# Limiti - lezione 2 dell'8/11/2010

Titolo nota 04/11/2010

## Teoremi algebrici sui limiti

<u>Teorema</u> $f, g : A \to \mathbb{R}$ due funzioni,

sia $x_0$ p.d.a. per $A$.

1) $\exists \lim_{x \to x_0} f(x) = p \quad \lim_{x \to x_0} g(x) = q \Rightarrow \exists \lim_{x \to x_0} (f+g)(x) = p+q$

2) " " $\Rightarrow \exists \lim_{x \to x_0} (f \cdot g)(x) = p \cdot q$

3) " " " & $g(x) \neq 0 \Rightarrow \exists \lim_{x \to x_0} \left(\frac{f}{g}\right)(x) = p/q$

purchè le scritte $p+q$, $p \cdot q$, $p/q$ abbiano senso
non siano indet.

<u>Nota bene</u> Nel Teorema preced. non si considerano i casi indeterminati, ovvero

$+\infty - \infty$ ; $0 \cdot \infty$ ; $0/0$ ; $\infty/\infty$ .

<u>Nota bene</u> Tutti i casi indeterminati sono tra loro equivalenti

in parole povere: somma di limite = limite somma
prodotto " " = " prodotto
quoziente " " = " quoziente

$\exists \lim_{x\to x_0} f = p \quad \lim_{x\to x_0} g = q \implies \exists \lim_{x\to x_0} (f\cdot g)(x) = p\cdot q$

dim

supponiamo $p, q \in \mathbb{R} \quad x_0 \in \mathbb{R}$

Hp $\begin{cases} \forall \varepsilon > 0 \ \exists \delta_1 > 0 \ \forall x \in A \setminus \{x_0\} \ |x-x_0| < \delta_1 \Rightarrow |f(x)-p| < \varepsilon \\ \forall \varepsilon > 0 \ \exists \delta_2 > 0 \ \forall x \in A \setminus \{x_0\} \ |x-x_0| < \delta_2 \Rightarrow |g(x)-q| < \varepsilon \end{cases}$

Ts $\forall \eta > 0 \ \exists \delta > 0 \ \forall x \in A \setminus \{x_0\} \ |x-x_0| < \delta \Rightarrow |f\cdot g - pq| < \eta$

$$|f(x)\cdot g(x) - p\cdot q| = |f(x)\cdot g(x) - \overline{f(x)\cdot q} + \overline{f(x)\cdot q} - p\cdot q|$$

$$= |f(x)\cdot(g(x)-q) + q(f(x)-p)|$$

$$\leq |f(x)\cdot(g(x)-q)| + |q(f(x)-p)|$$

$$= \underset{①}{|f(x)|} \cdot \underset{②}{|g(x)-q|} + \underset{③}{|q|} \cdot \underset{④}{|f(x)-p|}$$

1. $\lim_{x\to x_0} f(x) = p \Rightarrow$ prendo $\varepsilon = 1/4$, $\exists \delta_\varepsilon > 0$:

$\forall x \in A \setminus \{x_0\}$ $|x - x_0| < 1/4$ $p - 1/4 < f(x) < p + 1/4$

$\Rightarrow \exists \delta_3 > 0 : \forall x \in A \cap \{0 < |x - x_0| < \delta_3\}$

$$|f(x)| \leq \max\{|p - 1/4|, |p + 1/4|\}$$

2. dalla def. di $\lim_{x\to x_0} g(x) = q$ si ha

$\forall \varepsilon > 0 \ \exists \delta_2 > 0 : \forall x \in A \setminus \{x_0\}$ $|x - x_0| < \delta_2$

$$|g(x) - q| < \varepsilon$$

3. non devo fare nulla, in quanto $|q| \in [0, +\infty[$

4. dalla def di $\lim_{x\to x_0} f(x) = p$ si ha

$\forall \varepsilon > 0 \ \exists \delta_1 > 0 : \forall x \in A \setminus \{x_0\} \ |x - x_0| < \delta_1$

$$|f(x) - p| < \varepsilon$$

Adesso abbiamo tutti gli ingredienti, e prendendo $x \in A$ t.c.

