Integrazione per sostituzione

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

Regola di derivazione delle f.ni composte

da cui segue

l'insieme delle primitive dell'una coincide con l'insieme delle primitive dell'altra

$$\int g'(f(x)) \cdot f'(x)\,dx = \boxed{\int (g \circ f)'(x)\,dx}$$

la primitiva di una derivata

$$\int g'(f(x)) \cdot f'(x)\,dx = (g \circ f)(x) + c$$

Se ora indichiamo con G una primitiva di g, si trova

$$\boxed{\begin{aligned}\int g(f(x)) \cdot f'(x)\,dx &= G(f(x)) + c \\ &= (G \circ f)(x) + c\end{aligned}}$$

Integrazione per sostituzione

Oss

$$\int f'(x)\,dx = f(x) + c$$

$$\left(\int f(x)\,dx\right)' = f(x)$$

Notazione

$$\int f(\varphi(x))\cdot\varphi'(x)\,dx = \left(\int f(t)\,dt\right)_{t=\varphi(x)}$$

$\varphi(x)=t$

$$\frac{dt}{dx}=\frac{d\varphi(x)}{dx} \qquad \varphi'(x)\,dx=dt \qquad d\varphi(x)=dt$$

Esempio Calcolare $\int \operatorname{tg}x\,dx$

$$\int \frac{\operatorname{sen}x}{\cos x}\,dx = \left(\int \frac{-dt}{t}\right)_{t=\cos x}$$

$t=\cos x$

$\underbrace{dt=(-\operatorname{sen}x)\,dx}_{-dt=\operatorname{sen}x\,dx}$

osservo che $-\operatorname{sen}x=(\cos x)'$

$$= -\left(\int\frac{dt}{t}\right)_{t=\cos x} = -\left(\log|t|+c\right)_{t=\cos x} \qquad c\in\mathbb{R}$$

$$= -\log|\cos x| + c \qquad c\in\mathbb{R}$$

Verifica

$$\left(-\log|\cos x|+c\right)' = -\frac{1}{|\cos x|}\cdot\frac{\cos x}{|\cos x|}\cdot(-\operatorname{sen}x)$$

$$= \frac{\operatorname{sen}x}{\cos^2 x}\cdot\cos x$$

$$= \frac{\operatorname{sen}x}{\cos x}$$

Oss la derivata di $f(x) = |x|$ è

$$f'(x) = \frac{x}{|x|} \quad \forall x \neq 0$$

Infatti. se $x > 0$ allora $f(x) = x$ allora $f'(x) = 1$

se $x < 0$ " $f(x) = -x$ allora $f'(x) = -1$

allora
$$f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} = \frac{x}{|x|}$$

Esempio calcolare $\int \sqrt{1-x^2}\,dx$

Mi ricordo che $1-\operatorname{sen}^2 t = \cos^2 t$

$$\int \sqrt{1-x^2}\,dx =$$

$$\boxed{\begin{aligned} t &= \operatorname{arcsen} x \\ dt &= \frac{1}{\sqrt{1-x^2}}\,dx \end{aligned}} \rightarrow \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

$$\boxed{\begin{aligned} x &= \operatorname{sen} t \\ dx &= \cos t\,dt \end{aligned}}$$

$$\left(\int \sqrt{1-\operatorname{sen}^2 t}\cdot \cos t\,dt\right)_{t=\operatorname{arcsen} x} =$$

$$= \left(\int |\cos t|\cdot \cos t\,dt\right)_{t=\operatorname{arcsen} x} =$$

$$-\frac{\pi}{2} < t < \frac{\pi}{2}$$
$$\cos t > 0$$

$$= \left(\int (\cos t)^2\,dt\right)_{t=\operatorname{arcsen} x}$$

$$= \left(\frac{1}{2}\left(t + \operatorname{sen} t \cos t\right) + c\right)_{t=\operatorname{arcsen} x} = \frac{1}{2}\left(\operatorname{arcsen} x + x\sqrt{1-x^2}\right) + c$$

