$$\boxed{\int u'(x)\cdot v(x)\,dx = u(x)\cdot v(x) - \int u(x)\cdot v'(x)\,dx}$$

**Oss:** Data $f(x)$, se questa è derivabile, allora è SEMPRE possibile scrivere esplicitamente $f'(x)$ in termini di funzioni elementari

$$f = \ln\left(\operatorname{sen}\left(e^{x^2}\right)\right) \qquad f' = \frac{1}{\operatorname{sen}\left(e^{x^2}\right)}\cdot\cos\left(e^{x^2}\right)\cdot e^{x^2}\cdot 2x$$

il viceversa non è sempre possibile!

Data $f(x)$ $f: I \to \mathbb{R}$ $I$ intervallo ed $f$ continua su $I$, allora per il Teorema fondamentale del calcolo integrale (che vedremo) esiste $F: I \to \mathbb{R}$ primitiva di $f$ $\forall x \in I$ (cioè $F' = f$ $\forall x \in I$) PERÒ

non è sempre possibile esprimere $F(x)$ in termini di funzioni elementari

ovvero, per esempio

$f(x) = e^{x^2}$ è continua $\forall x \in \mathbb{R}$

esiste

$F(x)$ primitiva di $f(x) = e^{x^2}$

però non so come scrivere esplicitamente $F(x)$

$F(x) = \int_a^x f(t)\,dt$ ← non è una forma esplicita

$x^n \cdot e^x$ $\quad x^n \cdot \operatorname{sen} x$ esempi già fatti.

**Esercizio**

Calcolare $\int \operatorname{sen}^2(x)\,dx$

**dim**

$$\int \underset{u'}{\operatorname{sen} x}\cdot \underset{v}{\operatorname{sen} x}\,dx = \underset{u}{(-\cos x)}\cdot \underset{v}{\operatorname{sen} x} - \int \underset{u}{(-\cos x)}\cdot \underset{v'}{\cos x}\,dx$$

$$= -\operatorname{sen} x \cos x + \int \cos^2(x)\,dx$$

$$= -\operatorname{sen}(x)\cos(x) + \int \left[1 - \operatorname{sen}^2(x)\right]dx$$

$$= -\operatorname{sen}x\cos x + \int 1\cdot dx - \int \operatorname{sen}^2(x)\,dx$$

$$\Downarrow$$

$$2\int \operatorname{sen}^2(x)\,dx = x - \operatorname{sen}x\cos x + c \qquad c\in\mathbb{R}$$

$$\int \operatorname{sen}^2(x)\,dx = \frac{1}{2}\left(x - \operatorname{sen}x\cos x\right) + c \qquad c\in\mathbb{R}$$

verifica $\frac{d}{dx}\left[\frac{x}{2} - \frac{1}{2}\operatorname{sen}x\cos x + c\right] = \frac{1}{2} - \frac{1}{2}\cos^2 x - \frac{1}{2}\operatorname{sen}x\,(-\operatorname{sen}x)$

$$= \frac{1}{2} - \frac{1}{2}\cos^2 x + \frac{1}{2}\operatorname{sen}^2 x = \frac{1}{2} - \frac{1}{2}\left(1-\operatorname{sen}^2 x\right) + \frac{1}{2}\operatorname{sen}^2 x$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{2}\operatorname{sen}^2 x + \frac{1}{2}\operatorname{sen}^2 x = \operatorname{sen}^2 x \qquad \checkmark \qquad \blacksquare$$

**Esercizio**

Calcolare $\int \cos^2 x\,dx$

**dim**

$$\int \cos^2 x\,dx = \int (1-\operatorname{sen}^2 x)\,dx = \int 1\,dx - \int \operatorname{sen}^2 x\,dx =$$

$$= x - \left[\frac{1}{2}(x - \operatorname{sen}x\cos x)\right] + c$$

$$= \frac{x}{2} + \frac{1}{2}\operatorname{sen}x\cos x + c \qquad c\in\mathbb{R} \qquad \blacksquare$$

**Esempio**

Calcolare $\int \ln(x)\,dx$

**dim**

$$\int \ln(x)\,dx = \int \underset{u}{\ln(x)}\cdot \underset{v'}{1}\,dx = \underset{u}{\ln(x)}\cdot \underset{v}{x} - \int \underset{u'}{\frac{1}{x}}\cdot \underset{v}{x}\,dx$$

$$= x\cdot\ln(x) - \int 1\,dx = \boxed{x\cdot\ln x - x + c \qquad c\in\mathbb{R}} \quad \blacksquare$$

