Algebra dei limiti: efficace tranne che per le forme indeterminate

$$\infty-\infty\ ;\ \frac{\infty}{\infty}\ ;\ 0\cdot\infty\ ;\ 0/0\ ;\ 1^{\infty}$$

Quando non compaiono forme indeterminate, ci si riduce a introdurre $+\infty$ e $-\infty$ nell'usuale algebra su $\mathbb{R}$, ovvero estendiamo $+$ e $\cdot$ a $\overline{\mathbb{R}}$

$$\forall c\in\mathbb{R}\ \begin{cases} +\infty+c=+\infty & +\infty-c=+\infty \\ -\infty+c=-\infty & -\infty-c=-\infty \end{cases}$$

$$\forall c>0\ \begin{cases} c\cdot+\infty=+\infty & c\cdot-\infty=-\infty \\ \frac{+\infty}{c}=+\infty & \frac{-\infty}{c}=-\infty \\ \frac{c}{+\infty}=0^+ & \frac{c}{-\infty}=0^- \end{cases}$$

$$\forall c<0\ \begin{cases} c\cdot+\infty=-\infty & c\cdot-\infty=+\infty \\ \frac{+\infty}{c}=-\infty & \frac{-\infty}{c}=+\infty \\ \frac{c}{+\infty}=0^- & \frac{c}{-\infty}=0^+ \end{cases}$$

$$+\infty+\infty=+\infty \qquad -\infty-\infty=-\infty$$

$$\frac{+\infty}{0^+}=+\infty \qquad \frac{-\infty}{0^+}=-\infty$$

$$\frac{+\infty}{0^-}=-\infty \qquad \frac{-\infty}{0^-}=+\infty$$

$$\frac{0^-}{+\infty}=0^- \qquad \frac{0^-}{-\infty}=0^+$$

Teorema (limitata × infinitesima = infinitesima)

Siano date $\{a_n\}_n$ e $\{b_n\}_n$ successioni reali

i) $\lim_{n\to+\infty} a_n = 0$

ii) $\{b_n\}_n$ sia limitata $\left(\forall n \quad |a_n| \le K \quad K\in\,]0,+\infty[\right)$

Allora $\exists \lim_{n\to+\infty} a_n\cdot b_n = 0$

dim

Osservo che $\{b_n\}$ non è detto abbia limite

$\exists \lim_{n\to+\infty} a_n b_n = 0$

ovvero

$$\forall\varepsilon>0\ \exists\nu\in\mathbb{R}\ \forall n>\nu \quad |a_n\cdot b_n|<\varepsilon$$

Ipotesi
$$\begin{cases}\forall\varepsilon>0\ \exists\nu\in\mathbb{R} : \forall n>\nu \quad |a_n|<\varepsilon\\ \forall n \quad |b_n|\le K\end{cases}$$

$$\Rightarrow \forall\varepsilon>0\ \exists\nu\in\mathbb{R} : \forall n>\nu \quad |a_n\cdot b_n| \le |a_n|\,|b_n| < k\cdot\varepsilon$$

$$\Rightarrow \lim_{n\to+\infty} a_n b_n = 0$$

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Esercizio

Dato $x\in\mathbb{R}$: $\boxed{\forall\varepsilon>0 \quad |x|\le k\varepsilon}$ ipotesi, con $k>0$

Allora $x=0$

dim

per assurdo $x\neq 0 \Rightarrow |x|>0 \Rightarrow \dfrac{|x|}{K}>0$
$K>0$

$\Rightarrow \exists N\in\mathbb{N} : \dfrac{|x|}{K} > \dfrac{1}{N}$ e dunque, preso $\varepsilon = 1/N$

si ha che $\exists\varepsilon>0 : |x|>k\varepsilon$ ASSURDO ///

Esercizio

Calcolare, se esiste, $\lim_{n\to+\infty}\left(\underbrace{\frac{1}{\sqrt{n+0}}+\frac{1}{\sqrt{n+1}}+\dots+\frac{1}{\sqrt{2n}}}_{n+1 \text{ Termini}}\right)$

dim

$n=3 \quad \frac{1}{\sqrt{3+0}}+\frac{1}{\sqrt{3+1}}+\frac{1}{\sqrt{3+2}}+\frac{1}{\sqrt{3+3}} \longrightarrow 4$ Termini

