Successioni - Lez 4

Titolo nota 20/11/2010

La successione $\{a_n\}_n = \{q^n\}_n$ ha come limite

$$q^n \xrightarrow[n\to+\infty]{} \begin{cases} +\infty, & \text{se } q>1 \\ 1, & \text{se } q=1 \\ 0, & \text{se } 0\le q<1 \end{cases}$$

$$\sqrt[n]{q^n} = q \xrightarrow[n\to+\infty]{} q$$

$$\frac{q^{n+1}}{q^n} = q \xrightarrow[n\to+\infty]{} q$$

Osserviamo che $a_n \le q^n \ \forall n \geq n_0 \Rightarrow \sqrt[n]{a_n} \le q \ \forall n \geq n_0$

$$\Rightarrow \lim_n \sqrt[n]{a_n} \le q$$

e quindi, se $l = \lim \sqrt[n]{a_n} < 1$, allora

$a_n \to 0$

Viceversa, se $a_n \geq q^n \ \forall n \geq n_0 \Rightarrow \sqrt[n]{a_n} \geq q \ \forall n \geq n_0$

$$\Rightarrow \lim_n \sqrt[n]{a_n} \geq q$$

e quindi, se $l = \lim_n \sqrt[n]{a_n} > 1$ allora $a_n \to +\infty$

Teorema (Criterio radice per successioni $a_n \geq 0$)

$\{a_n\}_n$ t.c. $a_n \geq 0 \ \forall n$, ed $\exists \lim_{n\to+\infty} \sqrt[n]{a_n} = \ell$

1) $0 \leq \ell < 1 \Rightarrow a_n \xrightarrow[n\to+\infty]{} 0$

2) $1 < \ell \Rightarrow a_n \xrightarrow[n\to+\infty]{} +\infty$

$\ell = 1$????

Teorema (Criterio rapporto per successioni $a_n > 0$)

$\{a_n\}_n$ t.c. $a_n > 0 \ \forall n$, ed $\exists \lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \ell$

1) $0 \leq \ell < 1 \Rightarrow a_n \xrightarrow[n\to+\infty]{} 0$

2) $1 < \ell \Rightarrow a_n \xrightarrow[n\to+\infty]{} +\infty$

$\ell = 1$????

dim. 2) $\ell > 1 \Rightarrow \forall \varepsilon > 0 \ \exists \bar{n} : \forall n > \bar{n} \ \sqrt[n]{a_n} > \ell - \varepsilon$

posto $\boxed{\varepsilon = \ell - \frac{\ell+1}{2}}$ $\exists \bar{n} : \forall n > \bar{n} \ \sqrt[n]{a_n} > \frac{\ell+1}{2} > 1$

e quindi $\exists \bar{n} : \forall n \geq \bar{n} \quad a_n > \left(\frac{\ell+1}{2}\right)^n$

$\left(\frac{\ell+1}{2}\right)^n \xrightarrow[n\to+\infty]{} +\infty$ (essendo $\frac{\ell+1}{2} > 1$) $\Rightarrow$ $a_n \xrightarrow[n\to+\infty]{} +\infty$ Teorema Confronto

1) $0 \leq \ell < 1 \Rightarrow \forall \varepsilon > 0 \ \exists \bar{n} \ \forall n > \bar{n} \quad \ell - \varepsilon < \sqrt[n]{a_n} < \ell + \varepsilon$

$\boxed{\varepsilon = \frac{\ell+1}{2} - \ell}$ $\quad \exists \bar{n} : \forall n > \bar{n} \quad \sqrt[n]{a_n} < \frac{\ell+1}{2}$

$\exists \bar{n} \quad \forall n > \bar{n} \quad a_n < \left(\frac{\ell+1}{2}\right)^n$

$\left(\frac{\ell+1}{2}\right)^n \longrightarrow 0^+$ $\left(\text{essendo } \frac{\ell+1}{2} < 1\right) \implies a_n \longrightarrow 0^+$ $\blacksquare$

Teorema

$\{a_n\}_n$, $a_n > 0 \ \forall n$

Regola tre criteri

Se $\exists \lim\limits_{n\to+\infty} \frac{a_{n+1}}{a_n} = \ell$ allora $\exists \lim\limits_{n\to+\infty} \sqrt[n]{a_n} = \ell$.

