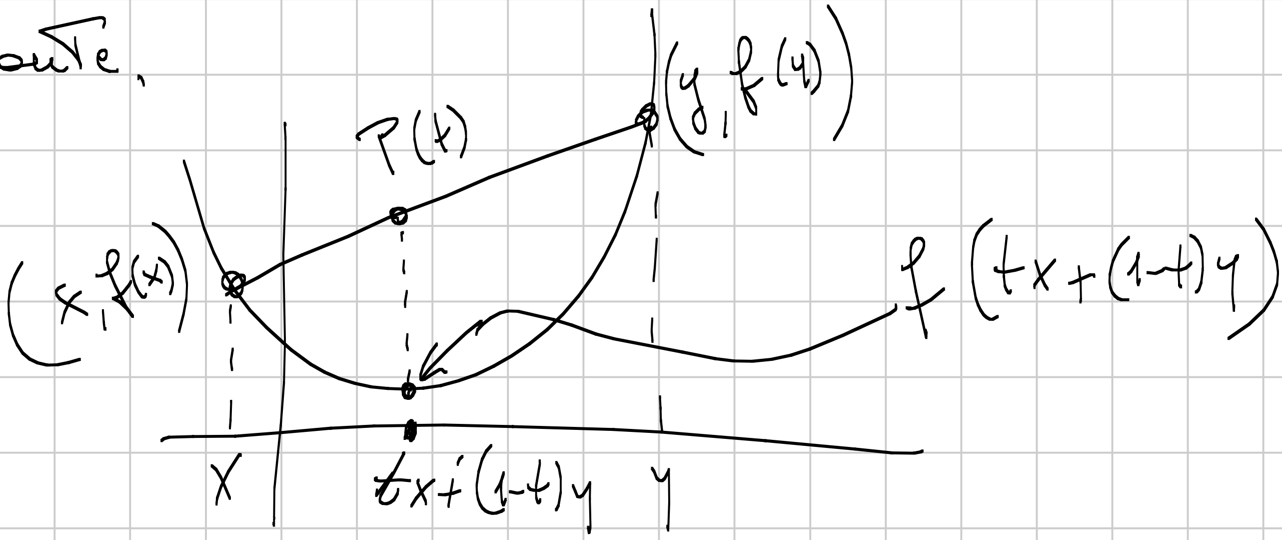
descrivono il segmento che congiunge

$(x,f(x))$ e $(y,f(y))$ che sono punti del grafico

di f. Dunque una funzione convessa

ha il grafico che sta "sotto" ogni sua

secante.



Problema $f(x)=x$ è convessa?

Def f concava se $-f$ convessa

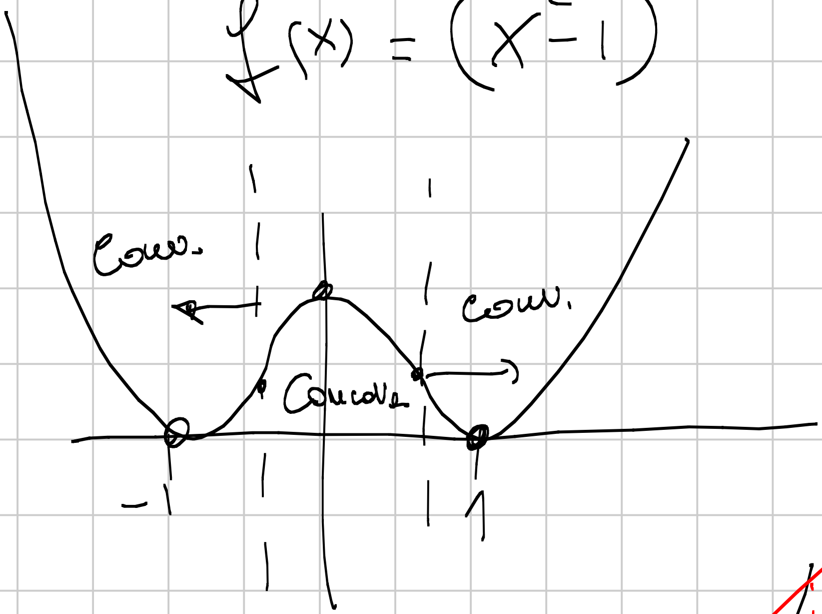
• Strette convessità $f(\cdots) < \cdots \quad \forall t \in]0,1[$

