2015-09-28-Am1-lez 2 lunedì 14.30-16.30 1

Def

$f: A \to \mathbb{R}$ questa si dice

debolmente crescente se $\forall x,y \in A \ [x<y \Rightarrow f(x) \le f(y)]$

strettamente " se " $[x<y \Rightarrow f(x) < f(y)]$

debolmente decrescente se " $[x<y \Rightarrow f(x) \geq f(y)]$

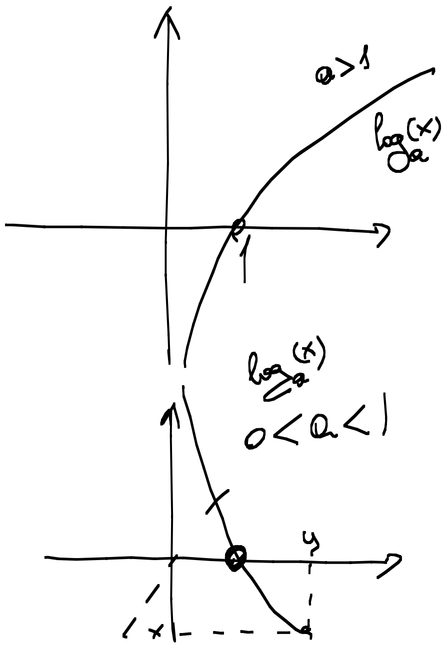
strettamente decrescente se " $[x<y \Rightarrow f(x) > f(y)]$

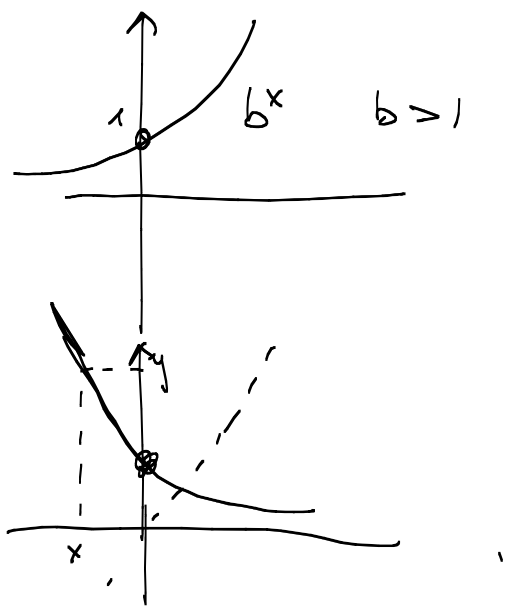
Esempio $f(x) = \log_a(x)$ strettamente crescente $a>1$