Ciascuno dei Termini Tende a 0 per $n\to+\infty$

ovvero $\lim_{n\to+\infty}\frac{1}{\sqrt{n}}=\lim_{n\to+\infty}\frac{1}{\sqrt{n+1}}=\lim_{n\to+\infty}\frac{1}{\sqrt{n+2}}=\dots=\lim_{n\to+\infty}\frac{1}{\sqrt{2n}}=0$

"La somma di "Tante" quantità infinitesime è infinitesima" ??

Dipende da "quanti" e dalla "velocità" con cui vanno a zero

Nel nostro caso il Tutto Tende a $+\infty$

$$\frac{(n+1)}{\sqrt{n}} \geq \frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}+\dots+\frac{1}{\sqrt{2n}} \geq \overbrace{\frac{1}{\sqrt{2n}}+\frac{1}{\sqrt{2n}}+\dots+\frac{1}{\sqrt{2n}}}^{n+1} = (n+1)\cdot\frac{1}{\sqrt{2}\cdot\sqrt{n}}$$

$$\begin{cases} a_n=\frac{n+1}{\sqrt{2}\cdot\sqrt{n}} \qquad c_n=\frac{n+1}{\sqrt{n}} \\ a_n\leq b_n\leq c_n \quad \forall n\geq 1 \end{cases}$$

questa è la situazione

$$\lim_{n\to+\infty}a_n=\lim_{n\to+\infty}\frac{n}{\sqrt{n}}\cdot\frac{1+1/n}{\sqrt{2}}=$$
$$=\lim_{n\to+\infty}\sqrt{n}\cdot\lim_{n\to+\infty}\frac{1+1/n}{\sqrt{2}}=+\infty\quad\frac{1+\frac{1}{+\infty}}{\sqrt{2}}=+\infty$$

$$\lim_{n\to+\infty}c_n=\lim_{n\to+\infty}\frac{n+1}{\sqrt{n}}=\lim_{n\to+\infty}\sqrt{n}\cdot\lim_{n\to+\infty}\frac{1+1/n}{1}=+\infty$$

x Thm 2 Carabinieri

$$\Longrightarrow \lim_{n\to+\infty}b_n=+\infty$$

Esercizio

Calcolare, se esiste, $\lim_{n\to+\infty}\left(\frac{1}{\sqrt{n^3+1}}+\frac{1}{\sqrt{n^3+2}}+\dots+\frac{1}{\sqrt{n^3+n}}\right)$

dim

voglio applicare il teorema dei 2 Carabinieri

$$\underbrace{\frac{1}{\sqrt{n^3+1}}+\dots+\frac{1}{\sqrt{n^3+1}}}_{=\frac{n}{\sqrt{n^3+1}}=c_n} \geq \underbrace{\frac{1}{\sqrt{n^3+1}}+\frac{1}{\sqrt{n^3+2}}+\dots+\frac{1}{\sqrt{n^3+n}}}_{b_n} \geq \underbrace{\frac{1}{\sqrt{n^3+n}}+\dots+\frac{1}{\sqrt{n^3+n}}}_{n} = \frac{n}{\sqrt{n^3+n}} = a_n$$

