Esercizio

Calcolare la derivata prima di $f(x) = x^8 = (x^2)^4$

dim

$f(x) = (h \circ g)(x)$ $\qquad x \xmapsto{g} x^2 = g(x) \xmapsto{h} h(g(x)) = (x^2)^4$

$f'(x_0) = (h \circ g)'(x_0) = h'(g(x_0)) \cdot g'(x_0)$ $\qquad g(x)=x^2 \quad g'(x_0) = 2 \cdot x_0$

$h(y) = y^4 \quad h'(y_0) = 4 y_0^3$

$= 4(g(x_0))^3 \cdot 2x_0$

$= 4(x_0^2)^3 \cdot 2x_0$

$= 8 \cdot x_0^{6+1} =$

$= 8 \; x_0^7$

Esercizio Calcolare $f'(x_0)$ quando

$f(x) = \operatorname{sen}(\log(1+x^3))$

dim

$x \xmapsto{g} x^3 = g(x) \xmapsto{h} \log(1+g(x)) = h(g(x)) \xmapsto{m} m(h(g(x)))$

$\log(1+x^3) \qquad \operatorname{sen}(\log(1+g(x))) \qquad \operatorname{sen}(\ln(1+x^3))$

$\operatorname{sen}(z) = m(z) \rightarrow m'(z_0) = \cos(z_0)$

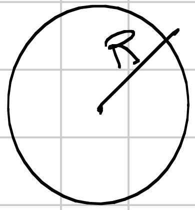
$\log(1+y) = h(y) \rightarrow h'(y_0) = \frac{1}{1+y_0}$

$x^3 = g(x) \longrightarrow g'(x_0) = 3x_0^2$

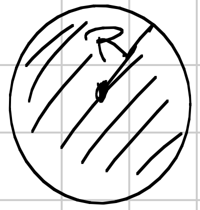
$f(x) = m(h(g(x)))$ $\qquad f'(x_0) = m'(h(g(x_0))) \cdot h'(g(x_0)) \cdot g'(x_0)$

$= \cos(h(g(x_0))) \cdot \frac{1}{1+g(x_0)} \cdot 3x_0^2$

$= \cos(\log(1+x_0^3)) \cdot \frac{1}{1+x_0^3} \cdot 3x_0^2$



Circonferenza di raggio R
$= 2\pi R$



Area cerchio raggio R
$A(R) = \pi R^2$

Superficie sfera
$S(R) = 4\pi R^2$

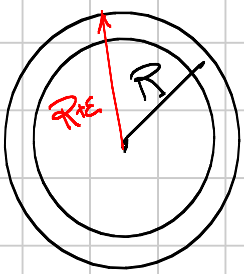
Volume sfera raggio R
$V(R) = \frac{4}{3}\pi R^3$

$$R \longrightarrow \pi R^2 = A(R)$$

$$\frac{dA}{dR}(R_0) = 2\pi R_0 = \text{Perim}(R_0)$$

$$R \longrightarrow \frac{4}{3}\pi R^3 = V_{ol}(R_0)$$

$$\frac{dV}{dR}(R_0) = \frac{4}{3}\pi\, 3R_0^2 = 4\pi R_0^2 = S(R_0)$$



$$\frac{A(R+\varepsilon) - A(R)}{\varepsilon} = \frac{(R+\varepsilon)^2 \cdot \pi - \pi \cdot R^2}{\varepsilon}$$

$$= \frac{\cancel{\pi R^2} + 2\varepsilon R\pi + \pi\varepsilon^2 - \cancel{\pi R^2}}{\varepsilon}$$

$$= 2\pi R + \pi\varepsilon \xrightarrow[\varepsilon \to 0]{} 2\pi R$$

Teorema (derivata della funzione inversa)

Sia $f:]a,b[\to \mathbb{R}$, strettamente monotona, f continua e derivabile in $x_0 \in]a,b[$ con $f'(x_0) \neq 0$

Allora

esiste la derivata di $f^{-1}: f(]a,b[) \to]a,b[$ nel punto $y_0 = f(x_0)$

e si ha $\quad (f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))} \qquad x \xrightarrow{f} y_0 = f(x_0)$

dim

$g(x) = f^{-1}(x)$

$$\begin{matrix} b \\ x_0 \\ a \end{matrix} \quad \underset{g=f^{-1}}{\overset{f}{\rightleftarrows}} \quad \begin{matrix} y_0 = f(x_0) \\ f(]a,b[) \end{matrix}$$

$$(f^{-1})'(y_0) = g'(y_0) \underset{\text{def}}{=} \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} \underset{\substack{y = f(x) \\ x = f^{-1}(y)}}{=} \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} =$$

$$= \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)} \qquad \blacksquare$$