" decrescente $0<a<1$

$f(x) = b^x$ strettamente crescente $b>1$

" decrescente $0<b<1$





$a^x < a^y \quad a > 1 \iff x < y$

$\uparrow$

$\log_a x < \log_a y \quad a > 1 \iff x < y$ $\quad$ $x > 0$, $y > 0$

Diseguaglianze esponenziali & logaritmiche

Proprietà fondamentale per risolvere disuguaglianze del tipo

$$e^{f(x)} > e^{g(x)} \Rightarrow f(x) > g(x)$$

numero di nepero $e = 2{,}71..$

è la stretta crescenza di a^x quando $a > 1$

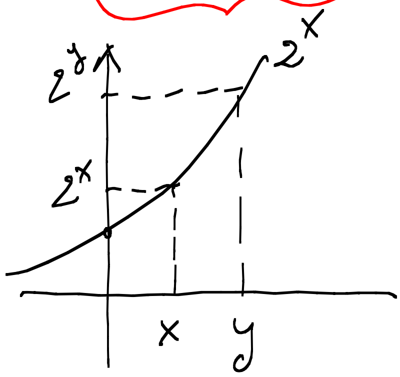
" decrescenza " a^x " $0 < a < 1$

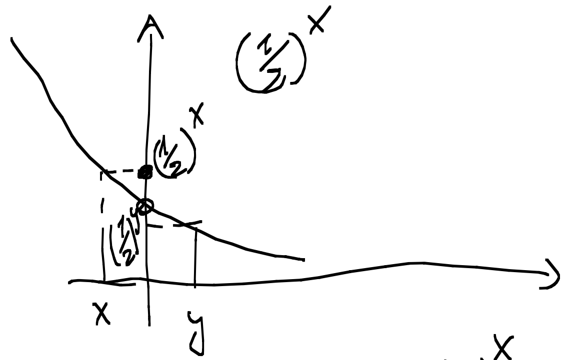
ovvero

$$\underbrace{2^{f(x)} > 2^{g(x)}} \iff \underbrace{f(x) > g(x)} \quad \text{(stretta crescenza)}$$

mentre

$$\underbrace{\left(\tfrac{1}{2}\right)^{f(x)} > \left(\tfrac{1}{2}\right)^{g(x)}} \iff f(x) < g(x) \quad \text{(stretta decrescenza)}$$

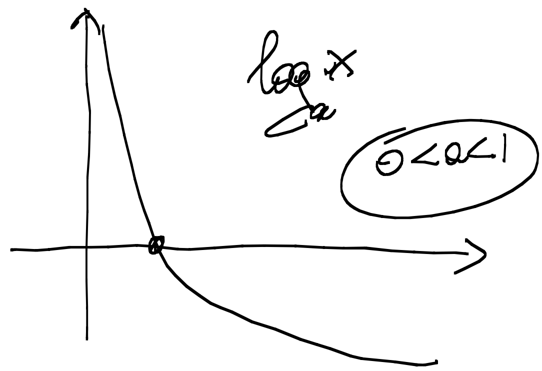




$x < y$ e $\left(\tfrac{1}{2}\right)^x > \left(\tfrac{1}{2}\right)^y$

$$\log_a x < \log_a y \iff x < y \qquad \text{quando } a > 1$$

$$\log_a x < \log_a y \iff x > y \qquad \text{quando } 0 < a < 1$$



Esercizio Per quali $x \in \mathbb{R}$ si ha

$$25^{2x^2-4} > 5^{10-2x^2-12x}$$

dim

Possiamo studiare la disuguaglianza $\forall x \in \mathbb{R}$

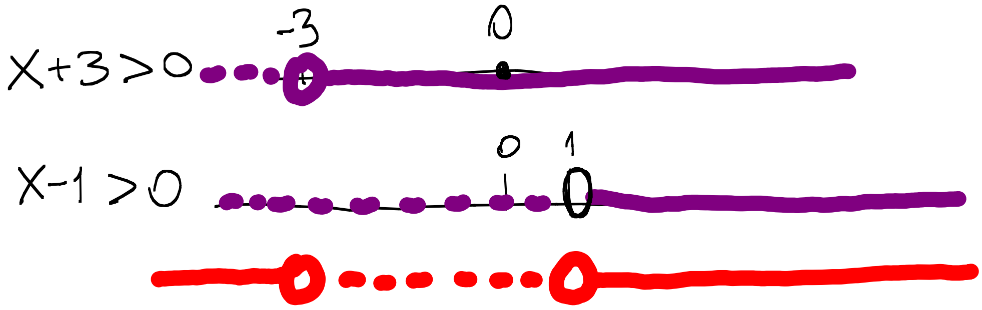
$$25^{2x^2-4} > 5^{2(5-x^2-6x)} = 25^{5-x^2-6x} \qquad \boxed{25>1}$$

$$\Updownarrow$$

$$2x^2-4 > 5-x^2-6x$$

$$3x^2+6x-9 = 3(x^2+2x-3) = 3(x+3)(x-1) > 0$$

$$\Updownarrow$$



$x^2+2x-3>0$ sse $x \in]-\infty,-3[\cup]1,+\infty[$

sse $x<-3$ o $1<x$ $\square$

OSSERVAZIONE Il numero e, o numero di Nepero, è la base privilegiata dei logaritmi

$$e = 2{,}781\dots = \lim_{n \to +\infty} \left(1+\frac{1}{n}\right)^n$$ (lo proveremo più avanti)

e quindi $\boxed{\log(x)}$ sta per $\log_e(x)$

(se si sottointende la base, allora la base è e)

Esercizio 5. Sia S l'insieme delle soluzioni della disequazione $\log(x+1)+\log(x-2)\le \log 10$. Allora

(A) $]2,4[\subset S$. Vero!