Oss Questi criteri non dicono nulla nel caso in cui $\ell = 1$

- $a_n = 1/n$ $\quad \sqrt[n]{a_n} = \frac{1}{\sqrt[n]{n}} \longrightarrow \frac{1}{1}$ ma $1/n \xrightarrow[n\to+\infty]{} 0$ $\|$
- $a_n = n$ $\quad \sqrt[n]{a_n} = \sqrt[n]{n} \longrightarrow 1$ ma $n \xrightarrow[n\to+\infty]{} +\infty$ $\|$

Esercizio $\sqrt[n]{n} \xrightarrow[n\to+\infty]{} 1$

dim $\sqrt[n]{n} = e^{\frac{\log n}{n}} \xrightarrow[n\to+\infty]{} e^0 = 1$

in quanto $\frac{\log n}{n} \xrightarrow[n\to+\infty]{} 0$ $\quad \left(\frac{n}{e^n} \xrightarrow[n\to+\infty]{} 0\right)$

Oss $\sqrt[n]{4} \xrightarrow[n\to+\infty]{} 1$, $\sqrt[n]{n^{500}} \xrightarrow[n\to+\infty]{} 1$

<u>Esempio</u> $\lim_{n\to+\infty} n! = +\infty$

<u>dim</u> $a_n = n!$ si ha che $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!}{n!} = (n+1) \xrightarrow[n\to+\infty]{} +\infty$

da cui segue la Tesi per il criterio del rapporto $\blacksquare$

<u>Esempio</u> $\lim_{n\to+\infty} n^n = +\infty$ $\qquad \left(n^n \geq n \quad \forall n \geq 1\right)$

<u>dim</u> $a_n = n^n$, si ha che

$$\sqrt[n]{n^n} = n \xrightarrow[n\to+\infty]{} +\infty$$ da cui la Ts per il criterio radice $\blacksquare$

Al tempo $t=0$, abbiamo un capitale $C = C(0)$

Dopo 1 anno abbiamo un capitale $C(\overline{T}) = C(0)(1+i)$

dove i è l'interesse

$$C(0) \longrightarrow C(\overline{T}) = C(0)(1+i)$$

Qualora si ricapitalizzi al tempo $T = \frac{\overline{T}}{2}$ si ha

$$C(0) \longmapsto C_2\left(\overline{T}/2\right) = C(0)\cdot\left(1+\frac{i}{2}\right) \longmapsto C_2(\overline{T}) = C(0)\left(1+\frac{i}{2}\right)^2$$

Ma si può ricapitalizzare 3 volte

$$C = C(0) \longmapsto C_3\left(\frac{\overline{T}}{3}\right) = C\cdot\left(1+\frac{i}{3}\right) \longmapsto C_3\left(\frac{2\overline{T}}{3}\right) = C\cdot\left(1+\frac{i}{3}\right)^2 \longmapsto C_3(\overline{T}) = C\cdot\left(1+\frac{i}{3}\right)^3$$

E si può iterare (x induzione) n volte

$$C(0) \longrightarrow C_n\left(\tfrac{T}{n}\right) = C\cdot\left(1+\tfrac{i}{n}\right)^n \longrightarrow \dots \longrightarrow C_n(T) = C\cdot\left(1+\tfrac{i}{n}\right)^n$$

Ci si convince in modo diretto (qualche conto)

che $\left(1+i\right) < \left(1+\frac{i}{2}\right)^2 < \left(1+\frac{i}{3}\right)^3 < \left(1+\frac{i}{4}\right)^4$

ma resta il dubbio se

$$\boxed{\left(1+\tfrac{i}{n}\right)^n < \left(1+\tfrac{i}{n+1}\right)^{n+1} \quad \forall n}$$