Esercizio

Calcolare $\int \arctan(x)\,dx$

dim

$$\int \underset{u}{\arctan(x)} \cdot \underset{v'}{1}\,dx = \underset{u}{\arctan(x)} \cdot \underset{v}{x} - \int \underset{u'}{\frac{1}{1+x^2}} \cdot \underset{v}{x}\,dx$$

$$= x \cdot \arctan(x) - \frac{1}{2} \int \frac{2x}{1+x^2}\,dx$$

$$= x \cdot \arctan(x) - \frac{1}{2} \int \frac{(1+x^2)'}{1+x^2}\,dx$$

$$= x \cdot \arctan(x) - \frac{1}{2} \ln(1+x^2) + c \qquad c \in \mathbb{R}$$

Verifica $\frac{d}{dx}\left(x \arctan(x) - \frac{1}{2}\ln(1+x^2)\right) = \arctan(x) + x \cdot \frac{1}{1+x^2} - \frac{1}{2}\,\frac{2x}{1+x^2}$

$$= \arctan(x) + \cancel{\frac{x}{1+x^2}} - \cancel{\frac{x}{1+x^2}}$$

CVD

# INTEGRAZIONE PER SOSTITUZIONE

$$\frac{d}{dx}(g \circ f)(x) = g'(f(x)) \cdot f'(x)$$

$$\Downarrow$$

$$\int \frac{d}{dx}(g \circ f)(x) = \int \frac{d}{dx}(g(f(x)))\,dx = \int g'(f(x)) \cdot f'(x)\,dx$$

$$\boxed{(g \circ f)(x) = \int g'(f(x)) \cdot f'(x)\,dx}$$

Integrazione per sostituzione

quando $G$ è una primitiva di $g$

$$(G \circ f)(x) = G(f(x)) = \int g(f(x)) \cdot f'(x)\,dx$$

$$\boxed{\int g(f(x))\cdot f'(x)\,dx \underset{\substack{y=f(x)\\ \frac{dy}{dx}=f'(x)\\ dy=f'(x)dx}}{=} \left(\int g(y)\cdot dy\right)_{y=f(x)}}$$

Esercizio

Calcolare $\int \cos(3x)\,dx$

dim

$$\underbrace{\int \cos(3x)\,dx}  \underset{\substack{y=3x\\ \frac{dy}{dx}=3\\ dy=3dx\\ \frac{dy}{3}=dx}}{=} \left(\int \cos(y)\,\frac{dy}{3}\right)_{y=3x}$$

devo saperlo
$\int \cos y\,dy$
$\shortparallel$
$\operatorname{sen} y + c \quad c\in\mathbb{R}$

$$= \left(\frac{1}{3}\operatorname{sen}(y) + c\right)_{y=3x} = \underbrace{\frac{1}{3}\operatorname{sen}(3x) + c} \quad c\in\mathbb{R}$$

$\square$

Esercizio

Calcolare $\int \cos(\ln(1+x))\cdot\frac{1}{1+x}\,dx$

dim

$$\int \cos(\ln(1+x))\cdot\frac{1}{1+x}\,dx = \int g(f(x))\cdot f'(x)\,dx$$

dove $f(x)=\ln(1+x)$

$$\underset{\substack{y=\ln(1+x)\\ \frac{dy}{dx}=\frac{1}{1+x}\\ x=e^y-1\\ \frac{dx}{dy}=e^y}}{=} \left(\int \cos(y)\cdot\frac{1}{\cancel{e^y}}\,\cancel{e^y}\,dy\right)_{y=\ln(1+x)}$$

$$= \left(\operatorname{sen}(y)+c\right)_{y=\ln(1+x)} \quad c\in\mathbb{R}$$

$$= \operatorname{sen}(\ln(1+x)) + c \quad c\in\mathbb{R}$$

$\square$

## Esercizio

Calcolare $\int \operatorname{Tg}(x)\,dx$

dim

$$\int \frac{\operatorname{sen}(x)}{\cos(x)}\,dx \underset{y=\cos x}{=} \left(\int \frac{\cancel{\sqrt{1-y^2}}}{y}\cdot -\frac{1}{\cancel{\sqrt{1-y^2}}}\cdot dy\right)_{y=\cos x}$$

$y=\cos x$

$x=\arccos(y)$

$\frac{dx}{dy}=\frac{1}{-\sqrt{1-y^2}}$

$dx=-\frac{1}{\sqrt{1-y^2}}\,dy$

$$\boxed{\begin{array}{l}\int \frac{1}{y}\,dy \\ =\ln(y)+c \\ c\in\mathbb{R}\end{array}}$$

$$=\left(-\int\frac{dy}{y}\right)_{y=\cos x}=\left(-\ln|y|+c\right)_{y=\cos x}=-\ln|\cos x|+c \qquad c\in\mathbb{R} \quad \blacksquare$$