$c \in \mathbb{R}$

$$(\cos t)_{t=\operatorname{arcsen} x} = \left(\sqrt{1-\operatorname{sen}^2 t}\right)_{t=\operatorname{arcsen} x} = \sqrt{1-x^2}$$

$$\int \cos^2 t\,dt = \int \underset{u'}{\cos t}\;\underset{v}{\cos t}\,dt = \underset{u}{\operatorname{sen} t}\;\underset{v}{\cos t} - \int \underset{u}{\operatorname{sen} t}\;\underset{v'}{(-\operatorname{sen} t)}\,dt$$

$$= \operatorname{sen} t \cos t + \int \left(1-\cos^2 t\right) dt$$

$$2\int \cos^2 t\,dt = t + \operatorname{sen} t \cos t + c \qquad c \in \mathbb{R}$$

$$\int \cos^2 t\,dt = \frac{1}{2}\left(t + \operatorname{sen} t \cos t\right) + c \qquad c \in \mathbb{R}$$

__Esempio__ calcolare $\int \mathrm{arctg}\, x \, dx$

$$\int \underbrace{\mathrm{arctg}\, x}_{v} \cdot \underbrace{1}_{u'} \, dx = \underbrace{\mathrm{arctg}\, x}_{v} \cdot \underbrace{x}_{u} - \int \underbrace{x}_{u} \underbrace{\frac{1}{1+x^2}}_{v'} \, dx$$

$$= x \cdot \mathrm{arctg}\, x - \frac{1}{2} \int \frac{2x \, dx}{1+x^2}$$

osservo che $(1+x^2)' = 2x$

$$\int \frac{2x}{1+x^2} \, dx \underset{\substack{t=(1+x^2) \\ dt = 2x\, dx}}{=} \left(\int \frac{dt}{t} \right)_{t=1+x^2}$$

$$= \left(\log|t| + c \right)_{t=1+x^2} \qquad c \in \mathbb{R}$$

$$= \log(1+x^2) + c \qquad c \in \mathbb{R}$$

$$\int \mathrm{arctg}\, x \, dx = \left[x \, \mathrm{arctg}\, x - \frac{1}{2} \log(1+x^2) + c \right] \quad c \in \mathbb{R}$$

__verifica__ $\left[\; ... \; \right]' =$

Integrazione delle funzioni razionali

<u>Problema</u> calcolare $\int \frac{P(x)}{Q(x)}\,dx$

<u>Inizialmente</u> ci si riduce al caso in cui

$$\text{grado}(P(x)) < \text{grado}(Q(x))$$

Infatti, se così non fosse, si scrive

$$\frac{P(x)}{Q(x)} = A(x) + \frac{P_1(x)}{Q(x)} \quad \text{con grado}(P_1(x)) < \text{grado}(Q(x))$$

e $A(x)$, polinomio, si integra subito con la decomposizione in somma

<u>Esempio</u>

grado $P(x) = 4$

$$\int \frac{3x^4+2x+1}{x^2+1}\,dx$$

grado $Q(x) = 2$

$$
\begin{array}{rrrrr|l}
3x^4 & 0x^3 & 0\cdot x^2 & +2x & +1 & x^2+1 \\
\cline{6-6}
3x^4 & & +3x^2 & & & 3x^2-3 \\
\cline{1-5}
\cancel{0} & & -3x^2 & +2x & +1 & \\
 & & -3x^2 & & -3 & \\
\cline{3-5}
 & & \cancel{0} & 2x & +\cancel{2}\,4 &
\end{array}
$$

$$\frac{3x^4+2x+1}{(x^2+1)} = \cancel{(x^2+1)}^{\,1}\cdot(3x^2-3) \quad \frac{+2x+4}{x^2+1}$$

$$\int \frac{3x^4+2x+1}{x^2+1}\,dx = \int\left(3x^2-3+\frac{2x+4}{x^2+1}\right)dx$$