$$0 < |x - x_0| < \bar{\delta} = \min\{\delta_1, \delta_2, \delta_3\}$$

le disuguaglianze ① ② e ④ sono soddisfatte ovvero

$$\underbrace{|f(x)|}_{①} \cdot \underbrace{|g(x) - q|}_{②} + \underbrace{|q|}_{③} \cdot \underbrace{|f(x) - p|}_{④} \leq$$

se $|x - x_0| < \bar{\delta}$

$$\underbrace{\max\{|p - 1/4|, |p + 1/4|\} \cdot \varepsilon + |q| \cdot \varepsilon}_{\eta}$$

$\forall \eta > 0 \ \exists \bar{\delta} > 0 \ \forall x \in A \setminus \{x_0\} \ |x - x_0| < \bar{\delta} \ \ |f \cdot g - p \cdot q| < \eta$

# Alcuni controesempi

$+\infty - \infty \overset{?}{=}$ 1

$0 \cdot \infty \overset{?}{=}$ 2

$\frac{0}{0} \overset{?}{=}$ 3

$\frac{\pm\infty}{\pm\infty} \overset{?}{=}$ 4

1 $f(x) = x^2 \quad g(x) = -x$

$\lim_{x\to+\infty} f(x) = +\infty \quad \lim_{x\to+\infty} g(x) = -\infty$

$\lim_{x\to+\infty} (f(x)+g(x)) \; (= +\infty - \infty) =$

$= \lim_{x\to+\infty} x(x-1)$

$= \lim_{x\to+\infty} x \cdot \lim_{x\to+\infty} (x-1)$

$= +\infty \cdot +\infty = +\infty$

però

$f(x) = x \quad g(x) = -x^2 \qquad f(x) \to +\infty \quad g(x) \to -\infty$

$\lim_{x\to+\infty} f(x) - g(x) = \lim_{x\to+\infty} x(1-x) =$

$= \lim_{x\to+\infty} x \cdot \lim_{x\to+\infty} (1-x) =$

$= +\infty \cdot (-\infty) = -\infty$

però $f(x) = x + \pi \qquad g(x) = -x$

$f(x) \longrightarrow +\infty \qquad g(x) \longrightarrow -\infty$

$\lim_{x\to+\infty} (f+g)(x) = \pi$

Teorema (della permanenza del segno)

$f: A \to \mathbb{R}$, $x_0$ p.d.a. per $A$.

Se $\exists \lim_{x \to x_0} f(x) = l > 0$ allora $\exists U \in \mathcal{U}_{x_0}$ t.c.

$f(x) > 0 \quad \forall x \in U \cap A$

dim

$x_0 = +\infty \qquad \lim_{x \to +\infty} f(x) = l > 0 \qquad l \in ]0, +\infty[$

$\Leftrightarrow \forall \varepsilon > 0 \; \exists M_\varepsilon \in \mathbb{R} : \forall x \in A \quad x > M \Rightarrow l - \varepsilon < f(x) < l + \varepsilon$

(devo fare in modo che $l - \varepsilon > 0$) sapendo che $l > 0$

$\Rightarrow$ preso $\varepsilon = l/2 \quad \exists M = M(l/2) : \forall x \in A \quad x > M$

$0 < \frac{l}{2} = l - \frac{l}{2} < f(x)$

$\blacksquare$

Oss Se $f(x) > 0 \quad \forall x \in [0, +\infty[$ ed $\exists \lim_{x \to +\infty} f(x) = l$

allora?? $l > 0$ ???

NO !! Controesempio $f(x) = 1/x > 0 \quad \forall x \in ]0, +\infty[$

ma $\lim_{x \to +\infty} 1/x = 0$

Teorema (del confronto)

$f, g : A \to \mathbb{R}$ due funzioni, $x_0$ p.d.a. per $A$

•) $\exists \lim_{x \to x_0} f(x) = p \in \overline{\mathbb{R}}$ $\exists \lim_{x \to x_0} g(x) = q \in \overline{\mathbb{R}}$

•) $\forall x \in A \quad f(x) \leq g(x)$

Allora $\quad p \leq q$

dim $\qquad x_0, p, q \in \mathbb{R}$

$\forall \varepsilon > 0 \ \exists \delta_1 > 0 \ : \forall x \in A \ 0 < |x - x_0| < \delta_1 \quad p - \varepsilon < f(x) < p + \varepsilon$