Esercizio

Calcolare la derivata di $f^{-1}(y)$, l'inversa di f, quando

$f(x) = x^n$, $f(x) = e^x$

dim

$f(x) = x^2 \quad f'(x) = 2x \neq 0$ se $x \neq 0$

$$g(y) = f^{-1}(y) = \sqrt{y} \qquad g'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{2(f^{-1}(y_0))} = \frac{1}{2\sqrt{y_0}}$$

quindi $(f^{-1})'(y_0) = \frac{1}{2\sqrt{y_0}}$ dove $f^{-1}(y) = \sqrt{y}$

$f(x) = x^3 \quad f'(x) = 3x^2 \neq 0$ se $x \neq 0$

$$g(y) = f^{-1}(y) = \sqrt[3]{y} \rightarrow g'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{3(f^{-1}(y_0))^2} = \frac{1}{3(\sqrt[3]{y_0})^2}$$

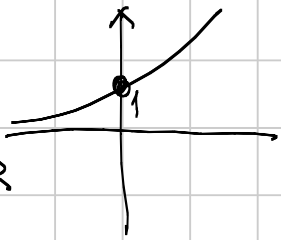
$$= \frac{1}{3(y_0)^{2/3}}$$

$f(x) = x^n \quad f'(x) = n x^{n-1} \neq 0$ se $x \neq 0$

$$g(y) = f^{-1}(y) = \sqrt[n]{y} \quad g'(y_0) = \frac{1}{f'(\underbrace{g(y_0)}_{f^{-1}(y_0)})} = \frac{1}{n(g(y_0))^{n-1}} =$$

$$= \frac{1}{n\left(\sqrt[n]{y_0}\right)^{n-1}} = \frac{1}{n\cdot\left(y_0\right)^{1-\frac{1}{n}}}$$

$f(x)=e^x$



$f'(x)=e^x \neq 0 \ \forall x\in\mathbb{R}$

$f^{-1}(y)=\ln(y)$

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'\left(f^{-1}(y_0)\right)} = \frac{1}{e^{\left(f^{-1}(y_0)\right)}} = \frac{1}{e^{\ln(y_0)}} = \frac{1}{y_0}$$

Teorema (derivata di $1/f(x)$)

Sia data $f: A\to\mathbb{R}$, $x_0\in A$ p.d.a. per A, $f(x_0)\neq 0$

f derivabile in x_0

Allora esiste $\left(\frac{1}{f}\right)'(x_0) = -\frac{f'(x_0)}{\left(f(x_0)\right)^2}$

dim

$g=\frac{1}{f(x)}$

$$g'(x_0)=\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{1/f(x) - 1/f(x_0)}{x-x_0} =$$

$$= \lim_{x\to x_0}\frac{f(x_0)-f(x)}{f(x)\cdot f(x_0)}\cdot\frac{1}{x-x_0} = -\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\cdot\lim_{x\to x_0}\frac{1}{f(x)f(x_0)}$$

$$= -f'(x_0)\,\frac{1}{\left(f(x_0)\right)^2}$$

Teorema (è un esercizio: derivata del rapporto f/g)

date $f,g: A\to\mathbb{R}$, $x_0\in A$ p.d.a. per A, $g(x_0)\neq 0$

se f e g sono derivabili in x_0

allora $\frac{f}{g}$ è derivabile in x_0 e si ha

$$\left(\frac{f}{g}\right)'(x_0)=\frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{\left(g(x_0)\right)^2}$$

dim

$$\left(\frac{f}{g}\right)'(x_0) = \left(f\cdot\frac{1}{g}\right)'(x_0) \underset{\text{Leibniz}}{=} f'(x_0)\cdot\frac{1}{g(x_0)} + f(x_0)\cdot\left(\frac{1}{g}\right)'(x_0)$$

$$\underset{\text{derivata del reciproco}}{=} \frac{f'(x_0)}{g(x_0)} + f(x_0)\left[-\frac{g'(x_0)}{(g(x_0))^2}\right] = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{(g(x_0))^2} \quad \square$$

