Analogamente

Esempio

$$(\cos x)'(x_0) = -\operatorname{sen} x_0 \qquad \forall x_0 \in \mathbb{R}$$

Esempio

$$f = Tg(x) \qquad f'(x) = \frac{1}{\cos^2 x} \underset{\uparrow}{=} 1 + Tg^2(x)$$

banale

dim

$$\left(\frac{\operatorname{sen} x}{\cos x}\right)' = \frac{(\operatorname{sen} x)' \cos x - \operatorname{sen} x \, (\cos x)'}{\cos^2 x}$$

$$= \frac{\cos^2 x - \operatorname{sen} x \, (-\operatorname{sen} x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \operatorname{sen}^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = 1 + Tg^2 x$$

Esempio

$$f(x) = \operatorname{arctg} x \longrightarrow f'(x) = \frac{1}{1+x^2} \quad x \in \mathbb{R}$$

dim

Per calcolare la derivata di $\operatorname{arctg} x$ ho bisogno

1- della derivata di $\operatorname{Tg} x$

2- del Teorema di derivazione dell'inversa

Posto $f(x) = \operatorname{arctg} x$ e $g(x) = \operatorname{Tg} x$

si ha

$$f'(x) = \frac{1}{g'(f(x))} = \frac{1}{1+\operatorname{Tg}^2(f(x))}$$

$$= \frac{1}{1+[\operatorname{Tg}(\operatorname{arctg} x)]^2}$$

$$= \frac{1}{1+x^2}$$

▨

f monotona crescente $\Rightarrow$ f' ? ←

$f' > 0$ $\Rightarrow$ f ? ←

Teorema $\left(f \nearrow^{=} \Rightarrow f' \geq 0 \right)$ $\left(f \searrow^{=} \Rightarrow f' \leq 0 \right)$

$f : A \to \mathbb{R}$ monotona debolmente crescente (decrescente)

f derivabile in $x_0 \in A$ & x_0 p.d.a. per A

allora si ha $f'(x_0) \geq 0$ $\quad (f'(x_0) \leq 0)$

dim

$x_0 \qquad R_{x_0}(x) = \dfrac{f(x) - f(x_0)}{x - x_0} \qquad f \nearrow^{=}$

$$\begin{cases} x < x_0 & \dfrac{\overbrace{f(x) - f(x_0)}^{\leq 0}}{\underbrace{x - x_0}_{< 0}} > 0 \\ \\ x > x_0 & \dfrac{\overbrace{f(x) - f(x_0)}^{\geq 0}}{\underbrace{x - x_0}_{> 0}} > 0 \end{cases}$$

$x \neq x_0 \qquad \dfrac{f(x) - f(x_0)}{x - x_0} > 0 \Rightarrow \lim\limits_{x \to x_0} R_x(x) = f'(x_0) \geq 0$

$\boxed{R.f.}$

Pb f è monotona strett. crescente $\Rightarrow$ $f'(x_0) > 0$
f derivabile in x_0 ??

No!

$f(x) = x^3$ è $\nearrow$, ma $f'(0) = 0$

(quindi soddisfa il Thm precedente, ma non si può concludere $f'(x_0) > 0$!!!)

✓

$$f(x) = x \longrightarrow f'(x) = 1$$

$f(x) = x^2 \qquad f'(x) = 2x$ infatti,

$$\begin{cases} \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} \frac{\cancel{(x-x_0)}(x+x_0)}{\cancel{x-x_0}} = 2x_0 \\ x_0 \xrightarrow{f'} 2x_0 = f'(x_0) \end{cases}$$

oppure

$$f(x) = x \cdot x \qquad f'(x) = (x \cdot x)' = 1 \cdot x + x \cdot 1 = 2x$$

* * *

$f(x) = x^3 \rightsquigarrow f'(x) = 3x^2$ infatti,

$$f(x) = x^2 \cdot x \qquad f' = 2x \cdot x + x^2 \cdot 1 = 3x^2$$

In generale $f(x)=x^n$ $\qquad f'(x) = n\,x^{n-1}$

$$f(x)=x^{\alpha} \qquad f'(x) = \alpha\, x^{\alpha-1} \quad \forall x>0,\ \alpha\in\mathbb{R}$$