$\forall \varepsilon > 0 \ \exists \delta_2 = \delta_2(\varepsilon) > 0 \ : \forall x \in A \ 0 < |x - x_0| < \delta_2 \quad q - \varepsilon < g(x) < q + \varepsilon$

$\bar{\delta} = \min\{\delta_1, \delta_2\} = \delta_1 \wedge \delta_2$

$\forall \varepsilon > 0 \ \exists \bar{\delta} = \bar{\delta}(\varepsilon) > 0 : \forall x \in A \ |x - x_0| < \bar{\delta} \quad \begin{cases} p - \varepsilon < f(x) < p + \varepsilon \\ q - \varepsilon < g(x) < q + \varepsilon \end{cases}$

$$q - \varepsilon - (p + \varepsilon) < g(x) - f(x) < q + \varepsilon - (p - \varepsilon)$$

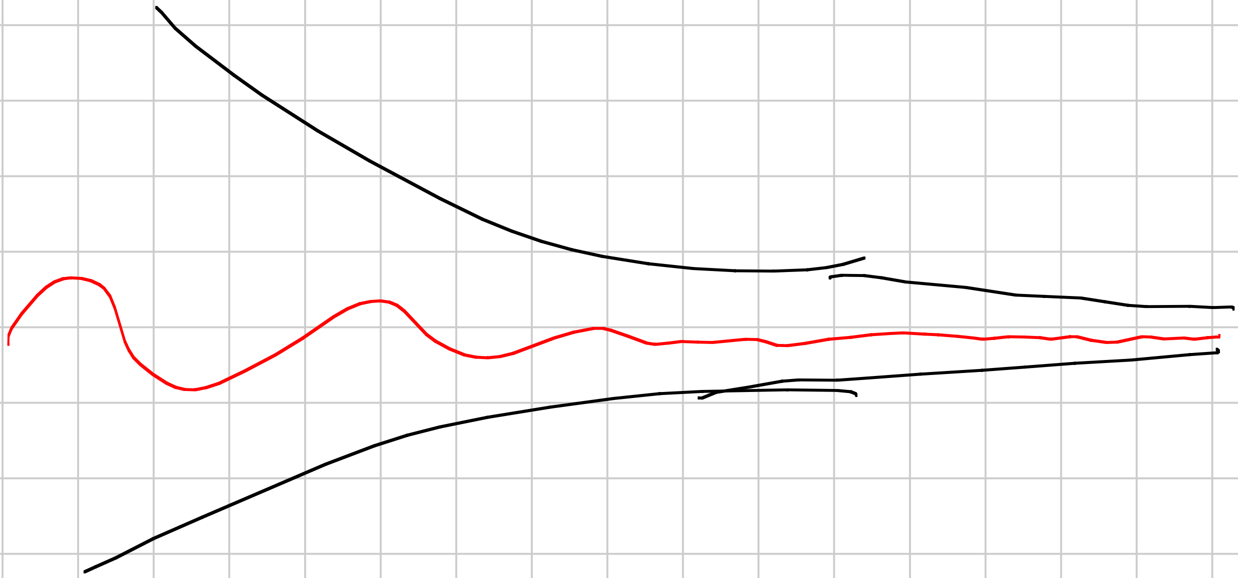
$$(q - p) - 2\varepsilon < g(x) - f(x) < (q - p) + 2\varepsilon$$

ma $g(x) - f(x) \geq 0 \implies \forall \varepsilon > 0 \quad q - p \geq -2\varepsilon$

