Analytic Semigroups and Reaction-Diffusion Problems

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Introduction

These lectures deal with the functional analytical approach to linear and nonlinear parabolic problems.

The simplest significant example is the heat equation, either linear

$$\begin{cases} u_t(t,x) = u_{xx}(t,x) + f(t,x), & 0 < t \le T, \quad 0 \le x \le 1, \\ u(t,0) = u(t,1) = 0, & 0 \le t \le T, \\ u(0,x) = u_0(x), & \qquad\qquad 0 \le x \le 1, \end{cases} \tag{1}$$

or nonlinear,

$$\begin{cases} u_t(t,x) = u_{xx}(t,x) + f(u(t,x)), & t > 0, \quad 0 \le x \le 1, \\ u(t,0) = u(t,1) = 0, & t \ge 0, \\ u(0,x) = u_0(x), & \qquad\qquad 0 \le x \le 1. \end{cases} \tag{2}$$

In both cases, u is the unknown, and f, u_0 are given. We will write problems (1), (2) as evolution equations in suitable Banach spaces. To be definite, let us consider problem (1), and let us set

$$u(t,\cdot) = U(t), \quad f(t,\cdot) = F(t), \quad 0 \le t \le T,$$

so that for every $t \in [0,T]$, $U(t)$ and $F(t)$ are functions, belonging to a suitable Banach space X. The choice of X depends on the type of the results expected, or else on the regularity properties of the data. For instance, if f and u_0 are continuous functions the most natural choice is $X = C([0,1])$; if $f \in L^p((0,T) \times (0,1))$ and $u_0 \in L^p(0,1)$, $p \ge 1$, the natural choice is $X = L^p(0,1)$, and so on.

Next, we write (1) as an evolution equation in X,

$$\begin{cases} U'(t) = AU(t) + F(t), & 0 < t \le T, \\ U(0) = u_0, \end{cases} \tag{3}$$

where A is the realization of the second order derivative with Dirichlet boundary condition in X (that is, we consider functions that vanish at $x = 0$ and at $x = 1$). For instance, if $X = C([0,1])$ then

$$D(A) = \{\varphi \in C^2([0,1]) : \varphi(0) = \varphi(1) = 0\}, \quad (A\varphi)(x) = \varphi''(x).$$

Problem (3) is a Cauchy problem for a linear differential equation in the space $X = C([0,1])$. However, the theory of ordinary differential equations is not easily extendable to this type of problems, because the linear operator A is defined on a proper subspace of X, and it is not continuous.

What we use is an important spectral property of A: the resolvent set of A contains a sector $S = \{\lambda \in \mathbb{C} : \lambda \neq 0, \ |\arg \lambda| < \theta\}$, with $\theta > \pi/2$ (precisely, it consists of a sequence of negative eigenvalues), and moreover

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda|}, \ \ \lambda \in S. \tag{4}$$

This property will allow us to define the solution of the homogeneous problem (i.e., when $F \equiv 0$), that will be called $e^{tA}u_0$. We shall see that for each $t \geq 0$ the linear operator $u_0 \mapsto e^{tA}u_0$ is bounded. The family of operators $\{e^{tA} : \ t \geq 0\}$ is said to be an *analytic semigroup*: semigroup, because it satisfies

$$e^{(t+s)A} = e^{tA}e^{sA}, \ t, s \geq 0, \ \ e^{0A} = I,$$

analytic, because the function $(0, +\infty) \mapsto \mathcal{L}(X)$, $t \mapsto e^{tA}$ is analytic.

Then we shall see that the solution of (3) is given by the variation of constants formula

$$U(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}F(s)ds, \ \ 0 \leq t \leq T,$$

that will let us study several properties of the solution to (3) and of u, recalling that $U(t) = u(t, \cdot)$.

We shall be able to study the asymptotic behavior of U as $t \to +\infty$, in the case that F is defined in $[0, +\infty)$. As in the case of ordinary differential equations, the asymptotic behavior depends heavily on the spectral properties of A.

Also the nonlinear problem (2) will be written as an abstract Cauchy problem,

$$\begin{cases} U'(t) = AU(t) + F(U(t)), \ \ t \geq 0, \\ \\ U(0) = u_0, \end{cases} \tag{5}$$

where $F : X \to X$ is the composition operator, or Nemitzky operator, $F(v) = f(v(\cdot))$. After stating local existence and uniqueness results, we shall see some criteria for existence in the large. As in the case of ordinary differential equations, in general the solution is defined only in a small time interval $[0, \delta]$. The problem of existence in the large is of particular interest in equations coming from mathematical models in physics, biology, chemistry, etc., where existence in the large is expected. Some sufficient conditions for existence in the large will be given.

Then we shall study the stability of the (possible) stationary solutions, that is all the $\overline{u} \in D(A)$ such that $A\overline{u} + F(\overline{u}) = 0$. We shall see that under suitable assumptions the Principle of Linearized Stability holds. Roughly speaking, $\overline{u}$ has the same stability properties of the null solution of the linearized problem

$$V'(t) = AV(t) + F'(\overline{u})V(t).$$

A similar study will be made in the case that F is not defined in the whole space X, but only in an intermediate space between X and $D(A)$. For instance, in several mathematical models the nonlinearity $f(u(t, x))$ in problem 2 is replaced by $f(u(t, x), u_x(t, x))$. Choosing again $X = C([0, 1])$, the composition operator $v \mapsto F(v) = f(v(\cdot), v'(\cdot))$ is well defined in $C^1([0, 1])$.

Chapter 1

Sectorial operators and analytic semigroups

1.1 Introduction

The main topic of our first lectures is the Cauchy problem in a general Banach space X,

$$\begin{cases} u'(t) = Au(t), \ t > 0, \\ u(0) = x, \end{cases} \tag{1.1}$$

where $A : D(A) \to X$ is a linear operator and $x \in X$. Of course, the construction and the properties of the solution depends upon the class of operators that is considered. The most elementary case, which we assume to be known to the reader, is that of a finite dimensional X and a matrix A. The case of a bounded operator A in general Banach space X can be treated essentially in the same way, and we are going to discuss it briefly in Section 1.2. We shall present two formulae for the solution, a power series expansion and an integral formula with a complex contour integral. While the first one cannot be generalized to the case of an unbounded A, the contour integral admits a generalization to the *sectorial* operators. This class of operators is discussed in Section 1.3. If A is sectorial, then the solution map $x \mapsto u(t)$ of (1.1) is given by an *analytic semigroup*. Sectorial operators and analytic semigroups are basic tools in the theory of abstract parabolic problems, and of partial differential equations and systems of parabolic type.

1.2 Bounded operators

Let $A \in \mathcal{L}(X)$. First, we give the solution of (1.1) as the sum of a power series of exponential type.