$$f(x)= e^{\alpha\log x} = (g\circ h)(x) \qquad \begin{cases} h(x)=\alpha\log x \\ g(y)=\exp(y) \end{cases}$$

$$x \longrightarrow h(x)=\alpha\log x \longrightarrow g(h(x))=e^{\alpha\log x}$$

$$f'(x) = \left[g(h(x))\right]' = g'(h(x))\cdot h'(x)$$

$$= e^{h(x)}\cdot\frac{\alpha}{x} =$$

$$= e^{\alpha\log x}\cdot\frac{\alpha}{x} = x^{\alpha}\cdot\frac{\alpha}{x} = \alpha\cdot x^{\alpha-1}$$

in x^{α} devo avere $\underline{\underline{x>0}}$

$$f(x)= q^{x} \qquad f'(x) = q^{x}\cdot\log q \quad \forall q>0\ \forall x$$

infatti $f(x)=\left(e^{\log q}\right)^{x} = e^{x\log q} = g(h(x))$

$$x \xmapsto{h} x\log q \xrightarrow{g} e^{x\log q}$$

$h(x) = x \log q \qquad g(y) = e^{y}$

$f'(x) = [g(h(x))]' = g'(h(x)) \cdot h'(x)$

$= e^{h(x)} \cdot \log q$

$= e^{x \log q} \cdot \log q = q^{x} \cdot \log q$

<u>Om</u> $(k \cdot f(x))' = k \cdot f'(x) \quad \forall k \in \mathbb{R}$

$k = \log q$

Def $f: A \to \mathbb{R}$ $\quad x_0 \in A$

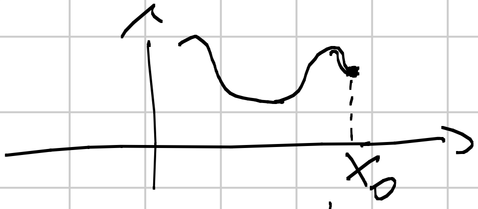
x_0 punto di minimo locale per f su A se $\exists U \in \mathcal{U}_{x_0} : \forall x \in U \cap A \; f(x_0) \le f(x)$

x_0 " " " " interno se " $: \forall x \in U \subset A \; f(x_0) \le f(x)$

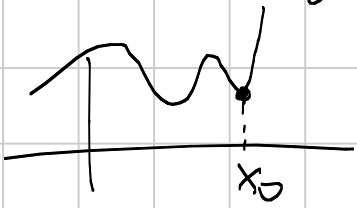
x_0 " " " " stretto se " $\forall x \in U \cap A \setminus \{x_0\} \; f(x_0) < f(x)$

e analogamente per i max

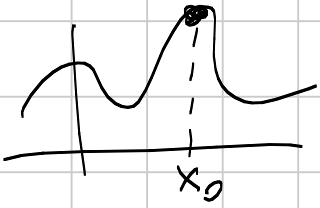
min locale



è p.to di minimo locale



p.to di min. locale interno



p.to max locale interno

Lemma (di Fermat) $\boxed{\text{FONDAMENTALE}}$

$$\left.\begin{array}{l} f: A \to \mathbb{R},\; x_0 \in A \;\&\; x_0 \text{ p.d.a. per } A \\ f \text{ derivabile in } x_0 \\ x_0 \text{ p.to di max (o min) locale interno per } f \end{array}\right\} \Longrightarrow f'(x_0) = 0$$

(tg orizzontale)

dim x_0 p.to max locale interno

$$\exists U \in \mathcal{U}_{x_0} : f(x) \leq f(x_0) \quad \forall x \in U$$

$x \in \{x < x_0\} \cap U$
$$\frac{\overbrace{f(x)-f(x_0)}^{\leq 0}}{\underbrace{x-x_0}_{<0}} \geq 0$$

$x \in \{x > x_0\} \cap U$
$$\frac{\overbrace{f(x)-f(x_0)}^{\leq 0}}{\underbrace{x-x_0}_{>0}} \leq 0$$

$$\lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} = f'_-(x_0) \geq 0$$

$$\lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} = f'_+(x_0) \leq 0$$

ma f è derivabile in x_0, ovvero

$$\exists\, f'(x_0) = f'_+(x_0) = f'_-(x_0) \implies f'(x_0) = 0 \quad \boxed{R/p}$$