$\iff q - p \geq 0$

Obs 1) $f(x) \to +\infty$, $f \le g \Rightarrow g(x) \to +\infty$

2) $g(x) \to -\infty$, $f \le g \Rightarrow f \to -\infty$



## Teorema (dei 2 carabinieri)

$f, g, h : A \to \mathbb{R}$, $x_0$ p.d.a. per $A$

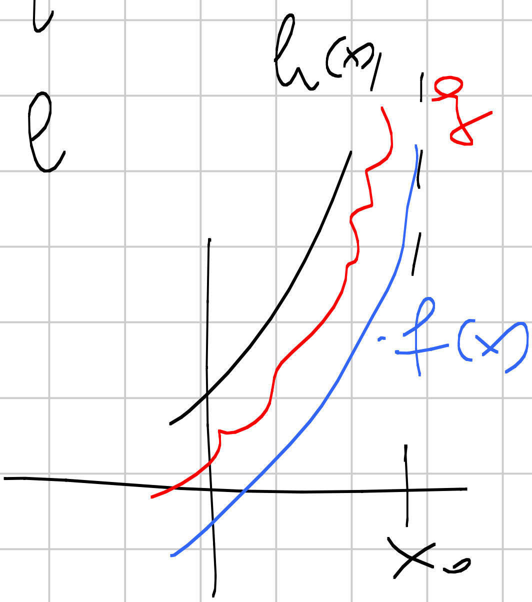
1) $f(x) \le g(x) \le h(x) \quad \forall x \in A$

2) $\exists \lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = \ell$

Allora $\exists \lim_{x \to x_0} g(x) = \ell$

dim

$x_0 \in \mathbb{R} \qquad \ell = +\infty$




1) $\forall M \in \mathbb{R} \ \exists \delta_1 > 0 : \forall x \in A \ 0 < |x - x_0| < \delta_1 \Rightarrow M < f(x)$

2) $\forall M \in \mathbb{R} \ \exists \delta_2 > 0 : \forall x \in A \ 0 < |x - x_0| < \delta_2 \Rightarrow M < h(x)$

3) $f(x) \le g(x) \le h(x)$

$$\bar{\delta} = \delta_1 \wedge \delta_2$$

$\forall M \in \mathbb{R} \ \exists \bar{\delta} = \bar{\delta}(M) > 0 : \forall x \in A \ 0 < |x - x_0| < \bar{\delta} \Rightarrow$

$$M < f(x) \le g(x) \le h(x)$$

$$\Downarrow$$

$\forall M \in \mathbb{R} \ \exists \bar{\delta} = \bar{\delta}(M) \ \forall x \in A \ 0 < |x - x_0| < \bar{\delta} \Rightarrow M < g(x)$

$\Longrightarrow \lim_{x \to +\infty} g(x) = +\infty$ $\blacksquare$

**Teorema** (limitato per infinitesimo = infinitesimo)

$f, g : A \to \mathbb{R}$, $x_0$ p.d.a. per $A$, t.c.

1) $\exists \lim_{x \to x_0} f(x) = 0$ (infinitesima)

2) $|g(x)| \leq M \quad \forall x \in A$ (limitata)

$\Longrightarrow \exists \lim_{x \to x_0} f(x) \cdot g(x) = 0$

**Esempio** $f(x) = 1/x \qquad g(x) = \operatorname{sen} x$

$\lim_{x \to +\infty} f(x) = 0 \qquad |g(x)| \leq 1 \quad \forall x \in \mathbb{R}$

$$\Longrightarrow \lim_{x \to +\infty} \frac{\operatorname{sen} x}{x} = \lim_{x \to +\infty} \underbrace{\left(\frac{1}{x}\right)}_{\text{infinitesimo}} \cdot \underbrace{(\operatorname{sen} x)}_{\text{limitato}} = 0$$

**dim** (del Teorema)

$x_0 \in \mathbb{R}$

(1) $\forall \varepsilon > 0 \ \exists \delta > 0 : \forall x \in A \quad \overbrace{0 < |x - x_0| < \delta}^{} \quad -\varepsilon < f(x) < +\varepsilon$

2) $\exists M>0 : |g(x)| \leq M \quad \forall x \in A$

$$-M \leq g(x) \leq M$$

$$\underset{\downarrow}{-M \cdot \varepsilon} < f(x)\cdot g(x) < \overset{\uparrow}{M \cdot \varepsilon}$$

$\implies \forall \varepsilon>0 \ \exists \delta>0 : \forall x \in A \quad 0<|x-x_0|<\delta$

$$|f(x)\cdot g(x)| < M \cdot \varepsilon$$

$\eta = M \cdot \varepsilon$

$\iff \forall \eta>0 \ \exists \delta>0 : \forall x \in A \quad 0<|x-x_0|<\delta$