Proposition 1.2.1 *Let $A \in \mathcal{L}(X)$. Then, the series*

$$\sum_{k=0}^{+\infty} \frac{t^k A^k}{k!}, \qquad t \in \mathrm{Re}, \tag{1.2}$$

converges in $\mathcal{L}(X)$ uniformly on bounded subsets of Re. *Setting $u(t) := \sum_{k=0}^{+\infty} t^k A^k x/k!$, the restriction of u to $[0, +\infty)$ is the unique solution of the Cauchy problem* (1.1).

Proof. *Existence.* Using Theorem A.3 as in the finite dimensional case, it is easily checked that solving (1.1) is equivalent to finding a continuous function $v : [0, +\infty) \to X$ which satisfies

$$v(t) = x + \int_0^t Av(s)ds, \quad t \geq 0. \tag{1.3}$$

In order to show that u solves (1.3), let us fix an interval $[0, T]$ and define

$$u_0(t) = x, \; u_{n+1}(t) = x + \int_0^t Au_n(s)ds, \; n \in \mathbb{N}. \tag{1.4}$$

We have

$$u_n(t) = \sum_{k=0}^{n} \frac{t^k A^k}{k!} x, \quad n \in \mathbb{N}.$$

Since

$$\left\| \frac{t^k A^k}{k!} \right\| \leq \frac{T^k \|A\|^k}{k!}, \; t \in [0, T],$$

the series $\sum_{k=0}^{+\infty} t^k A^k / k!$ converges in $\mathcal{L}(X)$, uniformly with respect to t in $[0, T]$. Moreover, the sequence $\{u_n(t)\}_{n \in \mathbb{N}}$ converges to $u(t)$ uniformly for t in $[0, T]$. Letting $n \to \infty$ in (1.4), we conclude that u is a solution of (1.3).
Uniqueness. If u, v are two solutions of (1.3) in $[0, T]$, we have by Proposition A.2(c)

$$\|u(t) - v(t)\| \leq \|A\| \int_0^t \|u(s) - v(s)\| ds$$

and from Gronwall's lemma (see Exercise 3 in §1.2.4 below), the equality $u = v$ follows at once. $\square$

As in the finite dimensional setting, we define

$$e^{tA} = \sum_{k=0}^{+\infty} \frac{t^k A^k}{k!}, \quad t \in \mathbb{R}, \tag{1.5}$$

In the proof of Proposition 1.2.1 we have seen that for every *bounded* operator A the above series converges in $\mathcal{L}(X)$ for each $t \in \mathbb{R}$. If A is unbounded, the domain of A^k may become smaller and smaller as k increases, and even for $x \in \bigcap_{k \in \mathbb{N}} D(A^k)$ it is not obvious that the series $\sum_{k=0}^{+\infty} t^k A^k x / k!$ converges. For instance, take $X = C([0, 1])$, $D(A) = C^1([0, 1])$, $Af = f'$.

Therefore, we have to look for another representation of the solution to (1.1) if we want to extend it to the unbounded case. As a matter of fact, it is given in the following proposition.

Proposition 1.2.2 *Let $A \in \mathcal{L}(X)$ and let $\gamma \subset \mathbb{C}$ be any circle with centre 0 and radius $r > \|A\|$. Then*

$$e^{tA} = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} R(\lambda, A)\, d\lambda, \qquad t \in \mathbb{R}. \tag{1.6}$$

Proof. From (1.5) and the power series expansion

$$R(\lambda, A) = \sum_{k=0}^{+\infty} \frac{A^k}{\lambda^{k+1}}, \qquad |\lambda| > \|A\|,$$

(see (B.10)), we have

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma e^{t\lambda}R(\lambda,A)\,d\lambda &= \frac{1}{2\pi i}\sum_{n=0}^{+\infty}\frac{t^n}{n!}\int_\gamma \lambda^n R(\lambda,A)\,d\lambda\\ &= \frac{1}{2\pi i}\sum_{n=0}^{+\infty}\frac{t^n}{n!}\int_\gamma \lambda^n\sum_{k=0}^{+\infty}\frac{A^k}{\lambda^{k+1}}\,d\lambda\\ &= \frac{1}{2\pi i}\sum_{n=0}^{+\infty}\frac{t^n}{n!}\sum_{k=0}^{+\infty}A^k\int_\gamma \lambda^{n-k-1}\,d\lambda = e^{tA},\end{aligned}$$

as the integrals in the last series equal $2\pi i$ if $n = k$, 0 otherwise. Note that the exchange of integration and summation is justified by the uniform convergence. □

Let us see how it is possible to generalize to the infinite dimensional setting the variation of constants formula, that gives the solution of the non-homogeneous Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t), & 0 \le t \le T,\\ u(0) = x, \end{cases} \tag{1.7}$$

where $A \in \mathcal{L}(X)$, $x \in X$, $f \in C([0,T];X)$ and $T > 0$.

Proposition 1.2.3 *The Cauchy problem* (1.7) *has a unique solution in* $[0,T]$*, given by*

$$u(t) = e^{tA}x + \int_0^t e^{(t-s)A}f(s)ds, \qquad t \in [0,T]. \tag{1.8}$$

Proof. It can be directly checked that u is a solution. Concerning uniqueness, let u_1, u_2 be two solutions; then, $v = u_1 - u_2$ satisfies $v'(t) = Av(t)$ for $0 \le t \le T$, $v(0) = 0$. By Proposition 1.2.1, we conclude that $v \equiv 0$. □

Exercises 1.2.4

1. Prove that $e^{tA}e^{sA} = e^{(t+s)A}$ for any $t, s \in \mathbb{R}$ and any $A \in \mathcal{L}(X)$.
2. Prove that if the operators $A, B \in \mathcal{L}(X)$ commute (i.e. $AB = BA$), then $e^{tA}e^{tB} = e^{t(A+B)}$ for any $t \in \mathbb{R}$.
3. Prove the following form of Gronwall's lemma:

   Let $u, v : [0,+\infty) \to [0,+\infty)$ be continuous functions, and assume that

   $$u(t) \le \alpha + \int_0^t u(s)v(s)ds$$

   for some $\alpha \ge 0$. Then, $u(t) \le \alpha\exp\{\int_0^t v(s)ds\}$, for any $t \ge 0$.
4. Check that the function u defined in (1.8) is a solution of problem (1.7).

1.3 Sectorial operators

Definition 1.3.1 *We say that a linear operator $A : D(A) \subset X \to X$ is* sectorial *if there are constants $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, $M > 0$ such that*

$$\begin{cases} (i) \quad \rho(A) \supset S_{\theta,\omega} := \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}, \\ \\ (ii) \quad \|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \dfrac{M}{|\lambda - \omega|}, \quad \lambda \in S_{\theta,\omega}. \end{cases} \tag{1.9}$$

Note that every sectorial operator is closed, because its resolvent set is not empty.

