$$\sum_n \frac{1}{n^\alpha} \qquad \int_1^{+\infty} \frac{dx}{x^\alpha}$$

C'è un legame tra serie e integrale improprio su dominio illimitato

Teorema (confronto tra serie e integrale improprio)

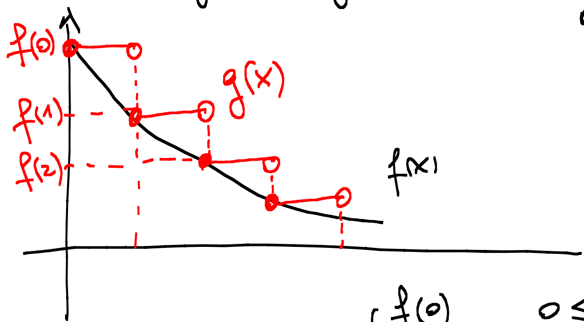
$f : [a,+\infty[\to \mathbb{R}$ $\quad a \in \mathbb{R}$

- $f(x) \geq 0 \quad \forall x \in [a,+\infty[$
- $f(x)$ sia debolmente decrescente su $[a,+\infty[$

Allora

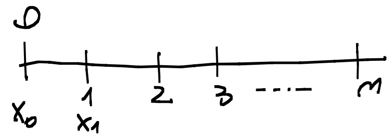
$\sum_n f(n)$ converge (diverge a $+\infty$) <u>sse</u> $\int_a^{+\infty} f(x)dx$ converge (diverge $+\infty$)

$\boxed{a=0}$



$f : [0,+\infty[\to \mathbb{R} \quad \longleftrightarrow \quad g(x) = \begin{cases} f(0) & 0 \leq x < 1 \\ f(1) & 1 \leq x < 2 \\ f(2) & 2 \leq x < 3 \\ \dots & \\ f(n) & n \leq x < n+1 \\ \dots & \end{cases}$

$$\sum_{n=0}^{\infty} f(n) = \int_0^{+\infty} g(x)dx$$



dim $\boxed{a=0}$ $\quad f : [0,+\infty[\to \mathbb{R}$

Prendo $n \in \mathbb{N}$ e considero la partizione

$$\mathcal{A} = \{0 < 1 < 2 < \dots < n-1 < n\} = \{x_k = k \ : k = 0,1,\dots,n\}$$

$$s(f,\mathcal{A}) = \sum_{k=0}^{n-1} (x_{k+1} - x_k) \cdot \inf f([x_k, x_{k+1}])$$

$$= \sum_{k=0}^{n-1} (\cancel{k}+1-\cancel{k}) \cdot f(x_{k+1}) = \sum_{k=0}^{n-1} f(k+1)$$

$$S(f,\mathcal{A}) = \sum_{k=0}^{n-1} (x_{k+1} - x_k) \sup f([x_k, x_{k+1}])$$

$$= \sum_{k=0}^{n-1} (\cancel{k}+1 - \cancel{k}) \cdot f(x_k)$$

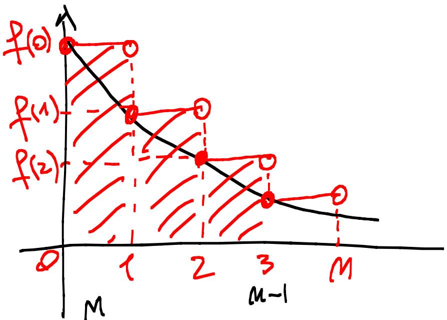
$$= \sum_{k=0}^{n-1} f(k)$$

$$\sum_{k=0}^{n-1} f(k+1) = s(f,A) \le \int_0^n f(x)\,dx \le S(f,A) = \sum_{k=0}^{n-1} f(k) \qquad \forall n \in \mathbb{N}$$