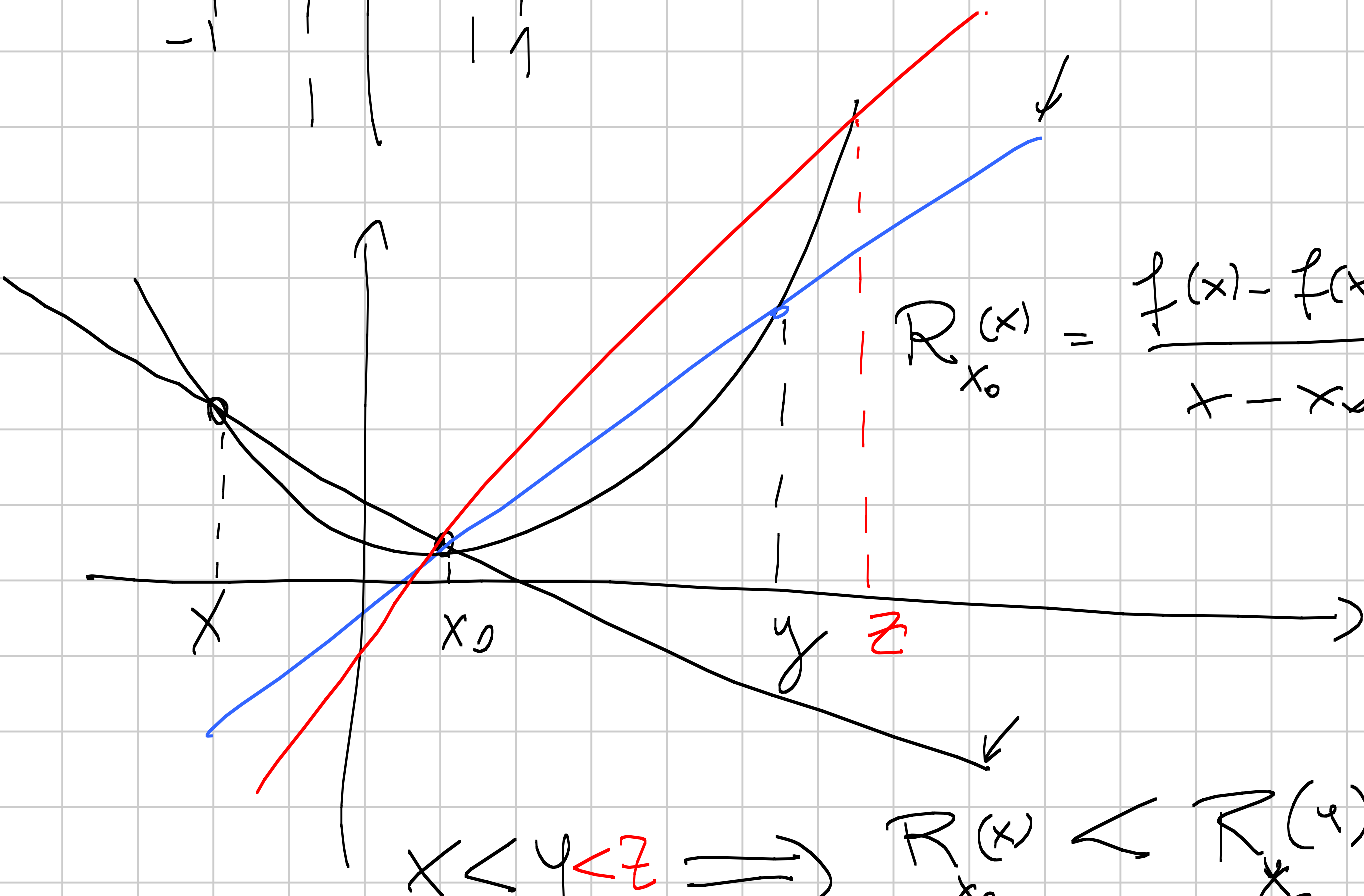
<u>Esempi</u> $f(x) = -\log x$ CONVESSA

$f(x) = e^x$ "

$f(x) = 1 - x^2$ CONCAVA

$f(x) = (x^2-1)^2$ NE' CONVESSA
NE' CONCAVA





$$R_{x_0}(x) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$x < y < z \implies R_{x_0}(x) < R_{x_0}(y) < R_{x_0}(z)$$