For every $t > 0$, the conditions (1.9) allow us to define a bounded linear operator e^{tA} on X, through an integral formula that generalizes (1.6). For $r > 0$, $\eta \in (\pi/2, \theta)$, let $\gamma_{r,\eta}$ be the curve

$$\{\lambda \in \mathbb{C} : |\arg \lambda| = \eta, \ |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg \lambda| \leq \eta, \ |\lambda| = r\},$$

oriented counterclockwise, as in Figure 1.

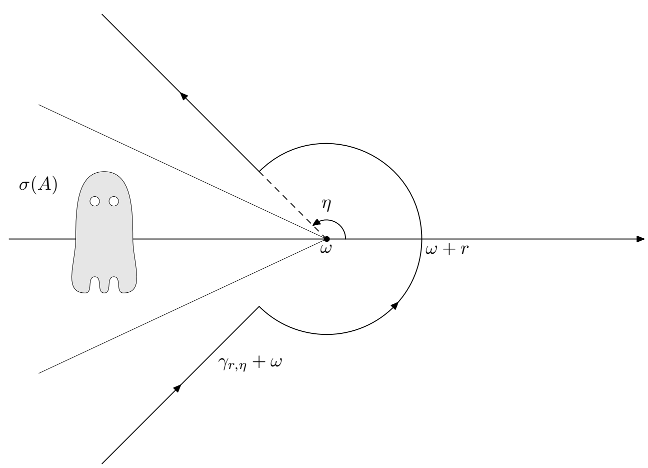

Figure 1.1: the curve $\gamma_{r,\eta}$.

For each $t > 0$ set

$$e^{tA} = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}+\omega} e^{t\lambda} R(\lambda, A)\, d\lambda, \quad t > 0. \tag{1.10}$$

Using the obvious parametrization of $\gamma_{r,\eta}$ we get

$$\begin{aligned} e^{tA} \quad = \quad & \frac{e^{\omega t}}{2\pi i}\bigg(- \int_r^{+\infty} e^{(\rho\cos\eta - i\rho\sin\eta)t} R(\omega + \rho e^{-i\eta}, A) e^{-i\eta} d\rho \\ & + \int_{-\eta}^{\eta} e^{(r\cos\alpha + ir\sin\alpha)t} R(\omega + re^{i\alpha}, A) ire^{i\alpha} d\alpha \\ & + \int_r^{+\infty} e^{(\rho\cos\eta + i\rho\sin\eta)t} R(\omega + \rho e^{i\eta}, A) e^{i\eta} d\rho \bigg), \end{aligned} \tag{1.11}$$

for every $t > 0$ and for every $r > 0$, $\eta \in (\pi/2, \theta)$.

Lemma 1.3.2 *If A is a sectorial operator, the integral in* (1.10) *is well defined, and it is independent of $r > 0$ and $\eta \in (\pi/2, \theta)$.*

Proof. First of all, notice that for each $t > 0$ the mapping $\lambda \mapsto e^{t\lambda}R(\lambda, A)$ is a $\mathcal{L}(X)$-valued holomorphic function in the sector $S_{\theta,\omega}$ (see Proposition B.4). Moreover, for any $\lambda = \omega + re^{i\theta}$, the estimate

$$\|e^{t\lambda}R(\lambda, A)\|_{\mathcal{L}(X)} \leq \exp(\omega t)\exp(tr\cos\eta)\frac{M}{r} \tag{1.12}$$

holds for each λ in the two half-lines, and this easily implies that the improper integral is convergent. Now take any $r' > 0, \eta' \in (\pi/2, \theta)$ and consider the integral on $\gamma_{r',\eta'} + \omega$. Let D be the region lying between the curves $\gamma_{r,\eta} + \omega$ and $\gamma_{r',\eta'} + \omega$ and for every $n \in \mathbb{N}$ set $D_n = D \cap \{|z - \omega| \leq n\}$, as in Figure 1.2. By Cauchy integral theorem A.9 we have

$$\int_{\partial D_n} e^{t\lambda}R(\lambda, A)\, d\lambda = 0.$$

By estimate (1.12), the integrals on the two arcs contained in $\{|z - \omega| = n\}$ tend to 0 as n tends to $+\infty$, so that

$$\int_{\gamma_{r,\eta}+\omega} e^{t\lambda}R(\lambda, A)\, d\lambda = \int_{\gamma_{r',\eta'}+\omega} e^{t\lambda}R(\lambda, A)\, d\lambda$$

and the proof is complete. □

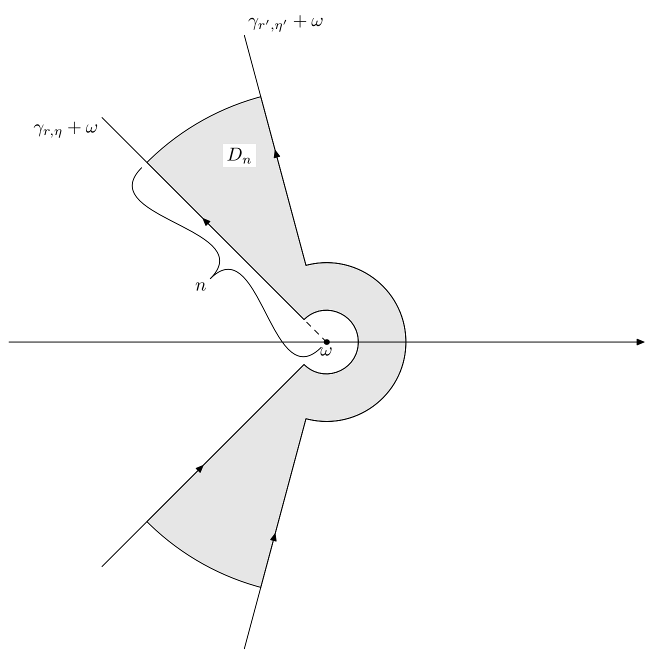

Figure 1.2: the region D_n.

Let us also set

$$e^{0A}x = x, \quad x \in X. \tag{1.13}$$

In the following theorem we summarize the main properties of e^{tA} for $t > 0$.

Theorem 1.3.3 *Let A be a sectorial operator and let e^{tA} be given by* (1.10)*. Then, the following statements hold.*

(i) $e^{tA}x \in D(A^k)$ *for all* $t > 0$, $x \in X$, $k \in \mathbb{N}$*. If* $x \in D(A^k)$*, then*

$$A^k e^{tA}x = e^{tA}A^k x, \;\; t \geq 0.$$

(ii) $e^{tA}e^{sA} = e^{(t+s)A}$ *for any* $t, \; s \geq 0$.