Teorema (di Rolle)

$f : [a,b] \longrightarrow \mathbb{R}$

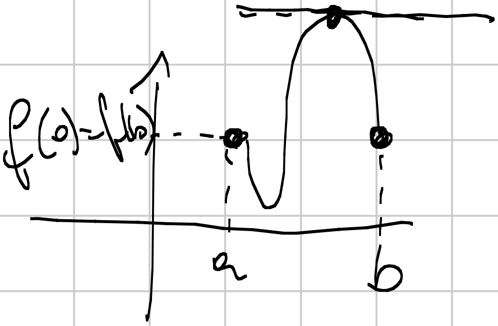
1) continua su $[a,b]$

2) derivabile su $]a,b[$

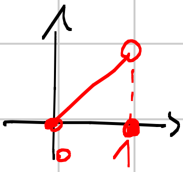
3) $f(a) = f(b)$

$\Longrightarrow$ $\exists\, x_0 \in \,]a,b[$ t.c.

$f'(x_0) = 0$

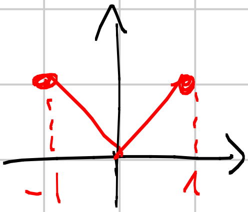


Oss



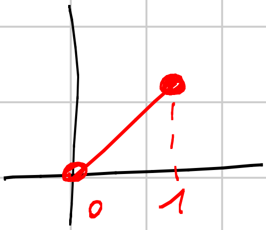
f non è continua in b

e il Teorema non vale



f non è derivabile in un p.to interno

e il Teorema non vale



$f(a) \neq f(b)$

e il Teorema non vale

dim

f continua su $[a,b]$ $\xrightarrow{\text{(Weierstrass)}}$ $\exists\, M = \max\limits_{[a,b]} f$ $\quad \exists\, m = \min\limits_{[a,b]} f$

① Se x_M p.to di massimo cadono in $\{a,b\}$

x_m " " minimo

allora $\min f = \max f$ allora $f = \text{cost.}$

allora $f'(x) = 0 \quad \forall x \in]a,b[$

② se $\{x_M, x_m\} \not\subseteq \{a,b\}$

(almeno uno dei due sta all'interno)

Supponiamo $x_M \in]a,b[\qquad M = \max f = f(x_M)$

essendo f derivabile in $x_M \overset{(\text{Fermat})}{\Longrightarrow} f'(x_M) = 0$

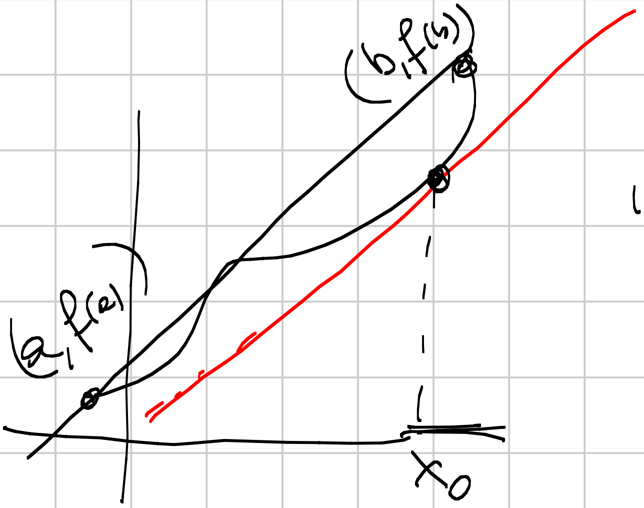
$\boxed{\text{Rp}}$

Teorema (di Lagrange)

$$\left.\begin{array}{l} f : [a,b] \longrightarrow \mathbb{R} \\ 1)\ \text{continua su } [a,b] \\ 2)\ \text{derivabile su }]a,b[\end{array}\right\} \Longrightarrow \begin{array}{l} \exists\, x_0 \in\,]a,b[\text{ t.c.} \\ f'(x_0) = \dfrac{f(b)-f(a)}{b-a} \end{array}$$

$\uparrow$

$\exists\, x_0 : f'(x_0)$ è il coeff. angolare della retta che passa per $(a, f(a))$ e $(b, f(b))$