Pb esempio di f.ni convesse e/o concave

$f(x) = x^2$ $\quad f(x) = \log x$ $\quad f(x) = |x|$

$f(x) = \operatorname{sen} x$ $\quad f(x) = x \cdot a + b$

Teorema $f : I \to \mathbb{R}$ $\quad I$ intervallo

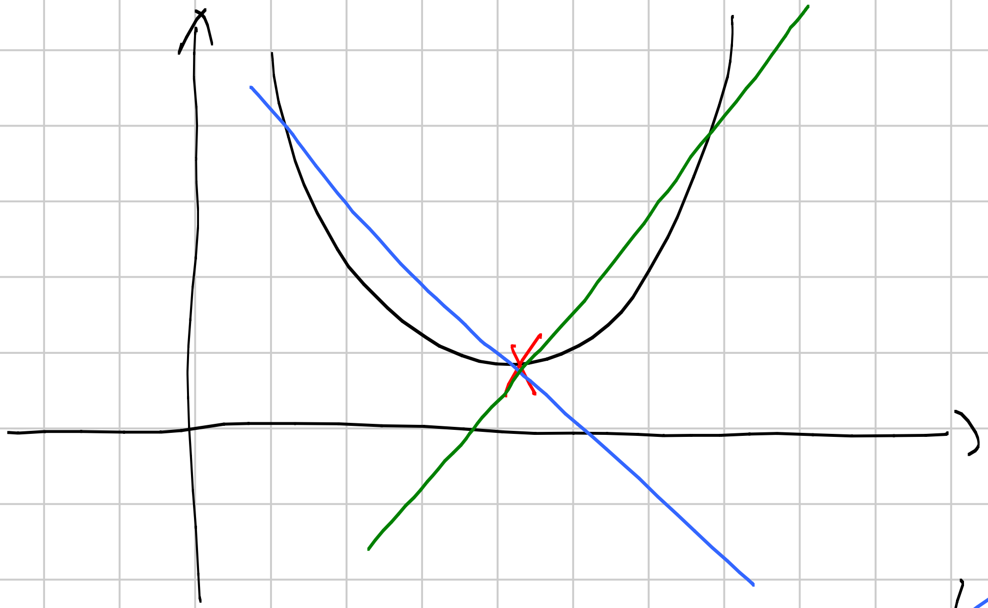
allora

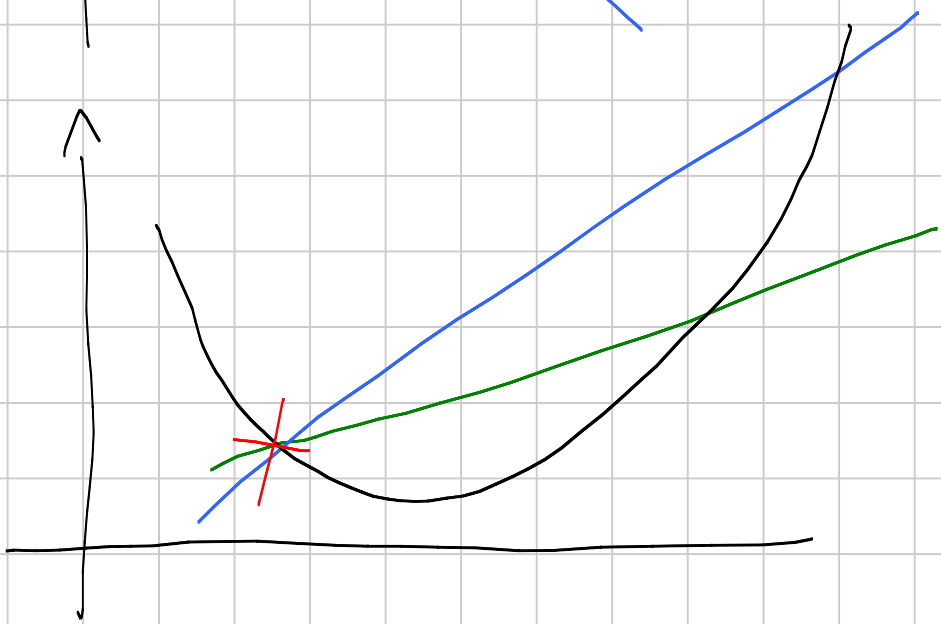
f convessa su I $\boxed{\text{sse}}$ $\forall x_0 \in I$ $\; R_{x_0}(x)$ è monotona deb. crescente

$$\frac{f(x) - f(x_0)}{x - x_0} \nearrow \implies \exists \lim_{x \to x_0^{\pm}} R_{x_0}(x)$$

$f \in C^1(I)$ $\qquad f$ convessa $\iff f' \nearrow_{=}$

$f \in C^2(I)$ $\qquad f$ convessa $\iff f' \nearrow_{=} \iff f'' \geq 0$





Teorema (fondamentale)

$f: I \to \mathbb{R}$ è convessa su I

se

$\forall x_0 \in I \quad \dfrac{f(x)-f(x_0)}{x-x_0}$ è crescente

dim

$\Leftarrow \quad x<y \quad \dfrac{f(x)-f(x_0)}{x-x_0} < \dfrac{f(y)-f(x_0)}{y-x_0}$