Questa disuguaglianza è vera, e si ha

che, preso $i=1$, la successione

$e_n = \left(1+\frac{1}{n}\right)^n$ $\nearrow$ è superiormente limitata

da $3-\frac{1}{12}$

Si pone $e \stackrel{\cdot}{=} \lim\limits_{n\to+\infty} e_n = \lim\limits_{n\to+\infty}\left(1+\frac{1}{n}\right)^n$

<u>Teorema</u> la successione $e_n = \left(1+\frac{1}{n}\right)^n$

- è strettamente crescente
- è limitata superiormente, e in particolare

$$2 \le e_n < 3 - \frac{1}{12} \quad \forall n$$

<u>dim</u> $\quad$ Dis Bernoulli $\quad$ $x \geq -1$ $\quad (1+x)^\alpha \geq 1+\alpha x$

- $e_n = \left(1+\frac{1}{n}\right)^n \geq 1 + n\cdot\frac{1}{n} = 2$ (Dis. di Bernoulli)
- $e_1 = \left(1+\frac{1}{1}\right)^1 = 2 < \frac{9}{4} = \left(1+\frac{1}{2}\right)^2 = e_2$

$$\text{TS} \quad e_n < e_{n+1} \quad \forall n$$

$$e_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} 1^{n-k} = \sum_{k=0}^{n} \frac{1}{k!} \frac{n!}{n^k (n-k)!}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \frac{\overbrace{n(n-1)\dots(n-k+1)}^{k \text{ Termini}}}{n^k}$$

$$\frac{n!}{n^k (n-k)!} = \frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)\cancel{(n-k)\dots 1}}{n^k \cdot \cancel{(n-k)\dots 1}}$$

$$= 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \quad 1 \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \ldots \cdot \frac{n-k+1}{n}$$

$$= 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \cdot \underbrace{1 \cdot \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n}\right)}_{= P_{k,n}}$$

$$P_{k,n} = \prod_{i=1}^{k-1}\left(1-\frac{i}{n}\right) = \left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right)$$

$$= 2 + \sum_{k=2}^{n} \frac{1}{k!} \underbrace{\boxed{1\cdot\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right)}}_{= P_{k,n}}$$

a) $0 < P_{k,n} < 1 \quad \forall k < n \quad \forall n$ (prodotto di termini <1)

b) $P_{k,n} < P_{k,n+1} \quad \forall k < n \quad \forall n$

infatti

$$P_{k,n} = \left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right) < \left(1-\frac{1}{n+1}\right)\cdot\left(1-\frac{2}{n+1}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n+1}\right) = P_{k,n+1}$$

dunque $P_{k,n}$ è monotona strett. $\nearrow$, è limitata superiormente e quindi

$$\exists \lim_{n\to+\infty} P_{k,n} = l \leq 1$$

In particolare $\boxed{l = 1}$ infatti

$$\lim_n P_{k,n} = \lim_n \left(1-\frac{1}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right) = 1\cdot 1\cdot\ldots\cdot 1 = 1$$

(limite di un prodotto = prodotto limiti)

Dunque

$$e_n = 2 + \sum_{k=2}^{n} \frac{1}{k!}\cdot P_{k,n} < 2 + \sum_{k=2}^{n} \frac{1}{k!} P_{k,n+1}$$

ma a sua volta

$$2+\sum_{k=2}^{n}\frac{1}{k!}P_{k,n+1} < 2+\sum_{k=2}^{n+1}\frac{1}{k!}P_{k,n+1}$$

$$\parallel$$

$$e_{n+1}$$

da cui segue la tesi $e_n < e_{n+1}\ \forall n$

Posto $b_n=\sum_{k=0}^{n}\frac{1}{k!}$, si sa che $k!\geq 2^{k-1}\ \forall k\in\mathbb{N}$

da cui segue $\frac{1}{k!}\leq\frac{1}{2^{k-1}}\ \forall k\in\mathbb{N}$ ovvero

$$b_n=\sum_{k=0}^{n}\frac{1}{k!}=1+1+\frac{1}{2}+\frac{1}{6}+\sum_{k=4}^{n}\frac{1}{k!}$$