$$= 3\int x^2\,dx - 3\int dx + \int \frac{2x+4}{x^2+1}\,dx$$

$$= x^3-3x + \int \frac{2x}{x^2+1}\,dx + 4\int \frac{1}{x^2+1}\,dx$$

$$= x^3-3x+\log(1+x^2)+4\,\mathrm{arctg}\,x + c \qquad c\in\mathbb{R}$$

<u>Esempio</u> calcolare $\int \frac{1}{1-x^2}\,dx$

$$\frac{1}{1-x^2} = \frac{1}{1-x}\cdot\frac{1}{1+x}$$

1 e −1 sono le radici di $1-x^2=0$

$$= \frac{A}{1-x} + \frac{B}{1+x}$$

$$= \frac{A(1+x)+B(1-x)}{1-x^2}$$

$$\Updownarrow$$

$$1 = (A+B) + (A-B)x \quad \forall x \neq \pm 1$$

$$\Updownarrow$$

$$\begin{cases} 1 = A+B \\ 0 = A-B \end{cases} \qquad \begin{cases} A = B \\ 2A = 1 \end{cases} \qquad A = B = \frac{1}{2}$$

$$\frac{1}{1-x^2} = \frac{1}{2}\,\frac{1}{1-x} + \frac{1}{2}\,\frac{1}{1+x}$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\int \frac{1}{1-x}\,dx + \frac{1}{2}\int\frac{1}{1+x}\,dx$$

$1 = -(1-x)'$ $\qquad$ $1 = (1+x)'$

$$= \frac{1}{2}\left(\int\frac{-dt}{t}\right)_{t=1-x} + \frac{1}{2}\left(\int\frac{dt}{t}\right)_{t=1+x}$$

$$= -\frac{1}{2}\left(\log|t|+c\right)_{t=1-x} + \frac{1}{2}\left(\log|t|+d\right)_{t=1+x} \quad \begin{matrix} c\in\mathbb{R} \\ d\in\mathbb{R}\end{matrix}$$

$$= -\frac{1}{2}\log|1-x| + c + \frac{1}{2}\log|1+x| + d \qquad c,d\in\mathbb{R}$$

$$= -\frac{1}{2}\log|1-x| + \frac{1}{2}\log|1+x| + c \qquad c\in\mathbb{R}$$

$$= \log\sqrt{\left|\frac{1+x}{1-x}\right|} + c \qquad c\in\mathbb{R}$$

<u>Verifica</u>

Esempio calcolare $\int \frac{1}{x^2+4}\, dx$

$$\int \frac{1}{x^2+4} dx = \frac{1}{4} \int \frac{1}{1+\left(\frac{x}{2}\right)^2} dx$$

$$\underset{\substack{y=\frac{x}{2} \\ dy=\frac{dx}{2}}}{=} \frac{1}{4} \left(\int \frac{2\,dy}{1+y^2} \right)_{y=\frac{x}{2}}$$

$$= \frac{1}{2} \left(\operatorname{arctg} y + c \right)_{y=x/2} \qquad c \in \mathbb{R}$$

$$= \frac{1}{2} \operatorname{arctg}\left(\frac{x}{2}\right) + c \qquad c \in \mathbb{R}$$

Esempio calcolare $\int \frac{1}{x^2+4x+6}\, dx$

voglio arrivare a

$$\int \frac{1}{1+y^2} \quad \text{o} \quad \int \frac{1}{1-y^2}$$

$$(x^2+4x)+6 = (x^2+4x+4)-4+6$$
$$= (x+2)^2+2$$

$$\int \frac{1}{x^2+4x+6}\, dx = \int \frac{dx}{(x+2)^2+2}$$
$$= \frac{1}{2} \int \frac{dx}{\left(\frac{x+2}{\sqrt{2}}\right)^2+1}$$

$$\underset{y=\frac{x+2}{\sqrt{2}}}{=} \left(\frac{1}{2} \int \frac{\sqrt{2}\, dy}{y^2+1}\right)_{y=\frac{x+2}{\sqrt{2}}}$$

$dy = \frac{dx}{\sqrt{2}}$

$$= \frac{1}{\sqrt{2}} \left(\operatorname{arctg} y + c\right)_{y=\frac{x+2}{\sqrt{2}}} \qquad c \in \mathbb{R}$$

$$= \frac{1}{\sqrt{2}} \operatorname{arctg}\left(\frac{x+2}{\sqrt{2}}\right) + c \qquad c \in \mathbb{R}$$