(iii) There are constants M_0, M_1, M_2, $\ldots$, *such that*

$$\begin{cases} (a) \quad \|e^{tA}\|_{\mathcal{L}(X)} \leq M_0 e^{\omega t}, \;\; t > 0, \\ \\ (b) \quad \|t^k (A - \omega I)^k e^{tA}\|_{\mathcal{L}(X)} \leq M_k e^{\omega t}, \;\; t > 0, \end{cases} \tag{1.14}$$

where ω is the number in (1.9)*. In particular, from* (1.14)*(b) it follows that for every* $\varepsilon > 0$ *and* $k \in \mathbb{N}$ *there is* $C_{k,\varepsilon} > 0$ *such that*

$$\|t^k A^k e^{tA}\|_{\mathcal{L}(X)} \leq C_{k,\varepsilon} e^{(\omega+\varepsilon)t}, \;\; t > 0. \tag{1.15}$$

(iv) The function $t \mapsto e^{tA}$ *belongs to* $C^\infty((0, +\infty); \mathcal{L}(X))$*, and the equality*

$$\frac{d^k}{dt^k} e^{tA} = A^k e^{tA}, \;\; t > 0, \tag{1.16}$$

holds for every $k \in \mathbb{N}$*. Moreover, it has an analytic continuation* e^{zA} *to the sector* $S_{\theta-\pi/2,0}$*, and, for* $z = \rho e^{i\alpha} \in S_{\theta-\pi/2,0}$, $\theta' \in (\pi/2, \theta - \alpha)$*, the equality*

$$e^{zA} = \frac{1}{2\pi i} \int_{\gamma_{r,\theta'}+\omega} e^{\lambda z} R(\lambda, A) d\lambda$$

holds.

Proof. Replacing A by $A - \omega I$ if necessary, we may suppose $\omega = 0$. See Exercise 1, §1.3.5.

Proof of (i). First, let $k = 1$. Recalling that A is a closed operator and using Lemma A.4 with $f(t) = e^{\lambda t} R(\lambda, A)$, we deduce that $e^{tA}x$ belongs to $D(A)$ for every $x \in X$, and that

$$Ae^{tA}x = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{t\lambda} AR(\lambda, A)x \, d\lambda = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} \lambda e^{t\lambda} R(\lambda, A)x \, d\lambda, \tag{1.17}$$

because $AR(\lambda, A) = \lambda R(\lambda, A) - I$, for every $\lambda \in \rho(A)$, and $\int_{\gamma_{r,\eta}} e^{t\lambda} d\lambda = 0$. Moreover, if $x \in D(A)$, the equality $Ae^{tA}x = e^{tA}Ax$ follows since $AR(\lambda, A)x = R(\lambda, A)Ax$. Iterating this argument, we obtain that $e^{tA}x$ belongs to $D(A^k)$ for every $k \in \mathbb{N}$; moreover

$$A^k e^{tA} = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} \lambda^k e^{t\lambda} R(\lambda, A) d\lambda,$$

and (i) can be easily proved by recurrence.

Proof of (ii). Since

$$e^{tA}e^{sA} = \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{r,\eta}} e^{\lambda t} R(\lambda, A) d\lambda \int_{\gamma_{2r,\eta'}} e^{\mu s} R(\mu, A) d\mu,$$

with $\eta' \in (\frac{\pi}{2}, \eta)$, using the resolvent identity it follows that

$$\begin{aligned}
e^{tA}e^{sA} &= \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{r,\eta}} \int_{\gamma_{2r,\eta'}} e^{\lambda t+\mu s} \frac{R(\lambda,A)-R(\mu,A)}{\mu-\lambda} d\lambda d\mu \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{r,\eta}} e^{\lambda t} R(\lambda,A) d\lambda \int_{\gamma_{2r,\eta'}} e^{\mu s} \frac{d\mu}{\mu-\lambda} \\
&\quad - \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{2r,\eta'}} e^{\mu s} R(\mu,A) d\mu \int_{\gamma_{r,\eta}} e^{\lambda t} \frac{d\lambda}{\mu-\lambda} = e^{(t+s)A},
\end{aligned}$$

where we have used the equalities

$$\int_{\gamma_{2r,\eta'}} e^{\mu s} \frac{d\mu}{\mu-\lambda} = 2\pi\, i e^{s\lambda}, \quad \lambda \in \gamma_{r,\eta}, \qquad \int_{\gamma_{r,\eta}} e^{\lambda t} \frac{d\lambda}{\mu-\lambda} = 0, \quad \mu \in \gamma_{2r,\eta'} \tag{1.18}$$

that can be easily checked (Exercise 2, §1.3.5).

Proof of (iii). Let us point out that if we estimate $\|e^{tA}\|$ integrating $\|e^{\lambda t}R(\lambda,A)\|$ over $\gamma_{r,\eta}$ we get a singularity near $t=0$, because the norm of the integrand behaves like $M/|\lambda|$ for $|\lambda|$ small. We have to be more careful. Setting $\lambda t = \xi$ in (1.10) and using Lemma 1.3.2, we get

$$\begin{aligned}
e^{tA} &= \frac{1}{2\pi i} \int_{\gamma_{rt,\eta}} e^{\xi} R\left(\frac{\xi}{t}, A\right) \frac{d\xi}{t} = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{\xi} R\left(\frac{\xi}{t}, A\right) \frac{d\xi}{t} \\
&= \frac{1}{2\pi i} \left(\int_r^{+\infty} e^{\rho e^{i\eta}} R\left(\frac{\rho e^{i\eta}}{t}, A\right) \frac{e^{i\eta}}{t} d\rho - \int_r^{+\infty} e^{\rho e^{-i\eta}} R\left(\frac{\rho e^{-i\eta}}{t}, A\right) \frac{e^{-i\eta}}{t} d\rho \right. \\
&\qquad \left. + \int_{-\eta}^{\eta} e^{re^{i\alpha}} R\left(\frac{re^{i\alpha}}{t}, A\right) ire^{i\alpha} \frac{d\alpha}{t} \right).
\end{aligned}$$

It follows that

$$\|e^{tA}\| \le \frac{1}{\pi} \left\{ \int_r^{+\infty} M e^{\rho\cos\eta} \frac{d\rho}{\rho} + \frac{1}{2} \int_{-\eta}^{\eta} M e^{r\cos\alpha} d\alpha \right\}.$$