$f(x)$ $\qquad$ $r(x) = f(a) + \dfrac{f(b)-f(a)}{b-a} \cdot (x-a)$

$g(x) = f(x) - r(x)$

$\rightarrow$ g è continua su $[a,b]$ (è differenza tra f.ni continue)

$\rightarrow$ g' è derivabile su $]a,b[$ (è differenza di f.ni derivabili)

$- \ g(a) = f(a) - f(a) - \dfrac{f(b)-f(a)}{b-a} \cdot (a-a) = 0$

$g(b) = f(b) - f(a) - \dfrac{f(b)-f(a)}{\cancel{b-a}} \cdot \cancel{(b-a)} = 0$

g soddisfa tutte le 3 ipotesi del teorema di Rolle

$$\Rightarrow \exists x_0 \in]a,b[\ : \ g'(x_0) = 0$$

ma $g'(x_0) = f'(x_0) - r'(x_0)$

$$= f'(x_0) - \frac{f(b)-f(a)}{b-a} = 0$$

$$\Rightarrow f'(x_0) = \frac{f(b)-f(a)}{b-a}$$

$\boxed{\cancel{\text{Rip}}}$

Teorema (di Cauchy)

$f, g : A \to \mathbb{R}$

1) continue in $[a,b]$

2) derivabili in $]a,b[$

$\Longrightarrow$ $\exists z \in]a,b[$ t.c.

$$\boxed{\begin{array}{c}(f(b)-f(a)) \cdot g'(z) \\ \parallel \\ (g(b)-g(a)) \cdot f'(z)\end{array}}$$

nel caso che siano $\neq 0$

$$\frac{f'(z)}{g'(z)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\$ \ \overset{\overset{]a,b[}{\cup}}{\exists z} ! \quad f'(z)\left(g(b)-g(a)\right) - g'(z)\left(f(b)-f(a)\right) = 0$$

F.ne ausiliaria $h(x) = f(x)\left(g(b)-g(a)\right) - g(x)\left(f(b)-f(a)\right)$

- $h(x)$ continua in $[a,b]$
- $h(x)$ derivabile in $]a,b[$
- $h(a) = f(a)\left(g(b)-\cancel{g(a)}\right) - g(a)\left(f(b)-\cancel{f(a)}\right) = f(a)g(b)-g(a)f(b)$

$$\parallel$$

$h(b) = f(b)\left(\cancel{g(b)}-g(a)\right) - g(b)\left(\cancel{f(b)}-f(a)\right) = f(a)g(b)-g(a)f(b)$

allora $h(x)$ soddisfa le 3 ipotesi Thm Rolle

$\Rightarrow \exists z \in]a,b[\; ; \; h'(z) = 0$

$$h'(z) = f'(z)\big(g(b)-g(a)\big) - g'(z)\big(f(b)-f(a)\big) = 0$$

$\boxed{\text{Bup}}$

Corollario (del Teorema Rolle)

$f : I \to \mathbb{R}$ continua, I intervallo

f derivabile $\forall x \in I \setminus \mathcal{F}(I)$, $f'(x) \neq 0$ $\Rightarrow$ f iniettiva su I

dim

Ts $\forall x, y \in I \quad x \neq y \Rightarrow f(x) \neq f(y)$

per assurdo $f(x) = f(y)$

f continua su $[x,y]$

f derivabile su $]x,y[$

$\Rightarrow \exists z \in]x,y[\; : \; f'(z) = 0$

Assurdo

Corollario (Lagrange)

f derivabile $\forall x \in I$, I intervallo

$f'(x) = 0 \quad \forall x \in I \setminus \mathcal{F}(I)$ $\Longrightarrow f(x) = c \quad \forall x \in \mathbb{R}$

dim

Ts $\forall x, y \in I, x \neq y, \quad f(x) = f(y)$

Prendo $x < y$, $x, y \in I$

- f continua su $[x, y]$
- f derivabile su $]x, y[$

$\xRightarrow{\text{Lagrange}} \quad \exists z \in]x, y[\ :$

$$f'(z) = \frac{f(y) - f(x)}{y - x}$$

ma $f'(z) = 0 \Longrightarrow \dfrac{f(y) - f(x)}{y - x} = 0$

$\Longrightarrow f(y) - f(x) = 0 \Longrightarrow f(y) = f(x) \quad \square$

Problema $f : A \to \mathbb{R}$: f derivabile $f'(x) = 0 \quad \forall x \in A$

$\Longrightarrow f(x) = \text{cost}$

NO

Infatti, prese $f(x) = \begin{cases} -2 & 1 \leq x < 2 \\ 1 & 3 < x < 4 \end{cases}$

non è costante ma $f'(x) = 0 \quad \forall x$

Corollario (Lagrange)