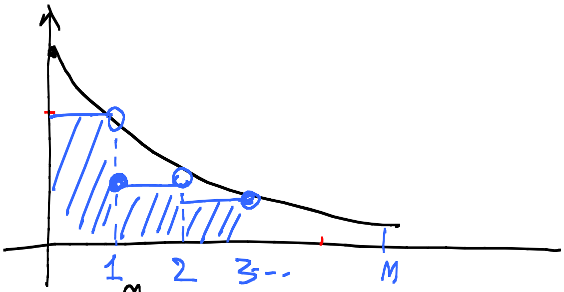
$$\sum_{k=1}^{n} f(k) \le \int_0^n f(x)\,dx \le \sum_{k=0}^{n-1} f(k) \qquad \forall n \in \mathbb{N}$$

i) Se $\sum_n f(n)$ converge allora $\int_0^{+\infty} f\,dx = \lim_{n\to+\infty} \int_0^n f(x)$ converge

ii) " $\int_0^{+\infty} f$ converge allora $\sum_1 {}_n f(n)$ converge



$$\int_0^n f(x)\,dx \le \sum_{k=0}^{n-1} f(k)$$



$$\int_0^n f(x)\,dx \ge \sum_{k=1}^{n} f(k)$$

Esercizio

Studiare la convergenza di $\sum_n \frac{1}{n^\alpha}$ $\alpha \in \mathbb{R}$

dire

$f(x) = \frac{1}{x^\alpha}$ è decrescente su $[1,+\infty[$ se $\alpha > 0$

$f(x) = \frac{1}{x^\alpha} > 0 \quad \forall x > 1 \qquad \forall \alpha$

Per il Teorema precedente

$\underbrace{\sum_n f(n) = \sum_n \frac{1}{n^\alpha}}$ converge sse $\int_1^{+\infty} f(x)\,dx = \underbrace{\int_1^{+\infty} \frac{dx}{x^\alpha}}$ converge

sse $\alpha > 1$

Se $\alpha \in (0,1]$ allora $\int_1^{+\infty} f\,dx$ diverge allora $\sum_n \frac{1}{n^\alpha}$ diverge

Se $\alpha \leq 0$ allora il Teorema non vale ($\frac{1}{x^\alpha}$ è crescente per $\alpha \leq 0$)

però $\lim_{n\to+\infty} \frac{1}{n^\alpha} \geq 1 \quad \forall \alpha \leq 0$ 3

allora $\sum_n 1/n^\alpha$ diverge (se $a_n \not\to 0$ allora $\sum_n a_n$ non converge ma $a_n \geq 0$ allora $\sum_n a_n$ diverge)

Esercizio

Studiare la convergenza di $\sum_n \frac{1}{n(\ln(n))^\alpha} \quad \alpha > 0$

dim

$f(x) = \frac{1}{x(\ln(x))^\alpha}$ è decrescente su $[2,+\infty[\quad \forall \alpha > 0$

$$\left(f'(x) = \frac{1}{x^2 (\ln(x))^{2\alpha}} \cdot \left((\ln(x))^\alpha + \cancel{x} \cdot \alpha (\ln(x))^{\alpha-1} \cdot \frac{1}{\cancel{x}} \right)(-1) \right.$$

$$= -\frac{(\ln(x))^{\alpha-1} \cdot (\ln(x) + \alpha)}{x^2 (\ln(x))^{2\alpha}} = -\frac{\ln(x)+\alpha}{x^2 (\ln(x))^{\alpha+1}} < 0$$

quando $x \geq 2$ e $\alpha > 0$.

e quindi $f(x)$ è decrescente $\Big)$

Considero $\int_2^{+\infty} \frac{dx}{x \cdot [\ln(x)]^\alpha}$

$$\int_2^\beta \frac{dx}{x[\ln(x)]^\alpha} = \int_{\ln(2)}^{\ln(\beta)} \frac{\cancel{e^y} dy}{\cancel{e^y} \cdot y^\alpha} = \int_{\ln(2)}^{\ln(\beta)} \frac{dy}{y^\alpha}$$