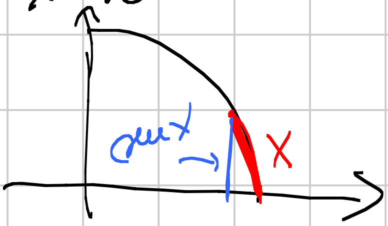
$$|(f\cdot g)(x)| < \eta$$

$\iff \lim_{x \to x_0} (f\cdot g)(x) = 0 \quad \blacksquare$

## Esercizio

1) $\lim_{x\to 0} \operatorname{sen} x = 0 \implies \lim_{x \to x_0} \operatorname{sen} x = \operatorname{sen} x_0$

$\lim_{x\to 0} \operatorname{sen} x = 0$ f.ni dispari




$0 \leq \operatorname{sen} x \leq x \quad \forall x>0$

$0 \leq \operatorname{sen}(y) \leq y \quad \forall x<0 \quad y=-x$

$$0 \le \operatorname{sen}(-x) \le -x \quad \forall x<0$$

$$0 \le -\operatorname{sen} x \le -x \quad \forall x<0$$

$$x \le \operatorname{sen} x \le 0 \quad \forall x<0$$

$$\Downarrow$$

$$\underset{f(x)}{0} \le \underset{g(x)}{|\operatorname{sen} x|} \le \underset{h(x)}{|x|} \quad \forall x \in \mathbb{R}$$

$$\lim_{x\to 0} f(x) = 0 \qquad \lim_{x\to 0} h(x) = 0$$

$$f(x) \le g(x) \le h(x) \quad \forall x \in \mathbb{R}$$

$\Longrightarrow$ (th. 2 carab.) $\displaystyle \lim_{x\to 0} g(x) = \lim_{x\to 0} \operatorname{sen} x = 0$

2ª parte

$$\lim_{x\to x_0} \operatorname{sen} x = \operatorname{sen} x_0$$

$$\operatorname{sen} x = \operatorname{sen}[(x-x_0)+x_0] =$$

$$= \underbrace{\operatorname{sen}(x-x_0)}_{\downarrow x\to x_0} \cdot \underbrace{\cos x_0}_{\text{limitata}} + \operatorname{sen} x_0 \underbrace{\cos(x-x_0)}_{\downarrow x\to x_0}$$

Voglio $\longrightarrow$ 0 ~~[scribbled]~~ 1

vero $|\cos x| \le 1$

significa $\operatorname{sen} y \underset{y\to 0}{\longrightarrow} 0$

Non lo so

$$\lim_{x\to x_0} \operatorname{sen} x = \lim_{x\to x_0} \operatorname{sen}(x-x_0)\cdot \lim_{x\to x_0}\cos x_0 + \lim_{x\to x_0} \operatorname{sen} x_0 \cdot \lim_{x\to x_0}\cos(x-x_0)$$

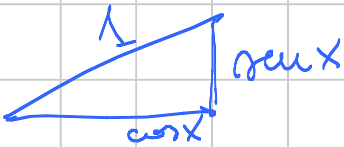
$$= 0\cdot\cos x_0 + \operatorname{sen} x_0 \cdot 1$$

$$= \operatorname{sen} x_0$$

2) $\lim_{x\to 0}\cos x = 1$ $\quad \left(\operatorname{sen}x+\cos x\geq 1 \ \& \ |\operatorname{sen}x|\leq x \ \& \ 1\geq|\cos x|\right)$