(Serve per determinare regioni monotonie)

f derivabile $\forall x \in I$

$f'(x) \geq 0 \ \forall x \in I \setminus \partial(I)$ $\Longrightarrow$ f è monotona debolmente crescente (decrescente) su I

$(f'(x) \leq 0)$

dim

TS $\quad x < y \in I \Rightarrow f(x) \leq f(y)$

Considero l'intervallino $[x,y]$

- f continua su $[x,y]$
- f derivabile su $]x,y[$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \xrightarrow{\text{Lagrange}} \exists\, z \in \,]x,y[\ \text{t.c.}$$

$$\boxed{\underbrace{f'(z)}_{\textcircled{1}} = \frac{\overbrace{f(y)-f(x)}^{\textcircled{2}}}{\underbrace{y-x}_{\textcircled{3}}}}$$

$$\underbrace{\textcircled{1} = \frac{\textcircled{2}}{\textcircled{3}} \qquad \textcircled{1} \geq 0 \qquad \textcircled{3} > 0}$$

$\Rightarrow \textcircled{2} \geq 0 \Rightarrow f(y) - f(x) \geq 0$

$\Rightarrow f(y) \geq f(x) \quad \square$

Corollario (Lagrange)

f derivabile $\forall x \in I$

$f'(x) > 0 \ \forall x \in I \setminus \mathcal{F}(I)$ $\Rightarrow$ f è monotona strettamente crescente (decrescente)

$(f'(x) < 0)$

Esercizio

Provare che $\operatorname{arctg} x + \operatorname{arctg}\left(\frac{1}{x}\right) = \begin{cases} \pi/2 & \forall x > 0 \\ -\pi/2 & \forall x < 0 \end{cases}$

dim

$x < 0$ (nel testo provare $x > 0$)

Ts $f(x) = \operatorname{arctg} x + \operatorname{arctg} \frac{1}{x} = -\pi/2 \quad \forall x \in]-\infty, 0[$

devo provare che f è costante in $]-\infty, 0[$

Suff. $f'(x) \equiv 0 \quad \forall x \in]-\infty, 0[$

$$f'(x) = \left(\operatorname{arctg} x\right)' + \left(\operatorname{arctg} \frac{1}{x}\right)'$$

$$= \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(\frac{1}{x}\right)'$$

$$= \frac{1}{1+x^2} - \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \frac{1}{x^2}$$

$$= 0 \quad \forall x$$

$$\Rightarrow f(x) \equiv \text{cost.} = f(-1) = \operatorname{arctg}(-1) + \operatorname{arctg}(-1)$$
$$= -\frac{\pi}{2}$$

Esercizio

$$e^x > \operatorname{sen} x + \cos x \quad \forall x \in]0, \pi/2[$$

dim

$$f(x) = e^x - \operatorname{sen} x - \cos x$$

Ts $f(x) > 0 \quad \forall x \in]0, \pi/2[$

$$f(0) = 1 - 0 - 1 = 0$$

$$f'(x) = e^x - \cos x + \operatorname{sen} x$$

$$f''(x) = e^x + \operatorname{sen} x + \cos x \geq 1 + 0 + 0 \quad \forall x \in]0, \pi/2[$$

$$f''(x) > 0 \ \forall x \in]0, \pi/2[\Rightarrow f'(x) \nearrow \ \& \ f'(0) = 1 - 1 + 0 = 0$$

$$\Rightarrow f'(x) > 0 \quad \forall x \in]0, \pi/2[$$

$$\Rightarrow f(x) \nearrow \ \& \ f(0) = 0$$

$$\Rightarrow f(x) > 0 \quad \forall x \in]0, \pi/2[$$