Esercizio Studiare la convergenza di

1) $\int_0^1 \left(\frac{1}{\operatorname{sen}x} - \frac{1}{x}\right) dx$

2) $\int_0^{+\infty} \frac{\sqrt{1+x^2} - x}{\sqrt{x}} dx$

3) $\int_0^1 \left(1 - \frac{\operatorname{sen}x}{x}\right)^\alpha dx$

4) Calcolare $\lim_{x \to +\infty} \frac{1}{x} \int_2^x \frac{1}{\log t} dt$

5) $\int_0^{+\infty} \frac{e^{\alpha x} - \cos x}{x^\alpha} dx$

$$\int_0^1 \underbrace{\left(\frac{1}{\operatorname{sen}x} - \frac{1}{x}\right)}_{f(x)} dx$$

$f(x)$ non è definita in $x = 0$

$f(x)$ è continua in $]0, 1]$

Ci interessa solo $\int_0^{10^{-30}} f(x) dx$

poiché $\int_{10^{-30}}^1 f(x) dx$ esiste finito

$$f(x) = \frac{1}{\operatorname{sen} x} - \frac{1}{x}$$ e vedere come si comporta "vicino" a $x=0$

$$= \frac{x - \operatorname{sen} x}{x \operatorname{sen} x}$$

$$= \frac{\cancel{x} - \left(\cancel{x} - \frac{x^3}{6} + o(x^4)\right)}{x\left(x - \frac{x^3}{6} + o(x^4)\right)}$$

$$= \frac{x^3/6 + o(x^4)}{x^2 - x^4/6 + o(x^5)}$$

$$= \frac{\cancel{x^2}}{\cancel{x^2}} \frac{x/6 + o(x^2)}{1 + o(1)}$$

$$\boxed{\sim} \quad \frac{x}{6}$$

$$\lim_{x\to 0} \frac{f(x)}{x/6} = 1 \implies \int_0^{10^{-30}} f(x)\,dx \in \mathbb{R}$$

$$\sim \int_0^{10^{-3}} \frac{x}{6}\,dx \in \mathbb{R}$$

$$\int_0^{+\infty} \underbrace{\frac{\sqrt{1+x^2}-x}{\sqrt{x}}}_{f(x)}\,dx \qquad f: \left]0,+\infty\right[\longrightarrow \mathbb{R}$$

$$\underbrace{\int_0^1 f(x)\,dx}_{1} + \underbrace{\int_1^{+\infty} f(x)\,dx}_{2}$$

Caso 1 $\int_0^1 f(x)\,dx$

devo capire come si comporta f quando $x \to 0$

$$f = \frac{\sqrt{1+x^2}-x}{\sqrt{x}} \sim \frac{1}{\sqrt{x}} \qquad x \to 0$$

infatti $\lim_{x\to 0} \frac{f}{1/\sqrt{x}} = \lim_{x\to 0} \sqrt{1+x^2}-x = 1$

ma $\int_0^1 \frac{1}{\sqrt{x}}\,dx \in \mathbb{R} \Longrightarrow \int_0^1 f(x)\,dx \in \mathbb{R}$

teorema conf. asintotico

Caso 2

$$\int_1^{+\infty} \underbrace{\frac{\sqrt{1+x^2}-x}{\sqrt{x}}}_{f(x)}\,dx$$

Come si comporta $f(x)$ in un intorno di $+\infty$

$$f(x) = \frac{\sqrt{1+x^2}-x}{\sqrt{x}} \cdot \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}+x} =$$

$$= \frac{1+\cancel{x^2}-\cancel{x^2}}{\sqrt{x}\left(\sqrt{1+x^2}+x\right)} = \frac{1}{\sqrt{x+x^3}+\sqrt{x^3}}$$