Verifica

verificare

$$\left(\frac{1}{\sqrt{2}} \operatorname{arctg}\left(\frac{x+2}{\sqrt{2}}\right) + c\right)' = \frac{1}{x^2+4x+6}$$

Calcolare $\int \frac{dx}{x^2+3x+2}$

$x^2+3x+2 = (x+2)(x+1)$ e quindi

$$\frac{1}{x^2+3x+2} = \frac{A}{x+2} + \frac{B}{x+1} = \frac{A(x+1)+B(x+2)}{x^2+3x+2}$$

$$\begin{cases} A+2B = 1 \\ A+B = 0 \end{cases} \quad \begin{cases} B = 1 \\ A = -B \end{cases} \quad \begin{cases} B = 1 \\ A = -1 \end{cases}$$

$$\int \frac{1}{x^2+3x+2}\,dx = \int \frac{-1}{x+2}\,dx + \int \frac{1}{x+1}\,dx$$

$$= -\int \frac{1}{x+2}\,dx + \int \frac{dx}{x+1}$$

$$= -\left(\int \frac{dt}{t}\right)_{\substack{t=x+2 \\ dt=dx}} + \left(\int \frac{dt}{t}\right)_{\substack{t=x+1 \\ dt=dx}}$$

$$= -\left(\log|t| + c\right)_{t=x+2} + \left(\log|t| + c\right)_{t=x+1}$$

$$= -\log|x+2| + \log|x+1| + c \qquad c \in \mathbb{R}$$

$$= \log\left|\frac{x+1}{x+2}\right| + c \qquad c \in \mathbb{R}$$

Verifica $\left(\log\left|\frac{x+1}{x+2}\right| + c\right)' = \ldots$

Esempio calcolare $\int \frac{dx}{(1+x)^4}$

$$\int \frac{1}{y^4}\,dy = \frac{y^{-4+1}}{-4+1} + c$$

$$\int \frac{dx}{(1+x)^4} \underset{\substack{t=1+x \\ dt=dx}}{\uparrow}{=} \left(\int \frac{dt}{t^4}\right)_{t=1+x}$$

$$= \left(\frac{t^{-4+1}}{-4+1} + c\right)_{t=1+x} \qquad c \in \mathbb{R}$$

$$= \left(-\frac{1}{3t^3} + c\right)_{t=1+x} \qquad c \in \mathbb{R}$$

$$= -\frac{1}{(1+x)^3} + c \qquad c \in \mathbb{R}$$

<u>Esempio</u> calcolare $\int \frac{1}{(x^2+1)^2}\,dx$

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1+x^2-x^2}{(1+x^2)^2}\,dx$$

$$= \int \frac{1}{(1+x^2)}\,dx + \int x \cdot \frac{x}{(1+x^2)^2}\,dx$$

$$= \operatorname{arctg} x + \underbrace{\int \frac{x}{2} \cdot \frac{2x}{(1+x^2)^2}\,dx}$$

$$\int \underset{v}{\frac{x}{2}} \cdot \underset{u'}{\frac{2x}{(1+x^2)^2}}\,dx = \underset{v}{\frac{x}{2}} \cdot \underset{u}{\left(-\frac{1}{(1+x^2)}\right)} - \int \underset{v'}{\frac{1}{2}} \cdot \underset{u}{\left(-\frac{1}{1+x^2}\right)}\,dx$$

$$= -\frac{1}{2}\,\frac{x}{1+x^2} + \frac{1}{2}\int \frac{1}{1+x^2}\,dx$$

$$= -\frac{1}{2}\,\frac{x}{1+x^2} + \frac{1}{2}\operatorname{arctg} x + c$$

Verifica $\left(-\frac{1}{2}\frac{x}{1+x^2}+\frac{1}{2}\operatorname{arctg} x+c\right)'$

$$= \frac{1}{(1+x^2)^2}$$

$$\int \frac{2x}{(1+x^2)^2}\,dx \underset{\substack{t=1+x^2\\ dt=2x\,dx}}{\uparrow=} \left(\int \frac{dt}{t^2}\right)_{t=1+x^2}$$