The estimate of $\|Ae^{tA}\|$ is easier, and we do not need the above procedure. Recalling that $\|AR(\lambda,A)\| \le M+1$ for each $\lambda \in \gamma_{r,\eta}$ and using (1.11) we get

$$\|Ae^{tA}\| \le \frac{M+1}{\pi} \int_r^{+\infty} e^{\rho t\cos\eta} d\rho + \frac{(M+1)r}{2\pi} \int_{-\eta}^{\eta} e^{rt\cos\alpha} d\alpha,$$

so that, letting $r \to 0$,

$$\|Ae^{tA}\| \le \frac{M+1}{\pi|\cos\eta| t} := \frac{N}{t}, \quad t > 0.$$

From the equality $Ae^{tA}x = e^{tA}Ax$, which is true for each $x \in D(A)$, it follows that $A^k e^{tA} = (Ae^{\frac{t}{k}A})^k$ for all $k \in \mathbb{N}$, so that

$$\|A^k e^{tA}\|_{\mathcal{L}(X)} \le (Nkt^{-1})^k := M_k t^{-k}.$$

Proof of (iv). This follows easily from Exercise A.6 and from (1.17). Indeed,

$$\frac{d}{dt} e^{tA} = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} \lambda e^{\lambda t} R(\lambda,A) d\lambda = Ae^{tA}, \qquad t > 0.$$

The equality

$$\frac{d^k}{dt^k}e^{tA} = A^k e^{tA}, \qquad t > 0$$

can be proved by the same argument, or by recurrence. Now, let $0 < \alpha < \theta - \pi/2$ be given, and set $\eta = \theta - \alpha$. The function

$$z \mapsto e^{zA} = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} e^{z\lambda}R(\lambda, A)d\lambda$$

is well defined and holomorphic in the sector

$$S_\alpha = \{z \in \mathbb{C} : z \neq 0,\ |\arg z| < \theta - \pi/2 - \alpha\},$$

because we can differentiate with respect to z under the integral, again by Exercise A.6. Indeed, if $\lambda = \xi e^{i\eta}$ and $z = \rho e^{i\phi}$, then $\mathrm{Re}(z\lambda) = \xi\rho\cos(\eta + \phi) \leq -c\xi\rho$ for a suitable $c > 0$.

Since the union of the sectors S_α, for $0 < \alpha < \theta - \pi/2$, is $S_{\theta-\frac{\pi}{2},0}$, (iv) is proved. $\square$

Statement (ii) in Theorem 1.3.3 tells us that the family of operators e^{tA} satisfies the *semigroup law*, an algebraic property which is coherent with the exponential notation. Statement (iv) tells us that $e^{\cdot A}$ is analytically extendable to a sector. Therefore, it is natural to give the following deefinition.

Definition 1.3.4 *Let A be a sectorial operator. The function from* $[0, +\infty)$ *to* $\mathcal{L}(X)$, $t \mapsto e^{tA}$ *(see* (1.10), (1.13)*) is called the* analytic semigroup generated by A *(in X).*

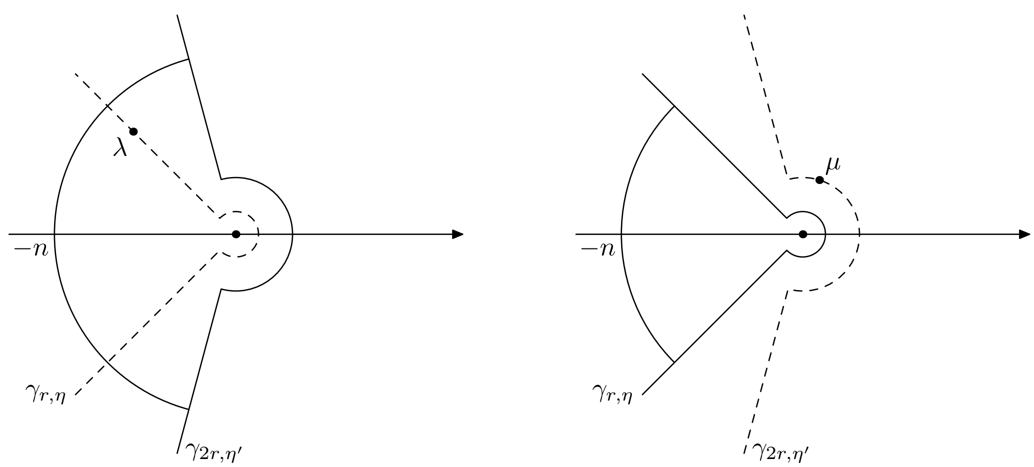

Figure 1.3: the curves for Exercise 2.

Exercises 1.3.5

1. Let $A : D(A) \subset X \to X$ be sectorial, let $\alpha \in \mathbb{C}$, and set $B : D(B) := D(A) \to X$, $Bx = Ax - \alpha x$, $C : D(C) = D(A) \to X$, $Cx = \alpha Ax$. Prove that the operator B is sectorial, and that $e^{tB} = e^{-\alpha t}e^{tA}$. Use this result to complete the proof of Theorem 1.3.3 in the case $\omega \neq 0$. For which α is the operator C sectorial?

2. Prove that (1.18) holds, integrating over the curves shown in Figure 1.3.

3. Let $A : D(A) \subset X \to X$ be sectorial and let $x \in D(A)$ be an eigenvector of A with eigenvalue λ.
   (a) Prove that $R(\mu, A)x = (\mu - \lambda)^{-1}x$ for any $\mu \in \rho(A)$.
   (b) Prove that $e^{tA}x = e^{\lambda t}x$ for any $t > 0$.

4. Prove that if both A and $-A$ are sectorial operators in X, then A is bounded.

Given $x \in X$, the function $t \mapsto e^{tA}x$ is analytic for $t > 0$. Let us consider its behavior for t close to 0.

Proposition 1.3.6 *The following statements hold.*

(i) If $x \in \overline{D(A)}$, then $\lim_{t\to 0^+} e^{tA}x = x$. Conversely, if $y = \lim_{t\to 0^+} e^{tA}x$ exists, then $x \in \overline{D(A)}$ and $y = x$.

(ii) For every $x \in X$ and $t \geq 0$, the integral $\int_0^t e^{sA}x\, ds$ belongs to $D(A)$, and

$$A\int_0^t e^{sA}x\, ds = e^{tA}x - x. \tag{1.19}$$

If, in addition, the function $s \mapsto Ae^{sA}x$ is integrable in $(0, \varepsilon)$ for some $\varepsilon > 0$, then

$$e^{tA}x - x = \int_0^t Ae^{sA}x\, ds, \quad t \geq 0.$$

(iii) If $x \in D(A)$ and $Ax \in \overline{D(A)}$, then $\lim_{t\to 0^+}(e^{tA}x - x)/t = Ax$. Conversely, if $z := \lim_{t\to 0^+}(e^{tA}x - x)/t$ exists, then $x \in D(A)$ and $Ax = z \in \overline{D(A)}$.