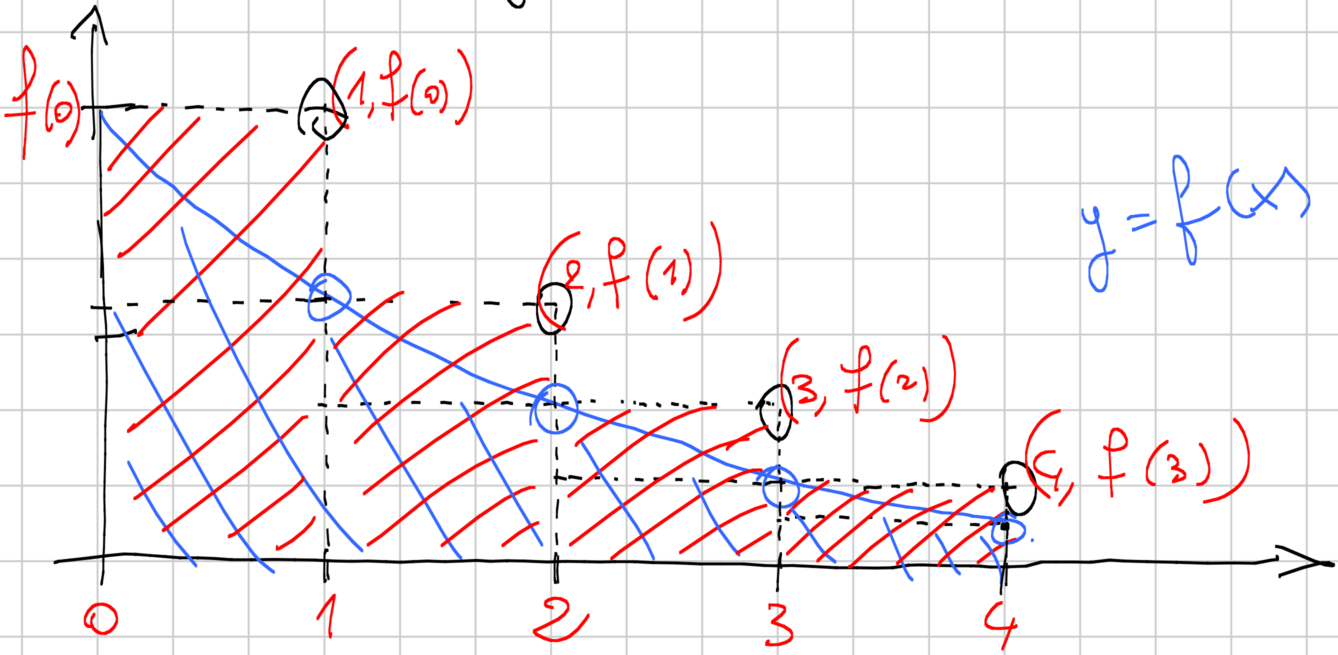
asintot.
eq.

$$\sim \frac{1}{2\sqrt{x^3}} = \frac{1}{2x^{3/2}} \qquad x \longrightarrow +\infty$$

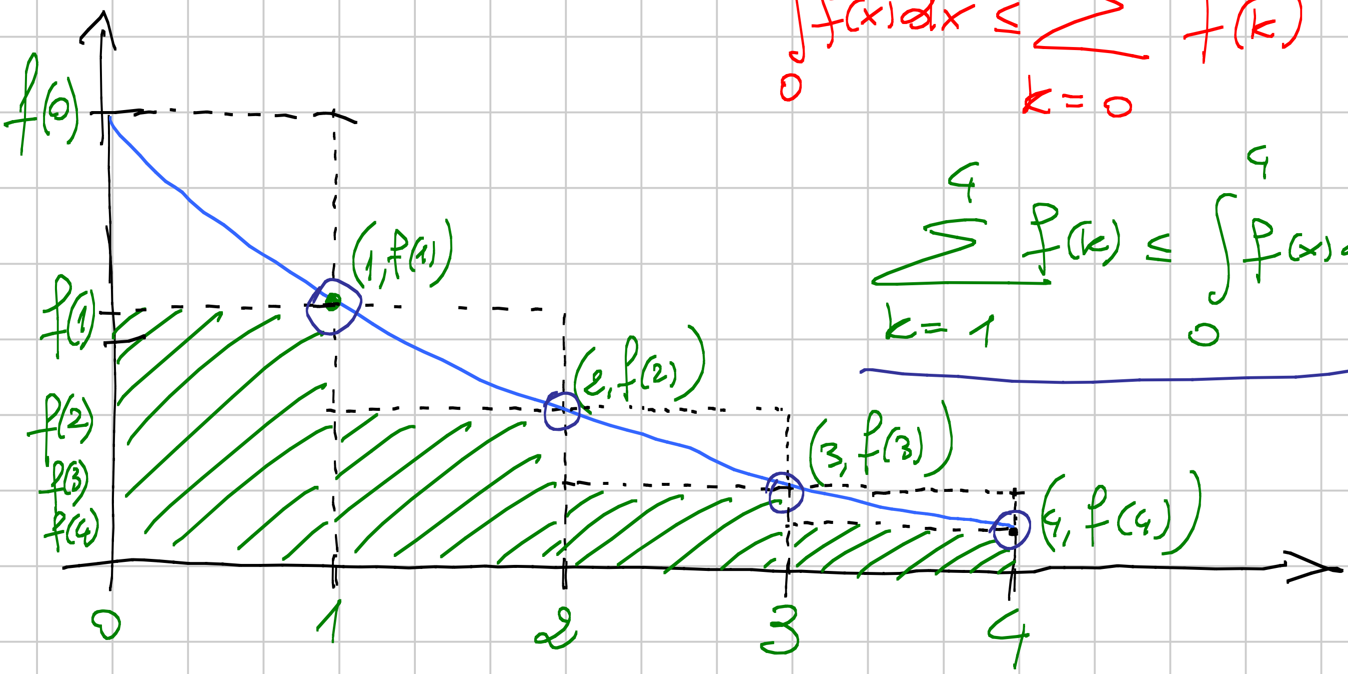
$$\int_1^{+\infty} \frac{1}{2x^{3/2}}\,dx \in \mathbb{R} \Longrightarrow \int_1^{+\infty} f(x)\,dx \in \mathbb{R}$$