Esercizio

Calcolare la derivata di $Tg(x) = \frac{sen\, x}{\cos x} = f(x)$

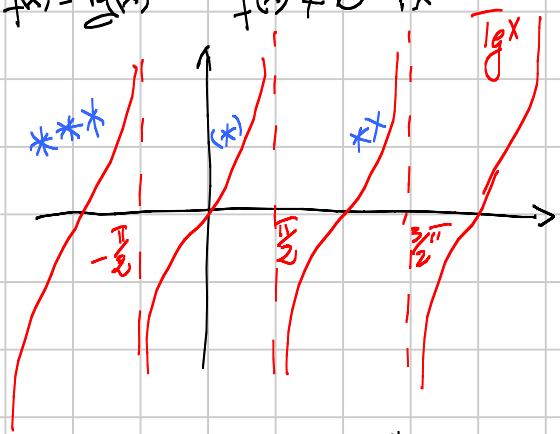
e la derivata di $f^{-1}(x) = arcTg(x)$

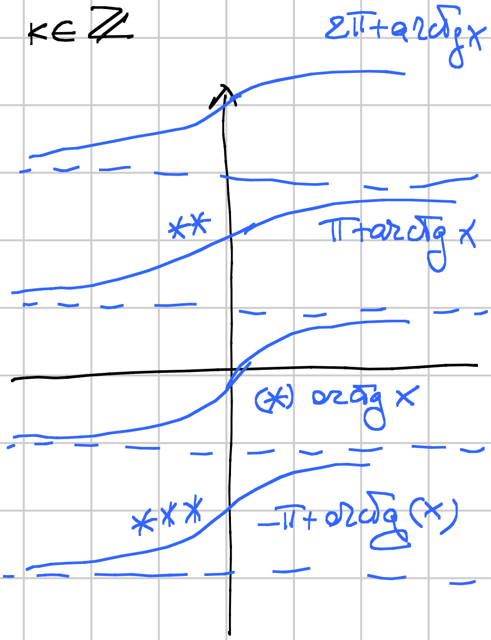
dim

$$(Tg)'(x_0) = \frac{\cos(x_0)\cdot\cos(x_0) - sen(x_0)\cdot(-sen(x_0))}{\cos^2(x_0)} = \frac{1}{\cos^2(x_0)} = 1+Tg^2(x_0)$$

questa è definita $\forall x_0 \neq \frac{\pi}{2}+k\pi \quad k\in\mathbb{Z}$

$f(x)=Tg(x) \qquad f'(x)\neq 0\ \forall x$





Considero $Tg(x)$ nell'intervallo $]-\frac{\pi}{2},\frac{\pi}{2}[$

$f = Tg:]-\frac{\pi}{2},\frac{\pi}{2}[\longrightarrow \mathbb{R} = Tg(]-\frac{\pi}{2},\frac{\pi}{2}[)$ continua e strett. ↗

$\downarrow$

$f^{-1} = arctg: \mathbb{R} \longrightarrow]-\frac{\pi}{2},\frac{\pi}{2}[$ continua e strett. ↗

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{1+Tg^2(arctg(y_0))} = \frac{1}{1+y_0^2}$$

Esercizio

Calcolare la derivata di $f^{-1}(y)$ quando

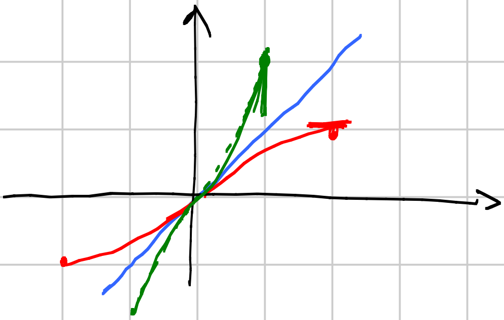
$f(x) = \operatorname{sen} x \qquad f(x) = \cos x$

dim

$f = \operatorname{sen} x$

$f: \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\longrightarrow \left]-1,1\right[$ e risulta essere invertibile

$f^{-1} = \operatorname{arcsen} : \left]-1,1\right[\longrightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$