(iv) If $x \in D(A)$ and $Ax \in \overline{D(A)}$, then $\lim_{t\to 0^+} Ae^{tA}x = Ax$.

Proof. *Proof of (i).* Notice that we cannot let $t \to 0^+$ in the Definition (1.10) of $e^{tA}x$, because the estimate $\|R(\lambda, A)\| \leq M/|\lambda - \omega|$ does not suffice to use any convergence theorem.

But if $x \in D(A)$ things are easier: indeed fix ξ, r such that $\omega < \xi \in \rho(A)$, $0 < r < \xi - \omega$, and set $y = \xi x - Ax$, so that $x = R(\xi, A)y$. We have

$$\begin{aligned}
e^{tA}x &= e^{tA}R(\xi, A)y = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}+\omega} e^{t\lambda}R(\lambda, A)R(\xi, A)y\, d\lambda \\
&= \frac{1}{2\pi i}\int_{\gamma_{r,\eta}+\omega} e^{t\lambda}\frac{R(\lambda, A)}{\xi - \lambda}y\, d\lambda - \frac{1}{2\pi i}\int_{\gamma_{r,\eta}+\omega} e^{t\lambda}\frac{R(\xi, A)}{\xi - \lambda}y\, d\lambda \\
&= \frac{1}{2\pi i}\int_{\gamma_{r,\eta}+\omega} e^{t\lambda}\frac{R(\lambda, A)}{\xi - \lambda}y\, d\lambda,
\end{aligned}$$

because the integral $\int_{\gamma_{r,\eta}+\omega} e^{t\lambda}R(\xi, A)y/(\xi - \lambda)\, d\lambda$ vanishes (why?). Here we may let $t \to 0^+$ because $\|R(\lambda, A)y/(\xi - \lambda)\| \leq C|\lambda|^{-2}$ for $\lambda \in \gamma_{r,\eta} + \omega$. We get

$$\lim_{t\to 0^+} e^{tA}x = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}+\omega} \frac{R(\lambda, A)}{\xi - \lambda}y\, d\lambda = R(\xi, A)y = x.$$

The second equality follows using Cauchy's Theorem with the curve $\{\lambda \in \gamma_{r,\eta} + \omega : |\lambda-\omega| \le n\} \cup \{|\lambda-\omega| = n,\ \arg(\lambda-\omega) \in [-\eta,\eta]\}$ and then letting $n \to +\infty$. Since $D(A)$ is dense in $\overline{D(A)}$ and $\|e^{tA}\|$ is bounded by a constant independent of t for $0 < t < 1$, then $\lim_{t\to 0^+} e^{tA}x = x$ for all $x \in \overline{D(A)}$, see Lemma A.1.

Conversely, if $y = \lim_{t\to 0^+} e^{tA}x$, then $y \in \overline{D(A)}$ because $e^{tA}x \in D(A)$ for $t > 0$, and we have $R(\xi,A)y = \lim_{t\to 0^+} R(\xi,A)e^{tA}x = \lim_{t\to 0^+} e^{tA}R(\xi,A)x = R(\xi,A)x$ as $R(\xi,A)x \in D(A)$. Therefore, $y = x$.

Proof of (ii). To prove the first statement, take $\xi \in \rho(A)$ and $x \in X$. For every $\varepsilon \in (0,t)$ we have

$$\begin{aligned}\int_\varepsilon^t e^{sA}x\,ds &= \int_\varepsilon^t (\xi - A)R(\xi,A)e^{sA}x\,ds\\ &= \xi\int_\varepsilon^t R(\xi,A)e^{sA}x\,ds - \int_\varepsilon^t \frac{d}{ds}(R(\xi,A)e^{sA}x)ds\\ &= \xi R(\xi,A)\int_\varepsilon^t e^{sA}x\,ds - e^{tA}R(\xi,A)x + e^{\varepsilon A}R(\xi,A)x.\end{aligned}$$

Since $R(\xi,A)x$ belongs to $D(A)$, letting $\varepsilon \to 0^+$ we get

$$\int_0^t e^{sA}x\,ds = \xi R(\xi,A)\int_0^t e^{sA}x\,ds - R(\xi,A)(e^{tA}x - x). \tag{1.20}$$

Therefore, $\int_0^t e^{sA}x ds \in D(A)$, and

$$(\xi I - A)\int_0^t e^{sA}x\,ds = \xi\int_0^t e^{sA}x\,ds - (e^{tA}x - x),$$

whence the first statement in (ii) follows. If in addition $s \mapsto \|Ae^{sA}x\|$ belongs to $L^1(0,T)$, we may commute A with the integral by Lemma A.4 and the second statement in (ii) is proved.

Proof of (iii). If $x \in D(A)$ and $Ax \in \overline{D(A)}$, we have

$$\frac{e^{tA}x - x}{t} = \frac{1}{t}A\int_0^t e^{sA}x\,ds = \frac{1}{t}\int_0^t e^{sA}Ax\,ds.$$

Since the function $s \mapsto e^{sA}Ax$ is continuous on $[0,t]$ by (i), then $\lim_{t\to 0^+}(e^{tA}x - x)/t = Ax$ by Theorem A.3.

Conversely, if the limit $z := \lim_{t\to 0^+}(e^{tA}x - x)/t$ exists, then $\lim_{t\to 0^+} e^{tA}x = x$, so that both x and z belong to $\overline{D(A)}$. Moreover, for every $\xi \in \rho(A)$ we have

$$R(\xi,A)z = \lim_{t\to 0^+} R(\xi,A)\frac{e^{tA}x - x}{t},$$

and from (ii) it follows

$$R(\xi,A)z = \lim_{t\to 0^+}\frac{1}{t}R(\xi,A)A\int_0^t e^{sA}x\,ds = \lim_{t\to 0^+}(\xi R(\xi,A) - I)\frac{1}{t}\int_0^t e^{sA}x\,ds.$$

Since $x \in \overline{D(A)}$, the function $s \mapsto e^{sA}x$ is continuous at $s = 0$, and then

$$R(\xi,A)z = \xi R(\xi,A)x - x.$$

In particular, $x \in D(A)$ and $z = \xi x - (\xi - A)x = Ax$.

Proof of (iv). Statement (iv) is an easy consequence of (i), since $Ae^{tA}x = e^{tA}Ax$ for $x \in D(A)$. □

Formula (1.19) is very important. It is the starting point of several proofs and it will be used throughout these lectures. Therefore, remind it!