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty}\frac{n}{n^{3/2}}\cdot\lim_{n\to+\infty}\frac{1}{\sqrt{1+1/n^2}} =$$

$$= \lim_{n\to+\infty}\frac{1}{\sqrt{n}}\cdot\frac{1}{\sqrt{1+\lim_{n\to+\infty}\frac{1}{n^2}}} = 0\cdot\frac{1}{\sqrt{1}} = 0$$

$$\lim_{n\to+\infty} c_n = \lim_{n\to+\infty}\frac{n}{n^{3/2}}\cdot\lim_{n\to+\infty}\frac{1}{\sqrt{1+1/n^3}} = \dots = 0$$

$$a_n \leq b_n \leq c_n \quad \forall n$$

Teorema 2 Carabinieri

$$\Longrightarrow \quad \lim_{n\to+\infty} b_n = 0$$

Esercizio

Calcolare, se esiste, $\lim_{n\to+\infty}\frac{n!}{n^n}$

dim

devo confrontare $n!$ con n^n : criterio rapporto (o radice)

oppure STIRLING

$$\frac{1}{n^n}\frac{n^n}{e^n} \leq \frac{n!}{n^n} \leq \frac{1}{\cancel{n^n}}\cdot n\cdot e\cdot\frac{\cancel{n^n}}{e^n} \quad \forall n\in\mathbb{N}$$

semplificando si ottiene

$$\frac{1}{e^n} \leq \frac{n!}{n^n} \leq n\cdot\frac{1}{e^{n-1}}$$

ma $\lim_{n\to+\infty} \frac{1}{e^n} = 0$ in quanto $e^n \xrightarrow[n\to+\infty]{} +\infty$ (*)

$\lim_{n\to+\infty} \frac{n}{e^{n-1}} = 0$ " " $\frac{e^{n-1}}{n} \xrightarrow[n\to+\infty]{} +\infty$ (*)

prossimo esercizio

Per il Teorema 2 Carabinieri

$$\Rightarrow \lim_{n\to+\infty} \frac{n!}{n^n} = 0$$

Teorema (Criterio della radice)

$\{a_n\}_n$ successione reale $a_n \geq 0 \ \forall n$

Sia $\ell = \lim_{n\to+\infty} \sqrt[n]{a_n}$

i) $0 \leq \ell < 1 \Rightarrow \lim_{n\to+\infty} a_n = 0$

ii) $1 < \ell \Rightarrow \lim_{n\to+\infty} a_n = +\infty$

Esercizio $\lim_{n\to+\infty} q^n = \begin{cases} +\infty & q > 1 \\ 0 & 0 \leq q < 1 \end{cases}$ Limite Fondamentale

dim

i) $q > 1 \Rightarrow q = 1 + h \quad h > 0$

$$\Rightarrow q^n = (1+h)^n \underset{\text{Bernoulli}}{\geq} 1 + nh \quad \forall n \in \mathbb{N}$$

ma $\lim_{n\to+\infty} (1+nh) = +\infty$ $\underline{\underline{h>0}}$

Confronto

$$\Rightarrow \lim_{n\to+\infty} q^n = +\infty$$

ii) $0 < q < 1 \Rightarrow \frac{1}{q} > 1 \Rightarrow$ (per il punto i) $\lim_{n\to+\infty} \frac{1}{q^n} = +\infty$

$$\Rightarrow \frac{1}{\lim_{n\to+\infty} q^n} = +\infty \Rightarrow \lim_{n\to+\infty} q^n = \frac{1}{+\infty} = 0$$

iii) $q = 0 \Rightarrow q^n = 0 \ \forall n \Rightarrow \lim_{n\to+\infty} 0^n = 0$ $\boxed{///}$

dimostro criterio della radice

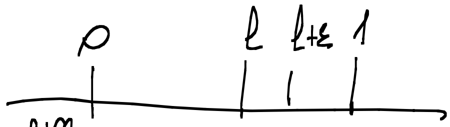
1) $\boxed{0 \le \ell < 1 \Rightarrow \lim_{n\to+\infty} a_n = 0}$ $\qquad \ell = \lim_{n\to+\infty}\sqrt[n]{a_n}$

$$\forall\varepsilon>0\ \exists\nu>0 : \forall n>\nu \quad \ell-\varepsilon<\sqrt[n]{a_n}<\ell+\varepsilon$$