(B) $]-3,-1[\subset S$. FALSA

(C) $]-1,2[\subset S$. FALSA

(D) S non è limitato superiormente. FALSA $4=\max S$

$$S=\{x:\ \log(x+1)+\log(x-2)\le \log 10\}$$

$$=\{x:\ x+1>0 \text{ e } x-2>0 \text{ e } \log(x+1)+\log(x-2)\le\log 10\}$$

$$=\{x:\ x>-1 \text{ e } x>2 \text{ e } \log[(x+1)(x-2)]\le\log 10\}$$

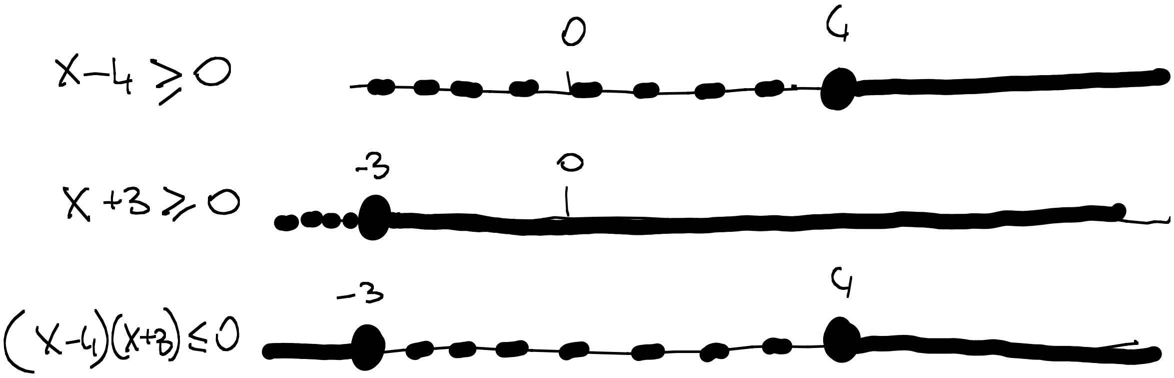
$$=\{x:\ x\in]-1,+\infty[\text{ e } x\in]2,+\infty[\text{ e } (x+1)(x-2)\le 10\}$$

$$=\{x:\ x\in]-1,+\infty[\,\cap\,]2,+\infty[\text{ e } x^2-x-2-10\le 0\}$$

$$=\{x:\ x\in]2,+\infty[\text{ e } (x-4)(x+3)\le 0\}$$

$$\Updownarrow$$

$$\begin{cases} x\in]2,+\infty[\\ (x-4)(x+3)\le 0\end{cases}$$



$$\begin{cases} x\in]2,+\infty[\\ x\in[-3,4]\end{cases} \iff \boxed{x\in]2,4]=S}$$

Esercizio Per quali valori di x è vero

$$\frac{7^{x-2}}{4} < \frac{7}{21+\sqrt{7^x}}$$

dim

$$\frac{7^{x-2}}{4} < \frac{7}{21+7^{x/2}} \qquad \underline{x \in \mathbb{R}}$$

$$7^{x-3} \cdot \left(21+7^{x/2}\right) < 4$$

$$7^{x-3} \left(3\cdot 7 + 7^{x/2}\right) - 4 < 0$$

$$3\cdot 7^{x-2} + 7^{\frac{3}{2}x-3} - 4 < 0$$

$$\frac{3}{7^2}\cdot 7^{2\frac{x}{2}} + \frac{7^{3\cdot\frac{x}{2}}}{7^3} - 4 < 0$$