$y = \ln(x)$

$x = e^y$

$x = 2 \to y = \ln(2)$

$x = \beta \to y = \ln(\beta)$

$\frac{dx}{dy} = e^y \Rightarrow \boxed{dx = e^y dy}$

$$\lim_{\beta\to+\infty} \int_2^\beta \frac{dx}{x(\ln x)^\alpha} = \lim_{\beta\to+\infty} \int_{\ln(2)}^{\ln(\beta)} \frac{dy}{y^\alpha} = \int_{\ln(2)}^{+\infty} \frac{dy}{y^\alpha} \begin{cases} \text{converge } \alpha > 1 \\ \text{diverge } \alpha \leq 1 \end{cases}$$

$$\Rightarrow \int_2^{+\infty} \frac{dx}{x(\ln(x))^\alpha} \begin{cases} \text{converge } \alpha > 1 \\ \text{diverge } \alpha \leq 1 \end{cases}$$

$$\Rightarrow \sum_n \frac{1}{n(\ln(n))^\alpha} \begin{cases} \text{converge } \alpha > 1 \\ \text{diverge } \alpha \leq 1 \end{cases}$$

Esercizio

Studiare la convergenza di $\sum_n \frac{1}{(n)^1\cdot(\ln(n))^1\cdot[\ln(\ln(n))]^\alpha}$

Esercizio

Studiare la convergenza di $\sum_n a_n$ dove

$$a_n = \int_n^{n+1/n^\alpha} t\cdot \operatorname{arctg}(t)\,dt \qquad \alpha\in\mathbb{R}$$

dim

Devo concentrare la attenzione su a_n

$$\int \underset{u'}{t}\,\underset{v}{\operatorname{arctg}(t)}\,dt = \underset{u}{\frac{t^2}{2}}\cdot\underset{v}{\operatorname{arctg}(t)} - \int \underset{u}{\frac{t^2}{2}}\cdot\underset{v'}{\frac{1}{1+t^2}}\,dt = \frac{t^2}{2}\operatorname{arctg} t - \frac{1}{2}\int\frac{t^2+1}{t^2+1} + \frac{1}{2}\int\frac{1}{t^2+1}$$

$$= \frac{t^2}{2}\operatorname{arctg}(t) - \frac{1}{2}t + \frac{1}{2}\operatorname{arctg}(t) + c \qquad c\in\mathbb{R}$$

$$2a_n = \Big[t^2\operatorname{arctg}(t) - t + \operatorname{arctg}(t)\Big]_{t=n}^{t=n+\frac{1}{n^\alpha}} = \Big[\operatorname{arctg}(t)(t^2+1) - t\Big]_{t=-}^{t=-}$$

$$= \operatorname{arctg}\left(n+\frac{1}{n^\alpha}\right)\cdot\left(\left(n+\frac{1}{n^\alpha}\right)^2+1\right) - \left(\cancel{n}+\frac{1}{n^\alpha}\right)$$

$$+ \cancel{(n)} - \operatorname{arctg}(n)\cdot(n^2+1)$$

$$= \operatorname{arctg}\left(n+\frac{1}{n^\alpha}\right)\left(n^2+2n^{1-\alpha}+\frac{1}{n^{2\alpha}}+1\right) - \operatorname{arctg}(n)(n^2+1) - \frac{1}{n^\alpha}$$

$$= n^2\left[\operatorname{arctg}\left(n+\frac{1}{n^\alpha}\right) - \operatorname{arctg} n\right] + 2n^{1-\alpha}\operatorname{arctg}\left(n+\frac{1}{n^\alpha}\right)$$

$$+\frac{1}{n^{2\alpha}}\operatorname{arctg}\left(n+\frac{1}{n^\alpha}\right) + \left(\operatorname{arctg}\left(n+\frac{1}{n^\alpha}\right) - 1\right) - \frac{1}{n^\alpha}$$

e non si sa come procedere!

Conviene maggiorare a_n !!