## Esercizio

Calcolare $\int\sqrt{1-x^2}\,dx$

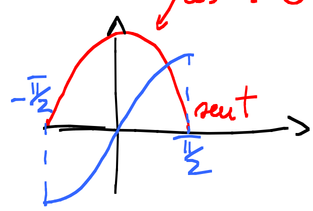
dim

$$\int\sqrt{1-x^2}\,dx \underset{x=\operatorname{sen}t}{=} \left(\int\sqrt{1-\operatorname{sen}^2 t}\cdot\cos t\cdot dt\right)_{t=\operatorname{arcsen}x}$$

$x=\operatorname{sen}t$

$\frac{dx}{dt}=\cos t$

$dx=\cos t\cdot dt$




$$=\left(\int|\cos t|\cdot\cos t\;dt\right)_{t=\operatorname{arcsen}x}$$

$$=\left(\int\cos^2 t\,dt\right)_{t=\operatorname{arcsen}x}$$

$$=\left(\frac{t}{2}+\frac{1}{2}\operatorname{sen}t\cos t+c\right)_{t=\operatorname{arcsen}x}$$

$$=\frac{1}{2}\operatorname{arcsen}x+\frac{1}{2}x\cdot\sqrt{1-x^2}+c \qquad c\in\mathbb{R}$$

Verifica $\left(\frac{1}{2}\operatorname{arcsen}x+\frac{1}{2}x\sqrt{1-x^2}+c\right)'=\frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}}+\frac{1}{2}\sqrt{1-x^2}+\frac{1}{\cancel{2}}x\cdot\frac{-\cancel{2}x}{2\sqrt{1-x^2}}$

$$=\frac{1}{2}\sqrt{1-x^2}+\frac{1}{2}\,\frac{1}{\sqrt{1-x^2}}-\frac{x^2}{2\sqrt{1-x^2}}$$

$$= \frac{1}{2}\sqrt{1-x^2} + \frac{1}{2}\,\frac{1-x^2}{\sqrt{1-x^2}} = \sqrt{1-x^2}$$ ✓



Per calcolare

$$\lim_{x\to 0} \frac{1-\cos x - \frac{x^2}{2}}{x^3} \quad \left(\frac{0}{0}\right)$$

come e quanto debbo sviluppare $\cos x$ ?

$$\cos x = 1 + o(x)$$
$$= 1 - \frac{x^2}{2} + o(x^3)$$
$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5) \qquad x\to 0$$

1ª scelta $\displaystyle \lim_{x\to 0} \frac{1-\cos x - x^2/2}{x^3} = \lim_{x\to 0} \frac{\cancel{1} - (\cancel{1} + o(x)) - x^2/2}{x^3}$

$$= \lim_{x\to 0} \frac{o(x)}{x^3}$$

non posso proseguire: $o(x)$ non mi dà sufficienti informazioni per concludere

infatti $\left.\begin{array}{ll} x^{1,1} = o(x) & x\to 0 \\ x^{1000} = o(x) & x\to 0 \end{array}\right\}$ ma quale scelgo?

devo prendere un maggior numero di termini dello sviluppo del $\cos x$ per $x\to 0$

2ª scelta : $\displaystyle \lim_{x\to 0} \frac{1-\cos x - x^2/2}{x^3} = \lim_{x\to 0} \frac{\cancel{1} - (\cancel{1} - \cancel{\frac{x^2}{2}} + o(x^3)) - \cancel{\frac{x^2}{2}}}{x^3}$

$$= \lim_{x\to 0} \frac{o(x^3)}{x^3} = \lim_{x\to 0} o(1) = 0$$

## Esercizio

Calcolare $\displaystyle\lim_{x\to 0} \frac{e^{3x}\cos 2x + \ln(1-3x) - (1-x^2)^2}{x - \operatorname{sen} x}$

**dim**

$x - \operatorname{sen} x = \cancel{x} - \left(\cancel{x} - \frac{x^3}{6} + o(x^4)\right) = \frac{x^3}{6} + o(x^4)$

se il denominatore è del 3° ordine, dovrò sviluppare il numeratore almeno fino al 3° ordine per poterli confrontare

$e^{3x} = 1 + (3x) + \frac{1}{2}(3x)^2 + \frac{1}{6}(3x)^3 + o\left((3x)^3\right) = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + o(x^3)$

$\cos(2x) = 1 - \frac{(2x)^2}{2} + o(x^3) = 1 - 2x^2 + o(x^3)$

$\ln(1-3x) = (-3x) - \frac{(-3x)^2}{2} + \frac{(-3x)^3}{3} + o(x^3) = -3x - \frac{9}{2}x^2 - 9x^3 + o(x^3)$

$e^{3x}\cdot\cos 2x + \ln(1-3x) = \left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3\right)\cdot\left(1-2x^2\right) - 3x - \frac{9}{2}x^2 - 9x^3 + o(x^3)$