$$\boxed{y = 7^{x/2}}$$

$$y^2 \cdot \frac{3}{7^2} + \frac{y^3}{7^3} - 4 < 0$$

$$7\cdot 3\cdot y^2 + y^3 - 4\cdot 7^3 < 0$$

$$\to y^3 + 3\cdot 7\cdot y^2 - 4\cdot 7^3 < 0$$

$$7^3 + 3\cdot 7\cdot 7^2 - 4\, 7^3 = 0$$

$$
\begin{array}{llll|l}
y^3 & +3\cdot 7 y^2 & + & -4 7^3 & y-7 \\
y^3 & -7y^2 & & & y^2+4\cdot 7 y + 4\cdot 7^2 \\
\hline
// & 4\cdot 7\cdot y^2 & & -4\cdot 7^3 & \\
 & 4\cdot 7 y^2 - 4 7^2 y & & & \\
\hline
 & / & 4\cdot 7^2 y & -4\cdot 7^3 &
\end{array}
$$

$$\boxed{\begin{aligned} & y^3 + 3\cdot 7 y^2 - 4\cdot 7^3 = \\ & = (y-7)\left(y^2+4\cdot 7\cdot y + 4\cdot 7^2\right) \end{aligned}}$$

$$y_{2,3} = -2\cdot 7 \pm \sqrt{4\cdot 7^2 - 4\cdot 7^2} = -2\cdot 7$$

$$y^3+3\cdot 7y^2-4\cdot 7^3=(y-7)\,(y-2\cdot 7)^2<0$$

$$\Updownarrow$$

$$\begin{cases} y-7<0 \\ y\neq 2\cdot 7=14 \end{cases} \iff y<7 \iff \boxed{y=7^{x/2}<7}$$

$$7^{x/2}<7=7^1$$

$$\Updownarrow$$

$$\frac{x}{2}=\log_7\left(7^{x/2}\right)<\log_7(7)=1$$

$$\Updownarrow$$

$$\boxed{x<2}$$

Esercizio Per quali $x \in \mathbb{R}$ vale

$$\log(x^2-3x+4) \geq \log(4x-6) \ ?$$

dim

$$\begin{cases} x^2-3x+4>0 \\ 4x-6>0 \\ \log(x^2-3x+4) \geq \log(4x-6) \end{cases} \iff \begin{cases} \left(x^2-2\cdot\frac{3}{2}\cdot x+\frac{9}{4}\right)-\frac{9}{4}+4>0 \\ x > 6/4 = 3/2 \\ x^2-3x+4 \geq 4x-6 \end{cases}$$

$$\iff \begin{cases} \left(x-3/2\right)^2+\frac{7}{4}>0 \\ x > 3/2 \\ x^2-7x+10 \geq 0 \end{cases} \iff \begin{cases} x > 3/2 \\ (x-5)(x-2) \geq 0 \end{cases}$$

$$\iff \begin{cases} x > 3/2 \\ x \leq 2 \ \text{ o } \ 5 \leq x \end{cases} \iff x \in \left]3/2,+\infty\right[\cap \left(\left]-\infty,2\right] \cup \left[5,+\infty\right[\right)$$

$$\iff x \in \left]3/2, 2\right] \cup \left[5,+\infty\right[$$

$$2 \quad \boxed{3} \quad 6$$

$$\log(x^2-3x+4) \geq \log(4x-6)$$

$2 \quad \log(4-6+4) \geq \log(8-6)$ ✓

$3 \quad \log(9-9+4) \geq \log(12-6)$ NO

$6 \quad \log(36-18+4) \geq \log(24-6)$

$\qquad 22 \qquad\qquad 18$

Non è

una garanzia di successo, ma aiuta a trovare errori grossolani

Esercizio Per quali valori di $x \in \mathbb{R}$ vale

$$3\log x - \frac{12}{\log x} < 5$$

dim

$$\begin{cases} x > 0 \\ x \neq 1 \\ 3\log x - \dfrac{12}{\log x} < 5 \end{cases} \iff \begin{cases} x > 0 \\ x \neq 1 \\ \dfrac{3\log^2 x - 5\log x - 12}{\log x} < 0 \end{cases}$$

$$\iff \underbrace{\begin{cases} 0 < x < 1 \quad (\log x < 0 !!) \\ 3\log^2 x - 5\log x - 12 > 0 \end{cases}}_{1} \quad \text{o} \quad \underbrace{\begin{cases} 1 < x \quad (\log x > 0) \\ 3\log^2 x - 5\log x - 12 < 0 \end{cases}}_{2}$$