It has several variants and consequences. For instance, if $\omega < 0$ we may let $t \to +\infty$ and, using (1.14)(a), we get $\int_0^{+\infty} e^{sA}x ds \in D(A)$ and

$$x = -A \int_0^{+\infty} e^{sA}x\, ds, \quad x \in X.$$

In general, if $\mathrm{Re}\,\lambda > \omega$, replacing A by $A - \lambda I$ and using (1.19) and Exercise 1, §1.3.5, we get

$$e^{-\lambda t}e^{tA}x - x = (A - \lambda I)\int_0^t e^{-\lambda s}e^{sA}x\, ds, \quad x \in X,$$

so that

$$x = (\lambda I - A)\int_0^{+\infty} e^{-\lambda s}e^{sA}x\, ds, \quad x \in X. \tag{1.21}$$

An important representation formula for the resolvent $R(\lambda, A)$ of A follows.

Proposition 1.3.7 *Let $A : D(A) \subset X \to X$ be a sectorial operator. For every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > \omega$ we have*

$$R(\lambda, A) = \int_0^{+\infty} e^{-\lambda t}e^{tA}dt. \tag{1.22}$$

Proof. The right hand side is well defined as an element of $\mathcal{L}(X)$ by estimate (1.14)(a). The equality follows applying $R(\lambda, A)$ to both sides of (1.21). □

Corollary 1.3.8 *For all $t \geq 0$ the operator e^{tA} is one to one.*

Proof. $e^{0A} = I$ is obviously one to one. If there are $t_0 > 0$, $x \in X$ such that $e^{t_0A}x = 0$, then for $t \geq t_0$, $e^{tA}x = e^{(t-t_0)A}e^{t_0A}x = 0$. Since the function $t \mapsto e^{tA}x$ is analytic, $e^{tA}x \equiv 0$ in $(0, +\infty)$. From Proposition 1.3.7 we get $R(\lambda, A)x = 0$ for $\lambda > \omega$, so that $x = 0$. □

Remark 1.3.9 Formula (1.22) is used to define the Laplace transform of the scalar function $t \mapsto e^{tA}$, if $A \in \mathbb{C}$. The classical inversion formula to recover e^{tA} from its Laplace transform is given by a complex integral on a suitable vertical line; in our case the vertical line has been replaced by a curve joining $\infty e^{-i\eta}$ to $\infty e^{i\eta}$ with $\eta > \pi/2$, in such a way that the improper integral converges by assumption (1.9).

Of course, the continuity properties of semigroups of linear operators are very important in their analysis. The following definition is classical.

Definition 1.3.10 *Let $(T(t))_{t\geq 0}$ be a family of bounded operators on X. If $T(0) = I$, $T(t + s) = T(t)T(s)$ for all $t, s \geq 0$ and the map $t \mapsto T(t)x$ is continuous from $[0, +\infty)$ to X then we say that $(T(t))_{t\geq 0}$ is a* strongly continuous semigroup.

By Proposition 1.3.6(i) we immediately see that the semigroup $\{e^{tA}\}_{t\geq 0}$ is strongly continuous in X if and only if $D(A)$ is dense in X.

In any case some weak continuity property of the function $t \mapsto e^{tA}x$ holds for a general $x \in X$; for instance we have

$$\lim_{t\to 0^+} R(\lambda, A)e^{tA}x = R(\lambda, A)x \tag{1.23}$$

for every $\lambda \in \rho(A)$. Indeed, $R(\lambda, A)e^{tA}x = e^{tA}R(\lambda, A)x$ for every $t > 0$, and $R(\lambda, A)x \in D(A)$. In the case when $D(A)$ is not dense in X, a standard way to obtain a strongly continuous semigroup from a sectorial operator A is to consider the part of A in $\overline{D(A)}$.

Definition 1.3.11 *Let $L : D(L) \subset X \to X$ be a linear operator, and let Y be a subspace of X. The* part of L in Y *is the operator L_0 defined by*

$$D(L_0) = \{x \in D(L) \cap Y : \ Lx \in Y\}, \ \ L_0x = Lx.$$

It is easy to see that the part A_0 of A in $\overline{D(A)}$ is still sectorial. Since $D(A_0)$ is dense in $\overline{D(A)}$ (because for each $x \in \overline{D(A)}$ we have $x = \lim_{t\to 0} e^{tA}x$), then the semigroup generated by A_0 is strongly continuous in $\overline{D(A)}$. By (1.10), the semigroup generated by A_0 coincides of course with the restriction of e^{tA} to $\overline{D(A)}$.

Coming back to the Cauchy problem (1.1), let us notice that Theorem 1.3.3 implies that the function

$$u(t) = e^{tA}x, \ \ t \geq 0$$

is analytic with values in $D(A)$ for $t > 0$, and it is a solution of the differential equation in (1.1) for $t > 0$. Moreover, u is continuous also at $t = 0$ (with values in X) if and only if $x \in \overline{D(A)}$ and in this case u is a solution of the Cauchy problem (1.1). If $x \in D(A)$ and $Ax \in \overline{D(A)}$, then u is continuously differentiable up to $t = 0$, and it satisfies the differential equation also at $t = 0$, i.e., $u'(0) = Ax$. Uniqueness of the solution to (1.1) will be proved in Proposition 4.1.2, in a more general context.

Let us give a sufficient condition, seemingly weaker than (1.9), in order that a linear operator be sectorial. It will be useful to prove that the realizations of some elliptic partial differential operators are sectorial in the usual function spaces.

Proposition 1.3.12 *Let $A : D(A) \subset X \to X$ be a linear operator such that $\rho(A)$ contains a halfplane $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \geq \omega\}$, and*

$$\|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} \leq M, \ \ \mathrm{Re}\,\lambda \geq \omega, \tag{1.24}$$

with $\omega \geq 0, \ M \geq 1$. Then A is sectorial.

Proof. By Proposition B.3, for every $r > 0$ the open disks with centre $\omega \pm ir$ and radius $|\omega + ir|/M$ is contained in $\rho(A)$. Since $|\omega + ir| \geq r$, the union of such disks and of the halfplane $\{\mathrm{Re}\,\lambda \geq \omega\}$ contains the sector $\{\lambda \in \mathbb{C} : \lambda \neq \omega, \ |\arg(\lambda - \omega)| < \pi - \arctan(M)\}$ and, hence, it contains $S = \{\lambda \neq \omega : \ |\arg(\lambda - \omega)| < \pi - \arctan(2M)\}$. If $\lambda \in S$ and $\mathrm{Re}\,\lambda < \omega$, we write $\lambda = \omega \pm ir - (\theta r)/(2M)$ for some $\theta \in (0, 1)$. Since by (B.4)

$$R(\lambda, A) = R(\omega \pm ir, A)\left(I + (\lambda - \omega \mp ir)R(\omega \pm ir, A)\right)^{-1}$$

and $\|(I+(\lambda-\omega\mp ir)R(\omega\pm ir,A))^{-1}\|\leq 2$, we have

$$\|R(\lambda,A)\|\leq \frac{2M}{|\omega\pm ir|}\leq \frac{2M}{r}\leq \frac{\sqrt{4M^2+1}}{|\lambda-\omega|}.$$

If $\lambda\in S$ and $\mathrm{Re}\,\lambda\geq\omega$, estimate (1.24) yields $\|R(\lambda,A)\|\leq M/|\lambda-\omega|$, and the statement follows. □

Next, we prove a useful perturbation theorem.