$x<x_0<y \qquad (y-x_0)f(x)-(y-x_0)f(x_0) > (x-x_0)f(y)-(x-x_0)f(x_0)$

$$(y-x_0)f(x_0)-(x-x_0)f(x_0) < (y-x_0)f(x)-(x-x_0)f(y)$$

$$\cancel{(y-x)}\; f(x_0) \le \underbrace{\frac{(y-x_0)}{y-x}}_{=t} f(x) + \underbrace{\frac{(x_0-x)}{y-x}}_{=1-t} f(y)$$

$$tx+(1-t)y = \frac{(y-x_0)x}{y-x} + \frac{(x-x_0)}{y-x}y$$

$=$

$x < y \qquad x_0 \in]x, y[\Rightarrow \exists t :$

$$x_0 = tx + (1-t)y$$

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x) - f(x_0)}{x - tx - (1-t)y} = \frac{f(x) - f(x_0)}{(1-t)(x-y)}$$

$$= \frac{1}{(1-t)} \frac{f(x_0) - f(x)}{y - x} \leq \frac{1}{1-t} \frac{tf(x) + (1-t)f(y) - f(x)}{y-x}$$

$$= \frac{1}{\cancel{1-t}} \frac{\cancel{(1-t)}\left[f(y) - f(x)\right]}{y-x} =$$

$$= \frac{tf(y) - tf(x)}{t(y-x)}$$

$$= \frac{tf(y) + f(y) - f(y) - tf(x)}{t(y-x)}$$

$$= \frac{f(y) - \left[(1-t)f(y) + tf(x)\right]}{t(y-x)} \leq \frac{f(y) - f(x_0)}{y - y + ty - tx}$$

$$= \frac{f(y) - f(x_0)}{y - \left[(1-t)y + tx\right]} = \frac{f(y) - f(x_0)}{y - x_0}$$

$x_0 < x < y$

$$\frac{f(x)-f(x_0)}{x-x_0} \le \frac{f(y)-f(x_0)}{y-x_0}$$

$$(y-x_0)\big(f(x)-f(x_0)\big) \le (x-x_0)\big(f(y)-f(x_0)\big)$$

$$(y-x_0)f(x) \le (x-x_0)f(y) + (y-x_0)f(x_0) + (x_0-x)f(x_0)$$

$$(y-x_0)f(x) \le (x-x_0)f(y) + (y-x)f(x_0)$$

$$f(x) \le \underbrace{\frac{x-x_0}{y-x_0}}_{=t} f(y) + \frac{y-x}{y-x_0} f(x_0)$$

$$f(x) \le t f(y) + (1-t) f(x_0)$$

$$f(x) = f(x - (1-t)x_0 + (1-t)x_0)$$

$$= f\left(t \cdot \frac{x-(1-t)x_0}{t} + (1-t)x_0\right)$$

$$\le t f\left(\frac{x-(1-t)x_0}{t}\right) + (1-t)f(x_0)$$

voglio $\frac{x-(1-t)x_0}{t} = y$ $\qquad \frac{x-x_0}{y-x_0} = t\,\cancel{(y-x_0)}$

$$x < y < x_0$$

$$\frac{f(x)-f(x_0)}{x-x_0} \leq \frac{f(y-f(x_0)}{y-x_0}$$

$$(y-x_0)\left(f(x)-f(x_0)\right) \leq (x-x_0)\left(\underline{f(y)-f(x_0)}\right)$$

$$(x_0-x)\,f(y) \leq (x_0-x)\,f(x_0) + (x_0-y)\,f(x) - (x_0-y)\,f(x_0)$$

$$\leq (x_0-\cancel{y})\,f(x) + (y-x)\,f(x_0)$$

$$f(y) \leq \underbrace{\frac{x_0-y}{x_0-x}}_{=t} f(x) + \frac{y-x}{x_0-x} f(x_0)$$

$$x < y < x_0$$

$$f(y) \leq t\,f(x) + (1-t)\,f(x_0)$$

$$f(y) = f\left(y - (1-t)x_0 + (1-t)x_0\right)$$

$$= f\left(t \cdot \frac{y-(1-t)x_0}{t} + (1-t)\,x_0\right)$$

$$\frac{y-(1-t)x_0}{t} \overset{?}{=} x \qquad y-(1-t)x_0 = tx$$

$$y - x_0 = t\,(x-x_0)$$

$$t = \frac{y-x_0}{x-x_0}$$

Esercizio: calcolare il seguente limite (se esiste)

$$\lim_{x\to+\infty} \frac{1}{x}\int_2^x \frac{1}{\log t}\,dt$$

$$= \lim_{x\to+\infty} \frac{\int_2^x \frac{1}{\log t}\,dt}{x} \overset{\text{Hopital}}{=} \lim_{x\to+\infty} \frac{1/\log x}{1} = 0$$

Esercizio Studiare, al variare di $\alpha\in\mathbb{R}$, la convergenza di $\int_0^1 \left(1-\frac{\operatorname{sen} x}{x}\right)^\alpha dx$