1

$$\begin{cases} y = \log x \\ y < 0 \\ 3y^2 - 5y - 12 > 0 \end{cases} \iff \begin{cases} y = \log x \\ y < 0 \\ y < -4/3 \text{ o } 3 < y \end{cases}$$

$$\left(y_{1,2} = \frac{5 \pm \sqrt{25+144}}{6} = \frac{5 \pm 13}{6} \begin{array}{l} \nearrow 3 \\ \searrow -4/3 \end{array} \right)$$

$$\iff \begin{cases} y = \log x \\ y < 0 \\ y < -4/3 \end{cases} \iff \log x < -4/3 \iff \boxed{x < e^{-4/3} \text{ e } x > 0}$$

2

$$\begin{cases} 1 < x \\ 3\log^2 x - 5\log x - 12 < 0 \end{cases} \iff \begin{cases} y = \log x \\ y > 0 \\ 3y^2 - 5y - 12 < 0 \end{cases}$$

$$\iff \begin{cases} y = \log x \\ y > 0. \\ y \in]-4/3, 3[\end{cases} \iff \begin{cases} y = \log x \\ y \in]0, 3[\end{cases} \iff \begin{cases} 0 < \log x < 3 \\ x > 1 \end{cases}$$

$$\iff \boxed{1 < x < e^3}$$

Dunque $x \in]0, e^{-4/3}[\cup]1, e^3[$

DISUGUAGLIANZE IRRAZIONALI

Sono di tre tipi (sarebbero due, in quanto il terzo tipo è simile al tipo 2)

① $\sqrt{f(x)} > g(x)$ (oppure $\geq$)

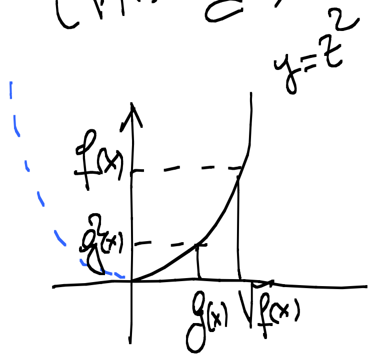
② $\sqrt{f(x)} < g(x)$ (oppure $\leq$)

③ $\sqrt{f(x)} > \sqrt{g(x)}$ (oppure $\geq$)

① $\sqrt{f(x)} > g(x)$

$$\Updownarrow$$

$$\begin{cases} f(x) \geq 0 \\ \sqrt{f(x)} > g(x) \end{cases} \iff \begin{cases} f(x) \geq 0 \\ g(x) < 0 \end{cases} \text{ O' } \begin{cases} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) > g^2(x) \end{cases}$$



② $\sqrt{f(x)} < g(x)$

$$\Updownarrow \begin{cases} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) < g^2(x) \end{cases}$$

① $\sqrt{f(x)} > g(x) \iff \begin{cases} f(x) \geq 0 \\ \sqrt{f(x)} > g(x) \end{cases}$

$\iff \begin{cases} f(x) \geq 0 \\ g(x) < 0 \end{cases}$ o $\begin{cases} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) > g^2(x) \end{cases}$

questo segue dal fatto $\forall x : f(x) \geq 0,\ \sqrt{f(x)} \geq 0$

questo segue dal fatto che $A \geq 0,\ B \geq 0,\ A > B^2 \Rightarrow A \geq 0\ B \geq 0\ \sqrt{A} > B$

② $\sqrt{f(x)} < g(x) \iff \begin{cases} f(x) \geq 0 \\ \sqrt{f(x)} < g(x) \end{cases} \iff \begin{cases} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) < g^2(x) \end{cases}$

③ $\sqrt{f(x)} > \sqrt{g(x)} \iff \begin{cases} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) > g(x) \end{cases}$