$\Downarrow$ $a_n>0$

$$\forall\varepsilon>0\ \exists\nu>0\ \forall n>\nu\quad 0<\sqrt[n]{a_n}<\ell+\varepsilon$$

$\Downarrow$



$$\varepsilon=\frac{1+\ell}{2}\ \exists\nu\in\mathbb{R}\ \forall n>\nu\quad 0<a_n<\left(\ell+\frac{1-\ell}{2}\right)^n$$

Scelgo piazzare $\ell+\varepsilon=\frac{\ell+1}{2}$

$\varepsilon=\frac{\ell+1}{2}-\ell=\frac{1-\ell}{2}$

$$\varepsilon=\frac{1-\ell}{2}\ \exists\nu\in\mathbb{R}\ \boxed{\forall n>\nu\quad 0<a_n<\left(\frac{1+\ell}{2}\right)^n}$$

$$\begin{cases}\lim_{n\to+\infty} 0 = 0 = \lim_{n\to+\infty}\left(\frac{1+\ell}{2}\right)^n & \text{poiché } \frac{1+\ell}{2}<1\\ \forall n>\nu \quad 0<a_n<\left(\frac{1+\ell}{2}\right)^n\end{cases}$$

2 carabinieri

$$\Longrightarrow \quad \lim_{n\to+\infty} a_n = 0$$

suppongo $\ell<+\infty$ ✓

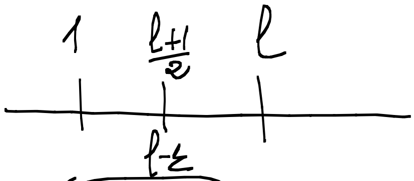
ii) $\ell>1 \Rightarrow \lim_{n\to+\infty} a_n = +\infty$ $\qquad \ell=\lim_{n\to+\infty}\sqrt[n]{a_n}$

$$\forall\varepsilon>0\ \exists\nu\in\mathbb{R}\ \forall n>\nu \quad \ell-\varepsilon<\sqrt[n]{a_n}<\ell+\varepsilon$$

$\Downarrow$

$$\forall\varepsilon>0\ \exists\nu\in\mathbb{R}\ \forall n>\nu \quad \ell-\varepsilon<\sqrt[n]{a_n}$$

$\Downarrow$



$$\varepsilon=\frac{\ell-1}{2}\ \exists\nu\in\mathbb{R}\ \forall n>\nu\quad \ell-\frac{\ell-1}{2}=\frac{\ell+1}{2}<\sqrt[n]{a_n}$$

$\ell-\varepsilon=\frac{\ell+1}{2}$ $\qquad \varepsilon=\ell-\frac{\ell+1}{2}=\frac{\ell-1}{2}$

$\Downarrow$

$$\begin{cases}\forall n>\nu \quad \left(\frac{\ell+1}{2}\right)^n<a_n\\ \text{e } \lim_{n\to+\infty}\left(\frac{\ell+1}{2}\right)^n=+\infty\end{cases} \quad \xRightarrow{\text{confronto}} \quad \lim_{n\to+\infty} a_n=+\infty$$

$\uparrow$ $\frac{\ell+1}{2}>1$

Teorema (Criterio Rapporto)

$\{a_n\}_n \quad a_n > 0 \ \forall n \qquad \ell = \lim_{n\to+\infty} \frac{a_{n+1}}{a_n}$

i) $0 \le \ell < 1 \Rightarrow \lim_{n\to+\infty} a_n = 0$

ii) $1 < \ell \Rightarrow \lim_{n\to+\infty} a_n = +\infty$

Esercizio

Calcolare $\lim_{n\to+\infty} \frac{n}{e^{n-1}}$

dim

essendo $a_n = \frac{n}{e^{n-1}}$ $\qquad \frac{a_{n+1}}{a_n} = \frac{n+1}{e^{\cancel{n}}} \cdot \frac{\cancel{e^{n-1}}}{n}$