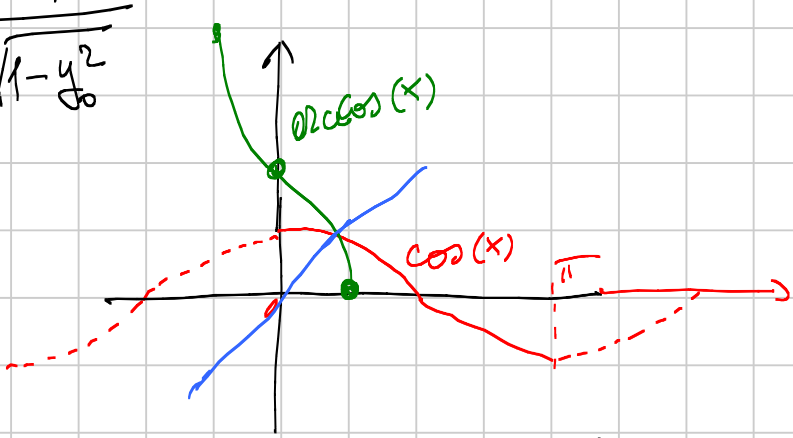
$f'(x) = \cos x > 0 \quad \forall x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

$f(x) = \operatorname{sen} x$

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'\left(f^{-1}(y_0)\right)} = \frac{1}{\cos\left(\operatorname{arcsen}(y_0)\right)} = \frac{1}{\sqrt{1-\operatorname{sen}^2\left(\operatorname{arcsen} y_0\right)}}$$

$$= \frac{1}{\sqrt{1-y_0^2}}$$

$f(x) = \cos x$



$f: \left]0,\pi\right[\to \left]-1,1\right[$

continua, strett. decrescente, $f' = -\operatorname{sen}(x) < 0 \quad \forall x \in \left]0,\pi\right[$

$\Downarrow$

$f^{-1}(y) = \arccos(y) \qquad \arccos : \left]-1,1\right[\longrightarrow \left]0,\pi\right[$

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'\left(f^{-1}(y_0)\right)} = \frac{1}{-\operatorname{sen}\left(\arccos(y_0)\right)} = \frac{1}{-\sqrt{1-\cos^2\left(\arccos(y_0)\right)}}$$

$$= \frac{1}{-\sqrt{1-y_0^2}}$$

Esercizio

Calcolare la derivata di $f(x) = a^x$ quando $a > 0$

Calcolare la derivata di $f(x) = x^x$

" " " " $f(x) = (\operatorname{sen} x)^{\operatorname{sen} x}$

dim

$$f = a^x = \left(e^{\ln(a)}\right)^x = e^{x\ln(a)}$$

$$g(x) = x\ln(a) \to g' = \ln(a)$$

$$x \xmapsto{g} x\ln(a) \xmapsto{h} e^{x\ln(a)}$$

$$h(y) = e^y \to h' = e^y$$

$$f'(x_0) = (h \circ g)'(x_0) = h'(g(x_0)) \cdot g'(x_0) = e^{x\ln(a)} \cdot \ln(a)$$

$$= a^x \cdot \ln(a)$$

$$f(x) = x^x = \left(e^{\ln(x)}\right)^x = e^{x\ln(x)}$$ è definita per $x > 0$

$$x \xmapsto{g} x\ln(x) \xmapsto{h} e^{x\ln(x)}$$

$$h(y) = e^y \to h'(y) = e^y$$

$$g(x) = x\ln x \to g'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

$$f'(x) = (h \circ g)'(x) = h'(g(x)) \cdot g'(x)$$

$$= e^{g(x)} \cdot \left(1 + \ln(x)\right) = e^{x\ln(x)} \cdot \left(1 + \ln(x)\right) = x^x\left(1 + \ln(x)\right)$$

$$f = (\operatorname{sen} x)^{\operatorname{sen} x} = e^{\operatorname{sen} x \ln(\operatorname{sen} x)}$$ definita per $0 < x < \pi$

$$f'(x) = e^{\operatorname{sen} x \ln(\operatorname{sen} x)} \cdot \left(\operatorname{sen} x \ln(\operatorname{sen} x)\right)' =$$

$$= (\operatorname{sen} x)^{\operatorname{sen} x} \cdot \left(\cos x \ln(\operatorname{sen} x) + \operatorname{sen} x \cdot \frac{1}{\operatorname{sen} x} \cdot \cos x\right)$$

$$= (\operatorname{sen} x)^{\operatorname{sen} x} \cdot \left(\cos x \ln(\operatorname{sen} x) + \cos x\right)$$