$f(x) = 1 - \frac{\operatorname{sen} x}{x}$ è continua su $\mathbb{R}$, purché venga estesa in $x=0$ con il valore 0

$$\lim_{x \to 0} f(x) = 1 - \lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 0$$

$$\tilde{f}(x) = \begin{cases} f(x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$ $\tilde{f}$ è continua

$\alpha \geq 0$ $\left(f(x)\right)^{\alpha} = \left(1 - \frac{\operatorname{sen} x}{x}\right)^{\alpha}$ è uniformemente

continua su $[0,1]$

allora $\left(f(x)\right)^{\alpha}$ è R-integrabile su $[0,1]$

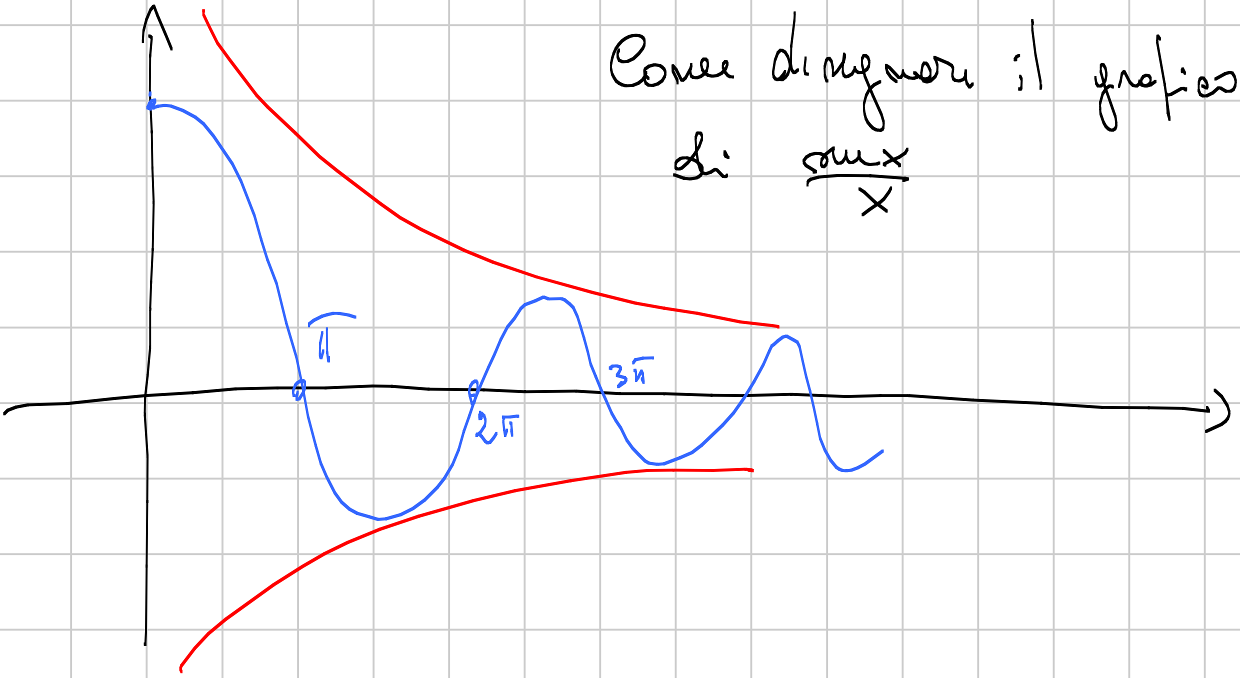
$\alpha < 0$ $\left(f(x)\right)^{\alpha}$ ha un asintoto verticale

in $x=0$

Osserviamo che $\lim_{x \to 0^+} \left(f(x)\right)^{\alpha} = +\infty$

quando $\alpha < 0$

Come disegnare il grafico di $\dfrac{\operatorname{sen} x}{x}$



$$1-\frac{\operatorname{sen} x}{x}=\frac{x-\operatorname{sen} x}{x}=\frac{\cancel{x}-\left(\cancel{x}-\frac{x^3}{6}+o(x^4)\right)}{x}$$

$$=\frac{x^2}{6}+o(x^3)\sim\frac{x^2}{6}\qquad(x\to 0)$$

$$\left(f(x)\right)^{\alpha}\sim\frac{x^{2\alpha}}{6^{\alpha}}\qquad(x\to 0)$$

$\displaystyle\int_0^{10^{-37}}\left(f(x)\right)^{\alpha}dx$ converge se $\displaystyle\int_0^{10^{-37}}\frac{6^{-\alpha}}{x^{-2\alpha}}dx$ converge

se $-2\alpha<1$

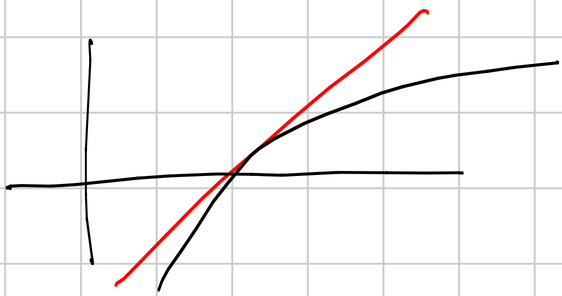
se $\alpha>-\frac{1}{2}$

$$\int_2^{\infty} \frac{dx}{\log x}$$ Dire se converge o non

Converge il seguente integrale improprio

dim.