$k-1=j$

$\frac{1}{k!}\leq\frac{1}{2^{k-1}}$

$$\leq\frac{8}{3}+\sum_{k=4}^{n}\left(\frac{1}{2}\right)^{k-1}=\frac{8}{3}+\sum_{j=3}^{n-1}\left(\frac{1}{2}\right)^{j}$$

$k=4\to j=3$

$k=n\to j=n-1$

$$=\frac{8}{3}+\frac{1}{8}\boxed{\sum_{j=0}^{n-4}\left(\frac{1}{2}\right)^{j}}=\frac{8}{3}+\frac{1}{8}\cdot\frac{1-\left(\frac{1}{2}\right)^{n-3}}{1-\frac{1}{2}}$$

$$\leq\frac{8}{3}+\frac{1}{8}\cdot\lim_{n\to+\infty}\frac{1-\left(\frac{1}{2}\right)^{n-3}}{1-\frac{1}{2}}=\frac{8}{3}+\frac{1}{4}$$

$$=3-\frac{1}{12}$$

$$b_n = \sum_{k=0}^{n} \frac{1}{k!}$$ è monotona strett. crescente

Infatti $b_n = \sum_{k=0}^{n} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \dots + \frac{1}{n!} < \frac{1}{0!} + \frac{1}{1!} + \dots + \frac{1}{n!} + \frac{1}{(n+1)!}$

$$\frac{1}{0!} + \frac{1}{1!} + \dots + \frac{1}{n!} + \frac{1}{(n+1)!} = \sum_{k=0}^{n+1} 1/k! = b_{n+1}$$

Quindi b_n è $\nearrow$ e limitata da $3 - 1/12$

Inoltre $e_n \nearrow$, $e_n \leq b_n \ \forall n$

$\Rightarrow e_n \nearrow$, $e_n \leq 3 - 1/12 \ \forall n$

$\Rightarrow \exists \lim_n e_n = \boxed{e} \in \,]2, 3 - 1/12[$

$\blacksquare$

Teorema: Data $b_n = \sum_{k=0}^{n} \frac{1}{k!}$, $b_n \nearrow e$

$\lim_{n\to+\infty} b_n = e$

dim

- b_n è monotona crescente, infatti

$$b_n = \sum_{k=0}^{n} \frac{1}{k!} < \sum_{k=0}^{n} \frac{1}{k!} + \frac{1}{(n+1)!} = b_{n+1}$$

- Nel Teorema precedente si è visto che

$$b_n < 3 - \frac{1}{12}$$

Quindi $\exists \lim_{n\to+\infty} b_n = l$

Inoltre, essendo $e_n \le b_n \Rightarrow \boxed{e \le l}$

Ma, fissato $m \geq 2$, $\forall n \geq m$

$$e_n = 2 + \sum_{k=0}^{n} \frac{1}{k!} P_{k,n} \geq 2 + \sum_{k=0}^{m} \frac{1}{k!} P_{k,n}$$

$$\Downarrow$$

$$e = \lim_n e_n \geq \lim_n \left(2 + \sum_{k=0}^{m} \frac{1}{k!} P_{k,n}\right)$$

$$= 2 + \sum_{k=0}^{m} \frac{1}{k!} = b_m$$

$$\Downarrow$$

$$\boxed{e \geq l = \lim_{m\to+\infty} b_m}$$

Ne segue che $e = l$ ∎

Formula di Stirling

$$\frac{n^n}{e^n} \leq n! \leq n \cdot e \cdot \frac{n^n}{e^n} \quad \forall n \in \mathbb{N}$$