$$= \left(\frac{t^{-2+1}}{-2+1}+c\right)_{t=1+x^2} \quad c\in\mathbb{R}$$

$$= -\frac{1}{1+x^2}+c \qquad c\in\mathbb{R}$$

Esempio calcolare $\int \frac{x^3}{(1+x^2)^2} dx$

$$\frac{x^3}{(1+x^2)(1+x^2)} = \frac{Ax+B}{1+x^2} + \frac{Cx+D}{(1+x^2)^2}$$

$$= \frac{(1+x^2)(Ax+B) + Cx+D}{(1+x^2)^2}$$

$$\frac{x^3}{(1+x^2)^2} = \frac{Ax+B}{1+x^2} + \frac{Cx+D}{(1+x^2)^2}$$

$$= \frac{Ax(1+x^2)+B(1+x^2)+Cx+D}{(1+x^2)^2}$$

$$= \frac{Ax^3+Bx^2+(A+C)x+B+D}{(1+x^2)^2}$$

$$\boxed{\begin{matrix} A=1 & A+C=0 \\ B=0 & B+D=0 \end{matrix}} \qquad \boxed{\begin{matrix} A=1 & B=D=0 \\ C=-1 & \end{matrix}}$$

$$\frac{x^3}{(1+x^2)^2} = \frac{x}{1+x^2} - \frac{x}{(1+x^2)^2}$$

$$\frac{1}{2}\int \frac{2x}{1+x^2}\,dx \underset{\substack{t=1+x^2 \\ dt=2x\,dx}}{=} \frac{1}{2}\left(\int \frac{dt}{t}\right)_{t=1+x^2}$$

$$= \frac{1}{2}\log(1+x^2) + c \qquad c\in\mathbb{R}$$

$$-\frac{1}{2}\int \frac{2x}{(1+x^2)^2}\,dx \underset{\substack{t=1+x^2 \\ dt=2x\,dx}}{=} -\frac{1}{2}\left(\int \frac{dt}{t^2}\right)_{t=1+x^2}$$

$$= -\frac{1}{2}\left(-\frac{1}{t}+c\right)_{t=1+x^2} \qquad c\in\mathbb{R}$$

$$= +\frac{1}{2}\frac{1}{1+x^2} + c \qquad c\in\mathbb{R}$$

$$\int \frac{x^3}{1+x^2}\,dx = \frac{1}{2}\log(1+x^2) + \frac{1}{2}\frac{1}{1+x^2} + c \quad c\in\mathbb{R}$$

<u>Caso radici semplici reali</u>

$$\int \frac{1}{(x-a_1)(x-a_2)\cdots(x-a_n)}\,dx$$

$$= \int \left(\frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}\right)dx$$

Caso radici compl. coniugate semplici

$$\int \frac{1}{(x^2+2a_1x+b_1)(x^2+2a_2x+b_2)\cdot\ldots\cdot(x^2+2a_mx+b_m)}\,dx$$

$$=\int\left(\frac{A_1x+B_1}{x^2+2a_1x+b_1}+\ldots+\frac{A_mx+B_m}{x^2+2a_mx+b_m}\right)dx$$

Caso radici reali molt. > 1

$$\int \frac{dx}{(x-a_1)^4 (x-a_2)^2}$$

$$= \int \frac{A_1}{x-a_1} + \int \frac{A_2}{(x-a_1)^2} + \int \frac{A_3}{(x-a_1)^3} + \int \frac{A_4}{(x-a_1)^4}$$

$$+ \int \frac{B_1}{(x-a_2)} + \int \frac{B_2}{(x-a_2)^2}$$

Radici complesse e coniugate molt > 1

$$\int \frac{dx}{(x^2+ax+b)^3} = \int \frac{A_1x+B_1}{x^2+ax+b} + \int \frac{A_2x+B_2}{(x^2+ax+b)^2}$$

$$+ \int \frac{A_3x+B_3}{(x^2+ax+b)^3}$$

$$\int \frac{x^4}{(x-1)^2 (x^2+4)^3 (x+1)}$$

reale molt $= 2$

compl coniug molt $= 3$

1 reale semplice

$$\frac{x^4}{(x-1)^2 (x^2+4)^3 (x+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$+ \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2} + \frac{Hx+I}{(x^2+4)^2}$$