$\log x \le x-1 \quad \forall x > 0$



$f(x) = x - 1 - \log x \qquad f(1) = 0$

$f'(x) = 1 - \frac{1}{x} > 0 \quad \forall x > 1$

$\Downarrow$

$f(x) \nearrow \qquad x > 1$

$\Downarrow$

$f(x) > 0 \qquad \forall x > 1$

$x - 1 - \log x > 0 \quad \forall x > 1$

$$x-1 > \log x \qquad \forall x>1$$

$$\frac{1}{x-1} < \frac{1}{\log x} \qquad \forall x>1$$

$$\vee$$

$$\frac{1}{x}$$

$$\int_2^{+\infty} \frac{dx}{\log x} > \int_2^{+\infty} \frac{dx}{x} = +\infty$$

$$\int_0^{\pi/2} \frac{\cos x}{(1-\operatorname{sen} x)^{2/3}}\, dx$$

Esercizio

Stabilire se converge o meno l'integrale improprio

$$\cos x = \cos\left(x-\frac{\pi}{2}+\frac{\pi}{2}\right) = \cancel{\cos\left(x-\frac{\pi}{2}\right)\cos\frac{\pi}{2}} - \operatorname{sen}\left(x-\frac{\pi}{2}\right)\operatorname{sen}\frac{\pi}{2}$$

$$= -\left[\left(x-\frac{\pi}{2}\right) - \frac{\left(x-\frac{\pi}{2}\right)^3}{6} + o\left(\left(x-\frac{\pi}{2}\right)^4\right)\right]$$

$$\sum_{k=0}^{4} (\cos x)^{(k)}\left(\frac{\pi}{2}\right) \cdot \frac{\left(x-\frac{\pi}{2}\right)^k}{k!}$$

$$1-\operatorname{sen} x = 1-\operatorname{sen}\left(x-\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= 1 - \left[\cancel{\operatorname{sen}\left(x-\frac{\pi}{2}\right)\cos\frac{\pi}{2}} + \underbrace{\operatorname{sen}\frac{\pi}{2}}_{=1}\cos\left(x-\frac{\pi}{2}\right)\right]$$

$$= 1-\cos\left(x-\frac{\pi}{2}\right) = 1-1+\frac{1}{2}\left(x-\frac{\pi}{2}\right)^2 + o\left(\left(x-\frac{\pi}{2}\right)^3\right)$$

$$(1-\operatorname{sen} x) \sim \frac{\left(x-\frac{\pi}{2}\right)^2}{2} \qquad x \to \pi/2$$

$$f(x) \sim \frac{-\left(x-\frac{\pi}{2}\right)}{\left[\frac{\left(x-\frac{\pi}{2}\right)^2}{2}\right]^{2/3}} \qquad x \to \frac{\pi}{2}$$

$$\parallel$$

$$-2^{2/3} \frac{1}{\left(x-\pi/2\right)^{1/3}}$$

$$f(x) \sim -2^{2/3} \cdot \frac{1}{\left(x-\frac{\pi}{2}\right)^{1/3}} \qquad x \to \pi/2$$

$\int_{\pi/2-10^{-17}}^{\pi/2} f(x)\,dx$ converge <u>sse</u> $\int_{\pi/2-10^{-17}}^{\pi/2} -2^{2/3} \cdot \frac{dx}{\left(x-\frac{\pi}{2}\right)^{\frac{1}{3}}}$

converge

Converge perché

$1/3 < 1$

$$\int_0^{+\infty} \frac{|\operatorname{sen} x|}{x^2}\,dx = I$$

$= f(x)$

Esercizio

studiare la convergenza dell'integrale

$$I = \int_0^1 f(x)\,dx + \int_1^{+\infty} f(x)\,dx$$

$|_{+\infty}$

$|_0$

$\boxed{x \to 0}$

$$\frac{|\operatorname{sen} x|}{x^2} \sim \frac{1}{x}$$

$$\int_0^1 \frac{|\operatorname{sen} x|}{x^2}\,dx \text{ conv. se } \int_0^1 \frac{1}{x}\,dx \text{ converge}$$

il che non è

infatti $\int_0^1 \frac{dx}{x} = +\infty$

$$\int_1^{+\infty} \frac{|\operatorname{sen} x|}{x^2}$$

$$\int_1^{+\infty} \frac{|\operatorname{sen} x|}{x^2} \leq \int_1^{+\infty} \frac{1}{x^2} < +\infty$$