A tal fine si osserva che

$$f(t)=t \nearrow \text{ e } g(t)=\operatorname{arctg}(t)\nearrow \quad \text{su } \left[n, n+\frac{1}{n^\alpha}\right]$$

$$\int_n^{n+1/n^\alpha} t \cdot \operatorname{arctg}(t)\,dt \underset{\text{Teorema Media}}{=} \left(n+\frac{1}{n^\alpha}-n\right) \cdot z_n \cdot \operatorname{arctg}(z_n) \qquad n < z_n < n+\frac{1}{n^\alpha}$$

$$= \frac{1}{n^\alpha} \cdot z_n \cdot \operatorname{arctg}(z_n) \qquad \operatorname{arctg} \leq \frac{\pi}{2} \ \text{e} \ z_n < n + 1/n^\alpha$$

$$\leq \frac{1}{n^\alpha} \cdot \left(n+\frac{1}{n^\alpha}\right) \cdot \frac{\pi}{2} = \frac{\pi}{2} \cdot \frac{1}{n^{\alpha-1}} + \frac{\pi}{2} \cdot \frac{1}{n^{2\alpha}} \quad \text{e quindi}$$

debbo studiare

$$\sum_n \frac{\pi}{2} \cdot \frac{1}{n^{\alpha-1}} + \sum_n \frac{\pi}{2} \cdot \frac{1}{n^{2\alpha}}$$

a) $\sum_n \frac{\pi}{2} \frac{1}{n^{\alpha-1}}$ converge se $\alpha - 1 > 1$ se $\alpha > 2$

b) $\sum_n \frac{\pi}{2} \cdot \frac{1}{n^{2\alpha}}$ converge se $2\alpha > 1$ se $\alpha > 1/2$

$$\boxed{\sum_n \frac{\pi}{2}\left(\frac{1}{n^{\alpha-1}} + \frac{1}{n^{2\alpha}}\right) \text{ converge se } \alpha > 2}$$

per il Teorema del confronto asintotico

se $\alpha > 2$ allora $\sum_n a_n$ converge

se $\boxed{\alpha = 2}$ $\qquad a_n = \frac{1}{n^\alpha} \cdot z_n \cdot \operatorname{arctg}(z_n) \qquad n < z_n < n + \frac{1}{n^\alpha}$

$$\geq \frac{1}{n^\alpha} \cdot n \cdot \operatorname{arctg} n = b_n$$

$$b_n = \frac{1}{n^{\alpha-1}} \cdot \operatorname{arctg}(n) \sim \frac{\pi}{2} \cdot \frac{1}{n^{\alpha-1}}$$

Ma $\sum_n \frac{\pi}{2} \frac{1}{n^{\alpha-1}}$ diverge per $\alpha \leq 2$

$\Rightarrow \sum_n b_n$ diverge per $\alpha \leq 2$

$\Rightarrow \sum_n a_n$ " " $\alpha \leq 2$

$\boxed{|||}$

Esercizio

Data $f_\alpha(x) = \ln\left(e^{\alpha x} - \operatorname{sen}(\alpha x)\right) + \cos(\alpha x) - 1$

- calcolare $P_4(x)$, il polinomio di Taylor di ordine 4 centrato in $x_0 = 0$

- Calcolare, al variare di $\alpha \in \mathbb{R}$, il limite
$$\lim_{x \to 0^+} \frac{f_\alpha(x) - 9x^3}{x^4}$$

dim

$g(y) = \ln\left(e^y - \operatorname{sen}(y)\right) + \cos y - 1 \qquad y \to 0$

$e^y = 1 + y + y^2/2 + y^3/6 + y^4/24 + o(y^4)$

$\operatorname{sen} y = y - y^3/6 + o(y^4)$

$\cos y = 1 - \frac{y^2}{2} + \frac{y^4}{24} + o(y^4)$

$\ln(1+z) = z - z^2/2 + z^3/3 - \frac{z^4}{4} + o(z^4) \quad \underline{z \to 0}$