$\Downarrow$

$$\lim_{x\to x_0}\cos x = \cos x_0$$




$\underline{x>0}$ $\quad \operatorname{sen}x+\cos x\geq 1$

$|\operatorname{sen}x|\leq x$

$|\cos x|\leq 1$

Obbiettivo $\quad f(x)\leq\cos x\leq h(x)$

$f(x)\downarrow 1 \qquad h(x)\downarrow 1 \qquad x\to 0$

$\boxed{h(x)=1}$ $\qquad \cos x+\operatorname{sen}x\geq 1 \quad \underline{x>0}$

$\cos x \geq 1-\operatorname{sen}x \quad x>0$

$-x\leq \operatorname{sen}x\leq x$

$x\geq -\operatorname{sen}x\geq -x$

$1-\operatorname{sen}x \leq \cos x$

$\Downarrow$

$1-x \leq \cos x \qquad x>0 \Rightarrow -x\leq \cos x-1$

$1+x \le \cos x$ $\qquad x<0 \Rightarrow x \le \cos x - 1$

$\underline{x>0}$ $\qquad 1-x \le \cos x \le 1$

$$\downarrow x\to 0^+ \qquad\qquad \downarrow x \to 0^+$$

$$1 \qquad\qquad 1$$

$$\lim_{x\to 0^+} \cos x = 1.$$

$x<0$ $\qquad 1+x \le \cos x \le 1$

$$\downarrow x\to 0^- \qquad\qquad \downarrow x \to 0^-$$

$$1 \qquad\qquad 1$$

$$\lim_{x\to 0^-} \cos x = 1 \longrightarrow \lim_{x\to 0} \cos x = 1$$

2) $\cos x \xrightarrow[x\to x_0]{} \cos x_0$

$$\cos x = \cos[(x-x_0)+x_0] = \underset{a}{\cos(x-x_0)}\,\underset{b}{\cos x_0} - \underset{c}{\sin(x-x_0)}\,\underset{d}{\sin x_0}$$

$$= a\cdot b - c\cdot d$$

(a) $\lim\limits_{x\to x_0} \cos(x-x_0) = \lim\limits_{y\to 0} \cos y = 1$

$y = x - x_0$

(b) $\lim\limits_{x\to x_0} \cos x_0 = \cos x_0$

(c) $\lim\limits_{x\to x_0} \operatorname{sen}(x-x_0) = \lim\limits_{y\to 0} \operatorname{sen} y = 0$

(d) $\lim\limits_{x\to x_0} \operatorname{sen} x_0 = \operatorname{sen} x_0$

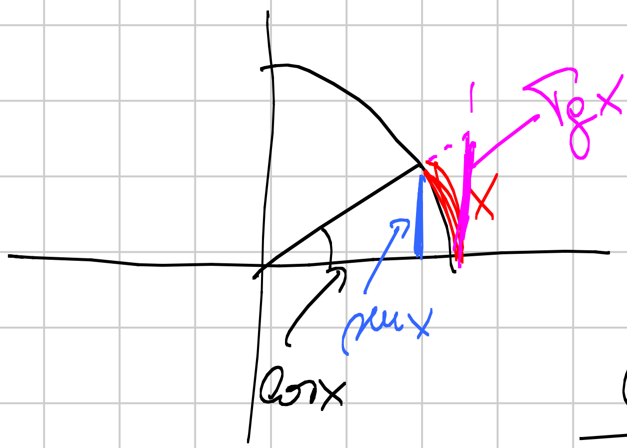
$$= \lim_{x\to x_0} (ab - cd) = 1\cdot\cos x_0 - 0\cdot\operatorname{sen} x_0 =$$

$$= \cos x_0$$

Esempio (importante)

$$\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = 1$$

dim




$x > 0$

$\operatorname{sen} x < x < \operatorname{Tg} x \qquad x > 0$

$$\boxed{1 \le \frac{x}{\operatorname{sen} x} \le \frac{1}{\cos x}} \qquad x > 0$$

$\frac{x}{\operatorname{sen} x}$ pari $\quad \frac{1}{\cos x}$ è pari $\quad$ 1 è pari

$$1 \leq \frac{x}{\operatorname{sen} x} \leq \frac{1}{\cos x} \quad x<0$$

$$\Downarrow$$

$$1 \geq \frac{\operatorname{sen} x}{x} \geq \cos x \quad \forall x \neq 0$$

$h(x) \qquad\qquad f(x)$

$$h(x) \xrightarrow[x\to 0]{} 1 \qquad f(x) \xrightarrow[x\to 0]{} 1 \implies \frac{\operatorname{sen} x}{x} \xrightarrow[x\to 0]{} 1$$

<u>**Oss**</u>: $f, g : A \to \mathbb{R}$, $x_0$ p.d.a.

$$\exists \lim_{x\to+\infty} f(x) = +\infty$$

$$|f(x)| \leq M \quad \forall x \in A \quad \not\Longrightarrow \quad \lim_{x\to x_0} f \cdot g = \infty$$

<u>**Perché?**</u>