Zagier's conjecture on special values of L-functions

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In this talk I will state Zagier's conjecture for number fields and discuss generalizations of this conjecture to curves over $\mathbf{Q}$. The formulation of these generalizations raises many deep and interesting questions.

Let F be a number field and $\zeta_F(s)$ the Dedekind zeta function of F. This function determines many arithmetical invariants of F, like the absolute value $|D_F|$ of the discriminant of F, and the number r_1 (resp. $2r_2$) of real (resp. complex) embeddings of F.

In particular we expect the special values of $\zeta_F(s)$ at positive integers to contain much arithmetical information about F.

If $F = \mathbf{Q}$ it is known since Euler that

$$\zeta_{\mathbf{Q}}(2m) = \zeta(2m) \in \pi^{2m}\mathbf{Q}^* \qquad (m \geq 1).$$

More generally, if F is totally real ($r_2 = 0$) a theorem of Klingen and Siegel says that

$$\zeta_F(2m) \in \frac{\pi^{2m[F:\mathbf{Q}]}}{|D_F|^{1/2}}\mathbf{Q}^* \qquad (m \geq 1).$$

Another case of interest is the case $s = 1$. The function $\zeta_F(s)$ has a simple pole at $s = 1$, and using the functional equation, a zero of order $r_1 + r_2 - 1$ at $s = 0$. A celebrated theorem of Dirichlet says the following.

Theorem 1 (Dirichlet) *The following equality holds*

$$\lim_{s \to 0} \zeta_F(s) s^{-(r_1+r_2-1)} = -\frac{h_F R_F}{\omega_F},$$

where h_F is the class number of F, ω_F is the number of roots of unity of F, and R_F is Dirichlet's regulator *associated to F.*

Here we are interested in the arithmetical nature of the regulator R_F. It is defined using the *regulator mapping*

$$\begin{aligned} \rho_F : \mathcal{O}_F^* &\to \mathbf{R}^{\Sigma} \\ x &\mapsto (\log |x|_\sigma)_{\sigma \in \Sigma}, \end{aligned}$$

where Σ is the set of standard archimedean absolute values of F (we have $|\Sigma| = r_1 + r_2$). The image of ρ_F is a lattice in a real hyperplane $H \subset \mathbf{R}^\Sigma$, and R_F is defined to be the covolume of this lattice with respect to the Lebesgue measure on H.

In particular, we see that Dirichlet's regulator R_F can be writen as a determinant of a matrix whose entries are logarithms of numbers in F. This matrix has size $r_1 + r_2 - 1$.

Now, for any $k \geq 1$, let

$$\mathrm{Li}_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k} \qquad (|z| < 1).$$

The function Li_k extends to a multivalued function on $\mathbf{C}\backslash\{0, 1\}$ (that is, Li_k is defined on the universal covering of $\mathbf{C}\backslash\{0, 1\}$). We have the obvious equality

$$\zeta(m) = \mathrm{Li}_m(1) \qquad (m \geq 2).$$

Dirichlet's theorem and a result of Zagier on 3-dimensional hyperbolic geometry suggest that for any number field F and any integer $m \geq 2$, the special value $\zeta_F(m)$ should be linked to the polylogarithm functions Li_k $(1 \leq k \leq m)$. The precise formulation of this link is the object of Zagier's conjecture.

Zagier defines for any $m \geq 2$ the single-valued real-analytic function D_m on $\mathbf{P}^1(\mathbf{C})\backslash\{0, 1, \infty\}$ by

$$D_m(z) := \Re_m\left(\sum_{r=0}^{m-1} \frac{(-1)^r}{r!} \log^r |z| \mathrm{Li}_{m-r}(z) - \frac{(-1)^m}{2m!} \log^m |z|\right)$$

where $\Re_m$ stands for $\Re$ or $\Im$ depending whether m is odd or even. This function generalizes the Bloch-Wigner function $D(z)$ which is the $m = 2$ case. Let us denote $r_+ = r_1 + r_2$ and $r_- = r_2$. We can now formulate Zagier's conjecture for number fields.

Conjecture 2 (Zagier's conjecture for number fields) *Let F be a number field and m be an integer ≥ 2. Let us denote $(-1)^m = \pm 1$. Then the special value $\zeta_F(m)$ is equal up to a non-zero rational factor to*

$$\pi^{mr_\pm} \det M$$

where M is a matrix of size $r_\mp$ whose entries are $\mathbf{Z}$-linear combinations of numbers $D_m(x)$ with $x \in F$.

Zagier's conjecture for number fields has been essentially proved by Zagier in the case $m = 2$, and by Goncharov in the case $m = 3$.

In my talk I will discuss the analogue of Zagier's conjecture for L- functions of elliptic curves over $\mathbf{Q}$ (the so-called elliptic Zagier's conjecture). It has been formulated by Wildeshaus in 1997 and proved in the case $m = 2$ by Goncharov and Levine in 1998.

I will also discuss the interesting problem of generalizing Zagier's conjecture to any curve X over $\mathbf{Q}$.