$$g(y) = \ln\left(1 + \cancel{y} + y^2/2 + y^3/6 + \frac{y^4}{24} - \cancel{y} + y^3/6 + o(y^4)\right) + \cancel{1} - y^2/2 + y^4/24 + o(y^4) - \cancel{1}$$
$$= \ln\left(1 + \left(y^2/2 + y^3/3 + y^4/24 + o(y^4)\right)\right) - y^2/2 + y^4/24 + o(y^4)$$
$$= \left(\cancel{y^2/2} + y^3/3 + y^4/24\right) - \frac{1}{2}\left(y^2/2\right)^2 - \cancel{y^2/2} + y^4/24 + o(y^4)$$
$$= y^3/3 + y^4/24 - y^4/8 + y^4/24 + o(y^4)$$
$$= y^3/3 - y^4/24 + o(y^4) \quad \underset{y \to 0}{} \qquad \Rightarrow \boxed{f_\alpha(x) = \frac{\alpha^3 x^3}{3} - \frac{\alpha^4 x^4}{24} + o(x^4)} \quad x \to 0$$

$$\lim_{x \to 0^+} \frac{f_\alpha(x) - 9x^3}{x^4} = \lim_{x \to 0^+} \frac{\frac{\alpha^3 x^3}{3} - \frac{\alpha^4 x^4}{24} + o(x^4) - 9x^3}{x^4}$$
$$= \lim_{x \to 0^+} \frac{x^3\left(\frac{\alpha^3}{3} - 9\right) - \frac{\alpha^4}{24}x^4 + o(x^4)}{x^4}$$

1) $\frac{\alpha^3}{3} - 9 < 0 \iff \alpha^3 < 27 \iff \alpha < 3 \Rightarrow \lim_{x \to 0^+} \frac{x^3\left(\frac{\alpha^3}{3} - 9\right) + o(x^3)}{x^4} = -\infty$

2) $\frac{\alpha^3}{3} - 9 = 0 \iff \alpha = 3 \implies \lim_{x\to 0^+} \frac{-\frac{3^4}{24}\cdot x^4 + o(x^4)}{x^4} =$

$$= \lim_{x\to 0^+} \frac{-\frac{81}{24} + o(1)}{1} = -\frac{81}{24} = -\frac{27}{8}$$

3) $\frac{\alpha^3}{3} - 9 > 0 \iff \alpha > 3 \implies \lim_{x\to 0^+} \frac{x^3\cdot\left(\frac{\alpha^3}{3} - 9\right) + o(x^3)}{x^4}$

$$= \lim_{x\to 0^+} \frac{\left(\frac{\alpha^3}{3} - 9\right) + o(1)}{x} = +\infty$$

Esercizio

Data la funzione $f = \arctan\left(\ln(1+x)\right) - \ln\left(1 + \arctan x\right)$

Calcolare ordine e pp di $f(x)$ per $x \to 0$

Calcolare, al variare di $\alpha \in \mathbb{R}$ $\lim_{x\to 0^+} \frac{f(x)}{x^4 + x^\alpha}$

scrivendo $f(x) = \left(\text{parte principale}\right) + o(x^\alpha)$ per $x \to 0$

Oss: utilizzando l'Hospital

$$\lim_{x\to 0^+} \frac{f(x)}{x^4 + x^\alpha} \underset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+(\ln(1+x))^2}\cdot\frac{1}{1+x} - \frac{1}{1+\arctan(x)}\cdot\frac{1}{1+x^2}}{4x^3 + \alpha\cdot x^{\alpha-1}}$$

Ho "complicato" il limite in luogo di "semplificarlo"

dim

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$$

per $x \to 0$

$$\arctan(x) = x - \frac{x^3}{3} + o(x^4)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + o(x^4)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + o(x^4)$$