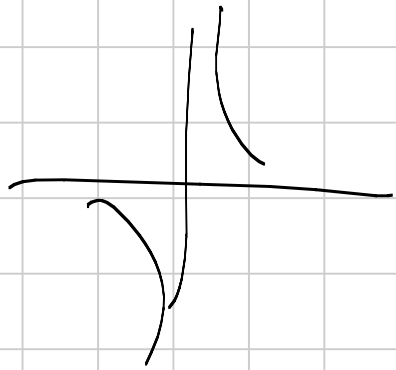
$$F(x) = \int_0^x \underbrace{\frac{e^{-t^2}}{\sqrt[3]{t}}}_{f(t)}\, dt$$

Studio di funzione integrale

$f(t)$ è continua su $\mathbb{R} \setminus \{0\}$

$f(t)$ ha un asintoto verticale in $t=0$

$$\lim_{t\to 0^\pm} f(t) = \pm\infty$$



$\int_0^1 \frac{e^{-t^2}}{\sqrt[3]{t}}\, dt$ converge in quanto

$$\frac{e^{-t^2}}{\sqrt[3]{t}} \sim \frac{1}{\sqrt[3]{t}} \qquad t\to 0$$

$\int_{-1}^0 \frac{e^{-t^2}}{\sqrt[3]{t}}\, dt$ converge

C.E. $F(x) \equiv \mathbb{R}$

$$F(x) = \int_0^x \underbrace{\frac{e^{-t^2}}{\sqrt[3]{t}}}_{f(t)}\, dt$$

$$f(-t) = \frac{e^{-(-t)^2}}{\sqrt[3]{-t}} = -\frac{e^{-t^2}}{\sqrt[3]{t}} = -f(t)$$

<u>Penso</u> che

$F(x)$ sia pari: proviamolo!

$$F(-x) = \int_0^{-x} f(t)\,dt = -\int_{-x}^{0} f(t)\,dt$$

$$= -\int_x^0 f(-y)\cdot(-dy)$$

$t=-y \qquad -x \to x$

$dt=-dy \qquad 0 \to 0$

$$= \int_x^0 f(-y)\,dy = -\int_x^0 f(y)\,dy$$

$$= \int_0^x f(y)\,dy = F(x)$$

F è pari

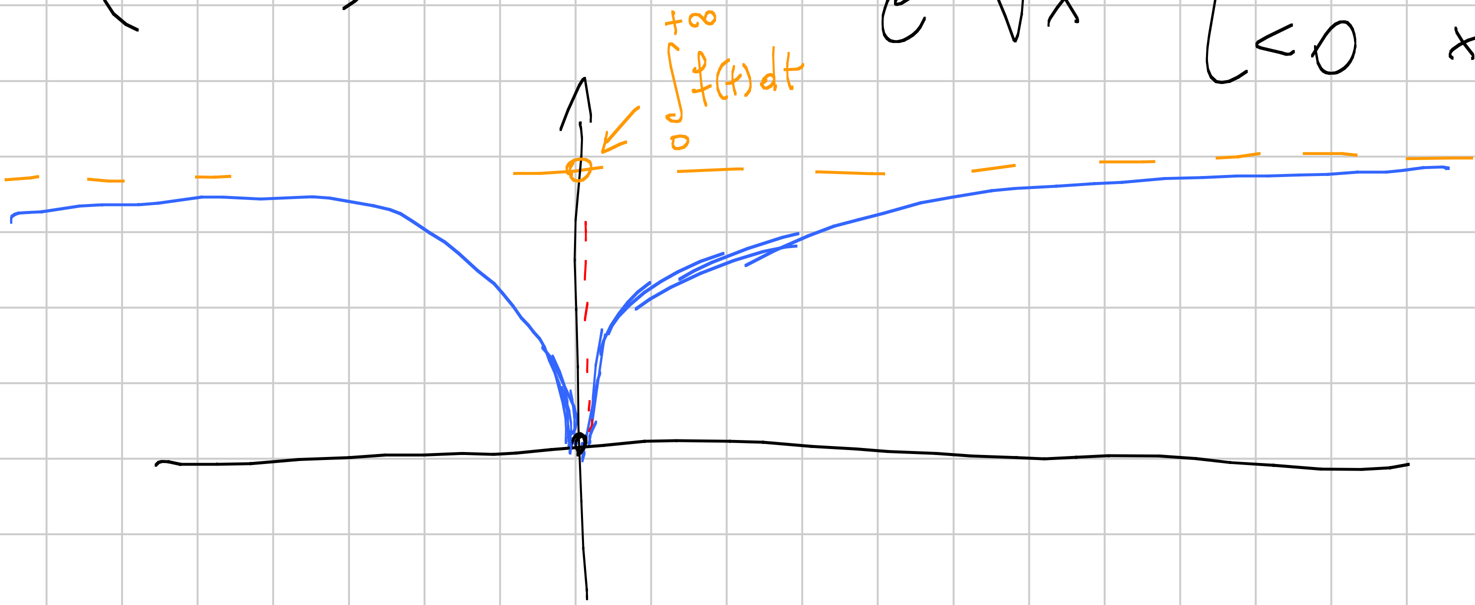
$$\lim_{x\to+\infty} \int_0^x \frac{e^{-t^2}}{\sqrt[3]{t}}\,dt = \lim_{x\to+\infty} \int_0^x \frac{dt}{e^{t^2}\cdot\sqrt[3]{t}}$$