$= \frac{n+1}{n} \cdot \frac{1}{e}$

$\lim_{n\to+\infty} \frac{n+1}{n} \cdot \frac{1}{e} = \frac{1}{e} \cdot \lim_{n\to+\infty} \frac{n+1}{n} = \frac{1}{e} \lim_{n\to+\infty} \frac{n}{n} \cdot \frac{1+1/n}{1}$

$= \frac{1}{e} \lim_{n\to+\infty} \frac{1+1/n}{1} = \frac{1}{e} \cdot \frac{1+1/+\infty}{1} = \frac{1}{e} < 1$

$\Rightarrow \lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{n}{e^{n-1}} = 0 \qquad \boxed{///}$

Oss: $\lim_{n\to+\infty} \frac{n}{e^{n-1}} = 0$ è $\frac{n}{e^{n-1}} > 0 \Rightarrow \lim_{n\to+\infty} \frac{e^{n-1}}{n} = +\infty$

Esercizio

Calcolare $\lim_{n\to+\infty} e^n$

dim

Voglio applicare criterio radice $\quad a_n = e^n \qquad e = 2,7\ldots$

$\sqrt[n]{a_n} = \sqrt[n]{e^n} = e \xrightarrow[n\to+\infty]{} e > 1$

Criterio Radice

$\Rightarrow \lim_{n\to+\infty} e^n = +\infty \qquad \boxed{///}$

Esercizio Calcolare $\lim_{n\to+\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$

dim

Non posso usare il Teorema Algebrico sui limiti poiché ho una forma indeterminata $\infty-\infty$

$$\left(\sqrt{n+1}-\sqrt{n}\right)\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$\lim_{n\to+\infty}\sqrt{n+1}-\sqrt{n}=\lim_{n\to+\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}=\lim_{n\to+\infty}\frac{1}{\sqrt{n}}\cdot\frac{1}{\sqrt{1+1/n}+1}$$

$$=\lim_{n\to+\infty}\frac{1}{\sqrt{n}}\cdot\lim_{n\to+\infty}\frac{1}{\sqrt{1+1/n}+1}=\frac{1}{+\infty}\cdot\frac{1}{2}=0 \quad ▥$$

Esercizio Calcolare $\lim_{n\to+\infty}\sqrt[n]{n}$

dim

Ieri avevamo provato $\sqrt[n]{a}\xrightarrow[n\to+\infty]{}1 \quad \forall a>0$

siamo nel caso $\infty^{1/\infty}$

Osservo $\sqrt[n]{\sqrt{n}}=1+h_n$ con $h_n\geq 0$

$\Downarrow$

elevo alla n $\quad \sqrt{n}=(1+h_n)^n\geq 1+n\cdot h_n \quad \forall n$

Bernoulli

$\Downarrow$

$$\boxed{\frac{\sqrt{n}-1}{n}\geq h_n}$$

Adesso $\boxed{1}\leq\sqrt[n]{n}=(1+h_n)^2=1+h_n^2+2h_n\leq\boxed{1+\left(\frac{\sqrt{n}-1}{n}\right)^2+2\,\frac{\sqrt{n}-1}{n}}$

$\boxed{1}\xrightarrow{n\to+\infty}1 \qquad\qquad \boxed{1+\left(\frac{\sqrt{n}-1}{n}\right)^2+2\,\frac{\sqrt{n}-1}{n}}\xrightarrow{n\to+\infty}1$

$$\left(\frac{\sqrt{n}-1}{n}\right)^2=\frac{n+1-2\sqrt{n}}{n^2}=\boxed{\frac{1}{n}}\cdot\frac{1+1/n-2/\sqrt{n}}{1}\xrightarrow[n\to+\infty]{}0$$

$$\frac{\sqrt{n}-1}{n} = \frac{1}{\sqrt{n}} \cdot \frac{1-1/\sqrt{n}}{1} \xrightarrow[n \to +\infty]{} 0$$

Teorema dei Carabinieri

$$\Longrightarrow \quad \lim_{n \to +\infty} \sqrt[n]{n} = 1$$