$x \to 0$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 + o(x^4)$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^5)$$

$$\operatorname{arctg}\left(\ln(1+x)\right) = \operatorname{arctg}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{5} + o(x^4)\right)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right) - \frac{1}{3}\left(x - \frac{x^2}{2}\right)^3 + o(x^4)$$

$$= x - \frac{x^2}{2} + \cancel{\frac{x^3}{3}} - \frac{x^4}{4} - \frac{1}{3}\left(\cancel{x^3} - \frac{3}{2}x^4\right) + o(x^4)$$

$$= x - \frac{x^2}{2} + x^4\left(-\frac{1}{4} + \frac{\overset{1}{\cancel{3}}}{\underset{2}{\cancel{6}}}\right) + o(x^4)$$

$$= x - \frac{x^2}{2} + \frac{x^4}{4} + o(x^4) \leftarrow$$

$$\ln\left(1 + \operatorname{arctg}(x)\right) = \ln\left(1 + \left[x - \frac{x^3}{3} + o(x^4)\right]\right) =$$

$$= \left(x - \frac{x^3}{3}\right) - \frac{1}{2}\left(x - \frac{x^3}{3}\right)^2 + \frac{1}{3}(x)^3 - \frac{1}{4}(x)^4 + o(x^4)$$

$$= x - \cancel{\frac{x^3}{3}} - \frac{1}{2}\left(x^2 - \frac{2}{3}x^4\right) + \cancel{\frac{x^3}{3}} - \frac{x^4}{4} + o(x^4)$$

$$= x - \frac{x^2}{2} + x^4\left(\frac{1}{3} - \frac{1}{4}\right) + o(x^4) = x - \frac{x^2}{2} + \frac{x^4}{12} + o(x^4) \leftarrow$$

$$f(x) = \left(\cancel{x} - \cancel{\frac{x^2}{2}} + \frac{x^4}{4}\right) - \left(\cancel{x} - \cancel{\frac{x^2}{2}} + \frac{x^4}{12}\right) + o(x^4) = \frac{x^4}{6} + o(x^4)$$

$$\text{ordine}(f) = 4 \qquad \text{p.p.}(f) = \frac{x^4}{6}$$

$$\lim_{x\to0^+}\frac{f(x)}{x^4+x^\alpha}=\lim_{x\to0^+}\frac{\frac{x^4}{6}+o(x^4)}{x^4+x^\alpha}=$$

$$\alpha=4\qquad \lim_{x\to0^+}\frac{\frac{x^4}{6}+o(x^4)}{x^4+x^4}=\frac{1}{12}$$

$$\alpha<4\qquad \lim_{x\to0^+}\frac{f(x)}{x^4+x^\alpha}=\lim_{x\to0^+}\frac{x^4/6+o(x^4)}{x^\alpha+o(x^\alpha)}=\lim_{x\to0^+}\frac{\frac{x^{4-\alpha}}{6}+o(x^{4-\alpha})}{1+o(1)}$$

$$=0$$

$$\alpha>4\qquad \lim_{x\to0^+}\frac{f(x)}{x^4+x^\alpha}=\lim_{x\to0^+}\frac{x^4/6+o(x^4)}{x^4+o(x^4)}=\lim_{x\to0^+}\frac{1/6+o(1)}{1+o(1)}=\frac{1}{6}$$

$$x^4+x^\alpha=x^4+o(x^4)\quad \text{se } \alpha>4$$
$$x^4+x^\alpha=x^\alpha+o(x^\alpha)\quad \text{se } 0<\alpha<4$$

Oss:

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+o(x^4)\qquad x\to0$$

$$\frac{1}{1+x}=1-x+x^2-x^3+x^4+o(x^4)=\frac{1}{1-(-x)}$$

$$-\frac{1}{(1+x)^2}=-1+2x-3x^2+4x^3-5x^4+o(x^4)$$

$$-(1+x)^{-2}=-\left[1+(-2)\cdot x+\frac{(-2)(-3)}{2}\cdot x^2+\frac{(-2)(-3)(-4)}{6}\cdot x^3+\frac{(-2)(-3)(-4)(-5)}{24}x^4+o(x^4)\right]$$

$$=-\left[1-2x+3x^2-4x^3+5x^4+o(x^4)\right]\qquad 6x^5\quad 7x^6$$

$$\frac{2}{(1+x)^3}=2-6x+12x^2-20x^3+30x^4+o(x^4)\qquad 42x^5$$

$$-\frac{6}{(1+x)^4}=-6+24x-60x^2+120x^3-220x^4+o(x^4)$$