confronto
asintotico

Vediamo ora un utile criterio di convergenza per le serie numeriche a termini non negativi. Questo si basa essenzialmente sul seguente disegno



$$\int_0^4 f(x)\,dx \le \sum_{k=0}^{3} f(k)$$



$$\sum_{k=1}^{4} f(k) \le \int_0^4 f(x)\,dx$$

Teorema (Criterio del confronto integrale per serie numeriche a termini positivi)

$f : [a, +\infty[\longrightarrow [0, +\infty[$ una funzione

continua, monotona debolmente decrescente ($= \searrow$)

tale che $\exists \lim_{x \to +\infty} f(x) = 0$

Allora

$$\int_a^{+\infty} f(x)\,dx \quad \text{ha lo stesso carattere di} \quad \sum_{k \geq [a]+1} f(k)$$

(o convergono entrambi o divergono a $+\infty$ entrambi)

dim

Si suppone $[a, +\infty[= [0, +\infty[$ x semplicità

$f(1) \leq f(x) \leq f(0) \qquad x \in [0,1]$

$f(2) \leq f(x) \leq f(1) \qquad x \in [1,2]$

$f(3) \leq f(x) \leq f(2) \qquad x \in [2,3)$

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$$f(n) \le f(x) \le f(n-1) \qquad x \in [n-1, n]$$

$\Downarrow$ Teorema confronto integrali Riemann

$$f(1) = \int_0^1 f(1)\,dx \le \int_0^1 f(x)\,dx \le \int_0^1 f(0)\,dx = f(0)\cdot(1-0)$$

$$f(2) = \int_1^2 f(2)\,dx \le \int_1^2 f(x)\,dx \le \int_1^2 f(1)\,dx = f(1)\cdot(2-1)$$

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$$f(n) = \int_{n-1}^{n} f(n)\,dx \le \int_{n-1}^{n} f(x)\,dx \le \int_{n-1}^{n} f(n-1)\,dx = f(n-1)$$

$$\Downarrow$$

$$f(1)+f(2)+\dots+f(n) \le \int_0^1 f(x)\,dx + \dots + \int_{n-1}^{n} f(x)\,dx \le f(0)+f(1)+\dots+f(n-1)$$

$$\sum_{k=1}^{n} f(k) \le \int_0^n f(x)\,dx \le \sum_{k=0}^{n-1} f(k)$$

$$\boxed{\sum_{k=0}^{n} f(k)} - f(0) \le \int_0^n f(x)\,dx \le \boxed{\sum_{k=0}^{n} f(k)} - f(n) \qquad \square$$

Esercizio

La serie $\sum_n 1/n^\alpha$

converge se $\alpha > 1$

dim

presa $f(x) = \left(\frac{1}{x}\right)^\alpha$

$$\int_1^{+\infty} \left(\frac{1}{x}\right)^\alpha dx \quad \text{converge } \underline{\text{se}} \quad \alpha > 1$$

se $f(x)$ è continua
monotona deb. decrescente $(\alpha > 1)$
positiva

$$\lim_{x \to +\infty} f(x) = 0 \quad (\forall \alpha > 1)$$

dunque $\sum_n f(n) = \sum_n 1/n^\alpha$

si comporta allo stesso modo, ovvero

converge se $\alpha > 1$

Esercizio

$$\sum_n \frac{1}{n(\log n)^\alpha} \quad \text{converge \underline{sse}} \quad \alpha > 1$$