$= 1 - 2x^2 + \cancel{3x} - 6x^3 + \cancel{\frac{9}{2}x^2} + \frac{9}{2}x^3 \cancel{-3x} - \cancel{\frac{9}{2}x^2} - 9x^3 + o(x^3)$

$= 1 - 2x^2 + x^3\left(-6 - \frac{9}{2}\right)x^3 + o(x^3) = 1 - 2x^2 - \frac{21}{2}x^3 + o(x^3)$

$\displaystyle\lim_{x\to 0} \frac{\cancel{1}\cancel{-2x^2} - \frac{21}{2}x^3 + o(x^3) - \left(\cancel{1} + x^4 \cancel{-2x^2}\right)}{\frac{x^3}{6} + o(x^4)} = \lim_{x\to 0} \frac{-\frac{21}{2}x^3 + o(x^3)}{\frac{x^3}{6} + o(x^4)}$

$\displaystyle= \lim_{x\to 0} \frac{-\frac{21}{2} + o(1)}{\frac{1}{6} + o(x)} = -63$ $\square$

## Esercizio

Data $f(x) = e^{-x+x^2} - (1+x-x^2)^{-1}$

1) calcolare lo sviluppo di ordine 3 di $f$ intorno a $x_0 = 0$

2) calcolare, al variare di $\alpha \in \mathbb{R}$,

$$\lim_{x\to 0} \frac{e^{-x+x^2} - (1+x-\alpha x^2)^{-1}}{x^3}$$

**dim**

$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + o(y^3) \Rightarrow e^{-x+x^2} = 1 + (-x+x^2) + \frac{1}{2}(-x+x^2)^2 + \frac{1}{6}(-x+x^2)^3 + o(x^3)$

$e^{-x+x^2} = 1 - x + x^2 + \frac{1}{2}(x^2 - 2x^3) + \frac{1}{6}(-x^3) + o(x^3)$

$= 1 - x + \frac{3}{2}x^2 - \frac{7}{6}x^3 + o(x^3)$

$$\left(1+x-x^2\right)^{-1} = \frac{1}{1-(x^2-x)} = 1+(x^2-x)+(x^2-x)^2+(x^2-x)^3+o(x^3)$$

$$= 1-x+x^2+\left(-2x^3+x^2\right)-x^3+o(x^3) = 1-x+2x^2-3x^3+o(x^3)$$

lo sviluppo al 3° ordine di f è

$$f(x) = \cancel{1}-\cancel{x}+\frac{3}{2}x^2-\frac{7}{6}x^3-\left(\cancel{1}-\cancel{x}+2x^2-3x^3\right)+o(x^3) = -\frac{1}{2}x^2+\frac{11}{6}x^3+o(x^3)$$

$$\frac{e^{-x+x^2}-\left(1+x-\alpha x^2\right)^{-1}}{x^3} = \frac{\left(1-x+\frac{3}{2}x^2-\frac{7}{6}x^3+o(x^3)\right)\left(1+x-\alpha x^2\right)-1}{x^3\left(1+x-\alpha x^2\right)}$$

$$= \frac{\cancel{1}+\cancel{x}-\alpha x^2-\cancel{x}-x^2+\alpha x^3+\frac{3}{2}x^2+\frac{3}{2}x^3-\frac{7}{6}x^3+o(x^3)-\cancel{1}}{x^3+o(x^3)}$$

$$= \frac{x^2\left(-\alpha+\frac{3}{2}-1\right)+x^3\left(\alpha+\frac{3}{2}-\frac{7}{6}\right)+o(x^3)}{x^3+o(x^3)}$$

$$= \frac{x^2\left(\frac{1}{2}-\alpha\right)+x^3\left(\alpha+\frac{1}{3}\right)+o(x^3)}{x^3+o(x^3)} \xrightarrow[x\to 0^+]{} \begin{cases} +\infty & \alpha<\frac{1}{2} \\ \frac{5}{6} & \alpha=\frac{1}{2} \\ -\infty & \alpha>\frac{1}{2} \end{cases}$$

$$\alpha<\frac{1}{2} \to \lim_{x\to 0^+} \cdots = \lim_{x\to 0^+} \frac{x^2\left(\frac{1}{2}-\alpha\right)+o(x^2)}{x^3+o(x^3)} = \lim_{x\to 0^+} \frac{1}{x}\cdot\frac{\left(\frac{1}{2}-\alpha\right)+o(1)}{1+o(1)} = +\infty$$

$$\alpha>\frac{1}{2} \to \cdots = \cdots = \lim_{x\to 0^+} \frac{1}{x}\cdot\frac{\left(\frac{1}{2}-\alpha\right)+o(1)}{1+o(1)} = -\infty$$