Questa formula serve x stimare il fattoriale

$$x^x = \begin{cases} e^{x \log x} & x > 0 \\ 0 & x = 0 \end{cases}$$

Esercizio

$$\frac{n^n}{e^n} \leq n!$$

$n=0 \quad \frac{1}{1} \leq 1$ ✓ $\qquad n=1 \quad \frac{1}{e} \leq 1$ ✓

$n=2 \quad \frac{4}{e^2} \leq 2$

suppongo $\frac{n^n}{e^n} \leq n!$ Hp induttiva

TS $\frac{(n+1)^{(n+1)}}{e^{n+1}} \leq (n+1)!$

$$(n+1)! = (n+1) \cdot n! \geq (n+1) \cdot \frac{n^n}{e^n} = \underbrace{\frac{n^{n+1}}{e^n} + \frac{n^n}{e^n}}_{\geq ?\ \frac{(n+1)^{n+1}}{e^{n+1}}}$$

$$\cancel{(n+1)}\,\frac{n^n}{e^n} > \frac{(n+1)^{\cancel{n+1}}}{e^{n+1}} \iff \frac{n^n}{\cancel{e^n}} > \frac{(n+1)^n}{e^{\cancel{n}+1}}$$

$$\iff n^n > \frac{(n+1)^n}{e}$$

$$\iff e \geq \left(\frac{n+1}{n}\right)^n$$

$$\iff e \geq \left(1+\frac{1}{n}\right)^n \quad \forall n$$

<u>Vero</u> perché

$$\left(1+\frac{1}{n}\right) \nearrow e$$

Adesso possiamo definire

$$\boxed{e^x := \lim_{n\to+\infty}\left(1+\frac{x}{n}\right)^n \qquad x\in\mathbb{R}} \; *$$

$$\left(\text{o anche } \boxed{e^x = \sum_{k=0}^{+\infty}\frac{x^k}{k!}} \quad \boxed{x\in\mathbb{R}}\right)$$

$$e^0 = \lim_{n\to+\infty}\left(1+\frac{0}{n}\right)^n = 1$$

Esercizio

$$n! \geq \frac{n^n}{e^n}$$

$$\lim_n \frac{\log(n!)}{n} = +\infty$$

$$\frac{n^n}{e^n} \leq n! \Rightarrow n\log\left(\frac{n}{e}\right) \leq \log(n!)$$

$$\log x \xrightarrow[x\to+\infty]{} +\infty$$

$$\Rightarrow \frac{1}{n}\left(n\log\left(\frac{n}{e}\right)\right) \leq \frac{\log(n!)}{n}$$

$$\Rightarrow \boxed{\log\left(\frac{n}{e}\right) \leq \frac{\log(n!)}{n}}$$

dal Thm confronto ho la tesi, poiché $\lim_n \log\left(\frac{n}{e}\right) = +\infty$ ∎

Oss $\lim_{n} \left(1+\frac{1}{n}\right)^{n^2} = +\infty$

1^{∞}

Infatti, $e_n = \left(1+\frac{1}{n}\right)^n \geq 2 \quad \forall n$

$\Rightarrow (e_n)^n = \left(1+\frac{1}{n}\right)^{n^2} \geq 2^n \quad \forall n$

$\lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^{n^2} = \lim_{n\to+\infty} (e_n)^n \geq \lim_{n\to+\infty} 2^n = +\infty$

$\lim_{n} \left(1+\frac{1}{n^2}\right)^n = 1 \qquad (2 \leq e < 3!)$

1^{∞}

$e_n = \left(1+\frac{1}{n}\right)^n \xrightarrow[n\to+\infty]{} e$

$e_{n^2} = \left(1+\frac{1}{n^2}\right)^{n^2} \xrightarrow[n\to+\infty]{} e$

$2 \leq e_{n^2} \leq 3$

$\sqrt[n]{2} \leq \sqrt[n]{e_{n^2}} \leq \sqrt[n]{3}$

$e^{\frac{\log 2}{n}} = \sqrt[n]{2} \leq \left(1+\frac{1}{n^2}\right)^n \leq \sqrt[n]{3} = e^{\frac{\log 3}{n}}$