$$= \int_0^{+\infty} \frac{dt}{e^{t^2}\sqrt[3]{t}} \in \mathbb{R}$$

poiché $e^{t^2} \geq 1+t^2 \quad \forall t>0$

e quindi $$\int \frac{1}{e^{t^2}\sqrt[3]{t}} \leq \int_1^{+\infty} \frac{1}{(1+t^2)\sqrt[3]{t}} \in \mathbb{R}$$

$$\left(F(x)\right)' = f(x) = \frac{1}{e^{x^2}\sqrt[3]{x}} \begin{cases} >0 & x>0 \\ <0 & x<0 \end{cases}$$



$$F(0) = 0$$

$$\left(f(t)\right)' = \left(\frac{1}{e^{t^2}\sqrt[3]{t}}\right)' = \frac{-2t\cancel{e^{t^2}}\sqrt[3]{t} - \frac{1}{3}t^{-\frac{2}{3}}\cancel{e^{t^2}}}{e^{\cancel{2}t^2}\, t^{2/3}}$$

$$= -\frac{2t^{4/3} + \frac{1}{3t^{2/3}}}{e^{t^2}\, t^{2/3}} < 0$$

$\forall t \neq 0$

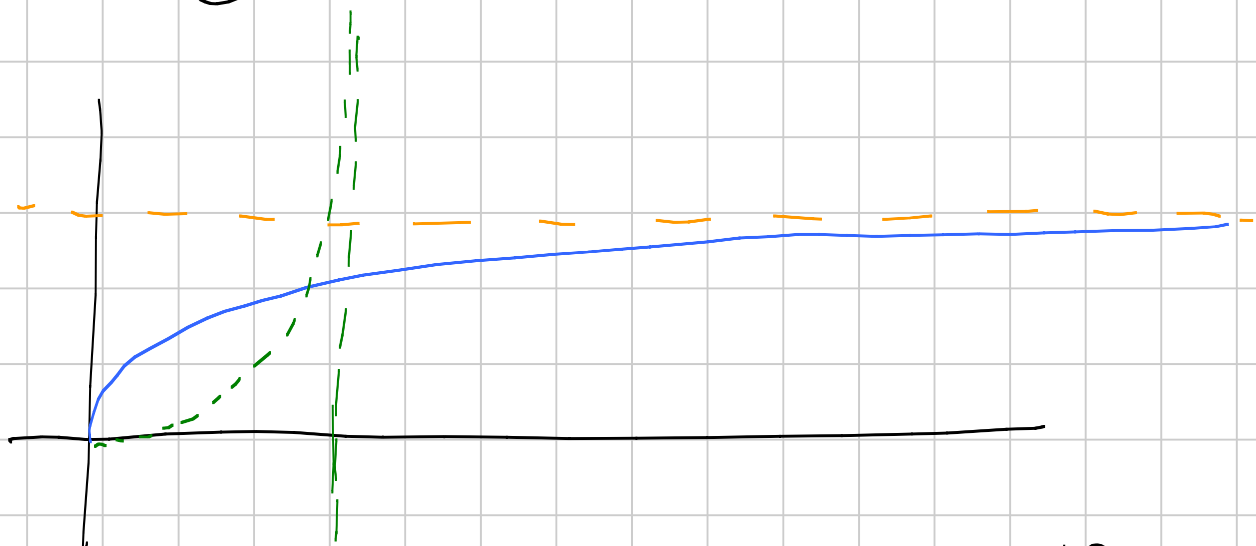
ovvero $F(x) \nearrow$ $x > 0$

$$G(x) = F(x)\big|_{[0,+\infty[}$$ questa è

monotona, strettamente crescente, continua

$\Longrightarrow$ è invertibile con inversa

$G^{-1}(x)$



$G(x)$ è crescente, definita da 0 a $\int_0^{+\infty} f(t)dt$,

ha derivata $= 0$ in $x = 0$

" " $+\infty$ in $x = \int_0^{+\infty} f(t)dt$

è convessa su $[0, \int_0^{+\infty} f(t)dt]$