Esercizio Per quali valori di x è vero che

$$\sqrt{x-1} > 2x-3$$

dim (è del Tipo 1)

$$\sqrt{f(x)} > g(x) \iff \begin{cases} f(x) \geq 0 \\ g(x) < 0 \end{cases} \lor \begin{cases} f \geq 0 \\ g \geq 0 \\ f \geq g^2 \end{cases}$$

$$\iff \begin{cases} x-1 \geq 0 \\ 2x-3 < 0 \end{cases} \lor \begin{cases} x-1 \geq 0 \\ 2x-3 \geq 0 \\ x-1 > (2x-3)^2 \end{cases}$$

$$\iff \begin{cases} x \geq 1 \\ x < 3/2 \end{cases} \lor \begin{cases} x \geq 1 \\ x \geq 3/2 \\ x-1 > 4x^2+9-12x \end{cases}$$

$$\iff x \in [1, 3/2[\ \lor \begin{cases} x \geq 3/2 \\ 4x^2-13x+10 < 0 \end{cases}$$

$$x_{1,2} = \frac{13 \pm \sqrt{169-160}}{8} \begin{matrix} \nearrow 5/4 \\ \searrow 2 \end{matrix}$$

$$\iff x \in [1, 3/2[\ \lor \begin{cases} x \geq 3/2 \\ x \in]5/4, 2[\end{cases}$$

$$\iff x \in [1, 3/2[\ \lor \ x \in [3/2, 2[$$

$$\iff x \in [1, 2[$$

Esercizio Per quali valori di $x \in \mathbb{R}$ si ha

$$\sqrt{2x+1} \leq x-3$$

dim (è del caso 2)

(2) $\sqrt{f(x)} \leq g(x)$

$$\Updownarrow \begin{cases} f(x) \geq 0 \\ g(x) \geq 0 \\ f(x) \leq g^2(x) \end{cases}$$

$$\begin{cases} 2x+1 \geq 0 \\ x-3 \geq 0 \\ (x-3)^2 \geq 2x+1 \end{cases} \iff \begin{cases} x \geq -\frac{1}{2} \\ x \geq 3 \\ x^2-6x+9-2x-1 \geq 0 \end{cases}$$

$$\iff \begin{cases} x \geq 3 \\ x^2-8x+8 \geq 0 \end{cases}$$

$$x_{1,2} = 4 \pm \sqrt{16-8}$$

$\to 4+2\sqrt{2}$

$\to 4-2\sqrt{2}$

$$\iff \begin{cases} x \geq 3 \\ x \leq 4-2\sqrt{2} \text{ or } 4+2\sqrt{2} \leq x \end{cases}$$

$$\iff x \geq 4+2\sqrt{2}$$

$$\iff x \in [4+2\sqrt{2}, +\infty[$$

$$4-2\sqrt{2} \leq 3$$

$$1 \leq 2\sqrt{2}$$

$$\begin{cases} 2x+1 \geq 0 \\ x-3 \geq 0 \\ 2x+1 \leq (x-3)^2 \end{cases} \iff \begin{cases} x \in [3, +\infty[\\ x^2-6x+9-2x-1 \geq 0 \end{cases}$$

$$\iff \begin{cases} x \in [3, +\infty[\\ x^2-8x+8 \geq 0 \end{cases} \qquad x_{1,2} = +4 \pm \sqrt{16-8} = +4 \pm 2\sqrt{2}$$

$$\iff x \in [3, +\infty[\cap \left( ]-\infty, 4-2\sqrt{2}] \cup [4+2\sqrt{2}, +\infty[\right)$$

$$\iff x \in [4+2\sqrt{2}, +\infty[$$

Esercizio

Per quali valori di x si ha $\sqrt{x^2-4x} > \sqrt{3-2x}$ (*)

dim (è il caso 3)

$$(*) \iff \begin{cases} x^2-4x > 0 \\ 3-2x > 0 \\ x^2-4x > 3-2x \end{cases} \iff \begin{cases} x \in]-\infty, 0[\cup]4, +\infty[\\ x \in]-\infty, 3/2[\\ x^2-2x-3 = (x-3)(x+1) > 0 \end{cases}$$

$$\iff \begin{cases} x \in]-\infty, 0[\\ x \in]-\infty, -1[\cup]3, +\infty[\end{cases} \iff x \in]-\infty, -1[$$