$\downarrow \qquad\qquad \downarrow n\to+\infty$

$1 \qquad\qquad 1$

Teorema (limiti f.ni vs limiti successioni)

$f: A \to \mathbb{R}$, x_0 p.d.a. per A. Le due proposizioni seguenti sono tra loro equivalenti

① $\exists \lim_{x \to x_0} f(x) = \ell$ $\Downarrow$ banale ($f_{|\{x_n\}}$

② $\forall \{x_n\} \subset A \setminus \{x_0\}$, $x_n \xrightarrow[n \to +\infty]{} x_0 \Rightarrow f(x_n) \xrightarrow[n \to +\infty]{} \ell$

dim

$\boxed{\boxed{x_0, \ell \in \mathbb{R}}}$

① $\Rightarrow$ ② è ovvio

② $\Rightarrow$ ① si ragiona per assurdo, e si suppone che

non $\left(\forall \varepsilon > 0 \ \exists \delta > 0 : \left[\forall x \in]x_0 - \delta, x_0 + \delta[\cap (A \setminus \{x_0\}) \Rightarrow |f(x) - \ell| < \varepsilon \right] \right)$

non $\left(\forall \varepsilon > 0 \ \exists \delta > 0 \quad \text{Pred}(\delta, \varepsilon) \text{ è vera} \right)$

$\exists \bar{\varepsilon} > 0 \quad \forall \delta > 0 \quad \text{Pred}(\delta, \varepsilon)$ è falsa

$\exists \bar{\varepsilon} > 0 \quad \forall \delta > 0 \quad \exists x_\delta \in]x_0 - \delta, x_0 + \delta[\cap (A \setminus \{x_0\})$ t.c. $|f(x_\delta) - \ell| \geq \bar{\varepsilon}$

$\exists \bar{\varepsilon} > 0 \quad \forall n > \bar{n} \quad \exists x_n \in]x_0 - \frac{1}{n}, x_0 + \frac{1}{n}[\cap (A \setminus \{x_0\})$ t.c. $|f(x_n) - \ell| \geq \bar{\varepsilon}$

Abbiamo costruito $\{x_n\}_n$

$\{x_n\} \subseteq A \setminus \{x_0\}$ $\quad x_n \xrightarrow[n \to +\infty]{} x_0$

ma $|f(x_n) - l| \geq \bar{\varepsilon}$ $\quad$ ASSURDO

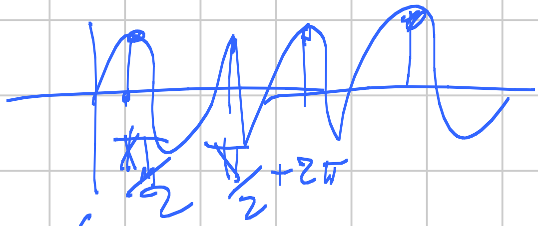
$\blacksquare$

Oss questo risultato, che è della forma

$A \Longleftrightarrow B$

si cerca nella forma non A sse non B

Esercizio $\nexists \lim_{\cancel{x} \to +\infty} \operatorname{sen} x$



dim

$\operatorname{sen}(x_n) = 1 \ \forall n$

Prese $x_n = \frac{\pi}{2} + 2n\pi$, si ha $\lim_{n \to +\infty} \operatorname{sen}(x_n) = 1$

$\neq$

$y_n = \frac{3}{2}\pi + 2n\pi$ " " $\lim_{n} \operatorname{sen}(y_n) = -1$

$\operatorname{sen}(y_n) = -1 \ \forall n$