Theorem 1.3.13 *Let $A: D(A)\to X$ be a sectorial operator, and let $B: D(B)\to X$ be a linear operator such that $D(A)\subset D(B)$ and*

$$\|Bx\|\leq a\|Ax\|+b\|x\|,\qquad x\in D(A). \tag{1.25}$$

There is $\delta>0$ such that if $a\in[0,\delta]$ then $A+B: D(A)\to X$ is sectorial.

Proof. Let $r>0$ be such that $R(\lambda,A)$ exists and $\|\lambda R(\lambda,A)\|\leq M$ for $\mathrm{Re}\,\lambda\geq r$. We write $\lambda-A-B=(I-BR(\lambda,A))(\lambda-A)$ and we observe that

$$\|BR(\lambda,A)x\|\leq a\|AR(\lambda,A)x\|+b\|R(\lambda,A)x\|\leq\left(a(M+1)+\frac{bM}{|\lambda|}\right)\|x\|\leq\frac{1}{2}\|x\|$$

if $a(M+1)\leq 1/4$ and $bM/|\lambda|\leq 1/4$. Therefore, if $a\leq\delta:=(4(M+1))^{-1}$ and for $\mathrm{Re}\,\lambda$ sufficiently large, $\|BR(\lambda,A)\|\leq 1/2$ and

$$\|(\lambda-A-B)^{-1}\|\leq\|R(\lambda,A)\|\,\|(I-BR(\lambda,A))^{-1}\|\leq\frac{2M}{|\lambda|}.$$

The statement now follows from Proposition 1.3.12. □

Corollary 1.3.14 *If A is sectorial and $B: D(B)\supset D(A)\to X$ is a linear operator such that for some $\theta\in(0,1)$, $C>0$ we have*

$$\|Bx\|\leq C\|x\|_{D(A)}^{\theta}\|x\|_X^{1-\theta},\quad x\in D(A),$$

then $A+B: D(A+B):=D(A)\to X$ is sectorial.

Remark 1.3.15 In fact the proof of Theorem 1.3.13 shows that if $A: D(A)\to X$ is a sectorial operator and $B: D(B)\to X$ is a linear operator such that $D(A)\subset D(B)$ and $\lim_{\mathrm{Re}\,\lambda\to+\infty,\ \lambda\in S_{\theta,\omega}}\|BR(\lambda,A)\|=0$, then $A+B: D(A)\to X$ is a sectorial operator.

The next theorem is sometimes useful, because it allows to work in smaller subspaces of $D(A)$. A subspace D as in the following statement is called a *core* for the operator A.

Theorem 1.3.16 *Let A be a sectorial operator with dense domain. If a subspace $D\subset D(A)$ is dense in X and $e^{tA}(D)\subset D$ for each $t>0$, then D is dense in $D(A)$ with respect to the graph norm.*

Proof. Fix $x \in D(A)$ and a sequence $(x_n) \subset D$ which converges to x in X. Since $D(A)$ is dense, then by Proposition 1.3.6(iii)

$$Ax = \lim_{t\to 0^+} \frac{e^{tA}x - x}{t} = \lim_{t\to 0^+} \frac{A}{t}\int_0^t e^{sA}x\,ds,$$

and the same formula holds with x_n in place of x. Therefore it is convenient to set

$$y_{n,t} = \frac{1}{t}\int_0^t e^{sA}x_n\,ds = \frac{1}{t}\int_0^t e^{sA}(x_n - x)\,ds + \frac{1}{t}\int_0^t e^{sA}x\,ds.$$

For each n, the map $s \mapsto e^{sA}x_n$ is continuous in $D(A)$ and takes values in D; it follows that $\int_0^t e^{sA}x_n ds$, being the limit of the Riemann sums, belongs to the closure of D in $D(A)$, and then each $y_{n,t}$ does. Moreover $\|y_{n,t} - x\|$ tends to 0 as $t \to 0^+$, $n \to +\infty$, and

$$Ay_{n,t} - Ax = \frac{e^{tA}(x_n - x) - (x_n - x)}{t} + \frac{1}{t}\int_0^t e^{sA}Ax\,ds - Ax.$$

Given $\varepsilon > 0$, fix τ so small that $\|\tau^{-1}\int_0^\tau e^{sA}Ax\,ds - Ax\| \le \varepsilon$, and then choose n large, in such a way that $(M_0e^{\omega\tau} + 1)\|x_n - x\|/\tau \le \varepsilon$. For such choices of τ and n we have $\|Ay_{n,\tau} - Ax\| \le 2\varepsilon$, and the statement follows. □

Theorem 1.3.16 implies that the operator A is the closure of the restriction of A to D, i.e. $D(A)$ is the set of all $x \in X$ such that there is a sequence $(x_n) \subset D$ with the property that $x_n \to x$ and Ax_n converges as $n \to +\infty$; in this case we have $Ax = \lim_{n\to+\infty} Ax_n$.

Remark 1.3.17 Up to now we have considered complex Banach spaces, and the operators e^{tA} have been defined through integrals over paths in $\mathbb{C}$. But in many applications we have to work in real Banach spaces.

If X is a real Banach space, and $A : D(A) \subset X \to X$ is a closed linear operator, it is however convenient to consider its complex spectrum and resolvent. So we introduce the complexifications of X and of A, defined by

$$\widetilde{X} = \{x + iy :\ x,\ y \in X\};\ \ \|x + iy\|_{\widetilde{X}} = \sup_{-\pi\le\theta\le\pi} \|x\cos\theta + y\sin\theta\|$$

and

$$D(\widetilde{A}) = \{x + iy :\ x,\ y \in D(A)\},\ \ \widetilde{A}(x + iy) = Ax + iAy.$$

With obvious notation, we say that x and y are the real and the imaginary part of $x + iy$. Note that the “euclidean norm” $\sqrt{\|x\|^2 + \|y\|^2}$ is not a norm, in general. See Exercise 5 in §1.3.18.

If the complexification $\widetilde{A}$ of A is sectorial, so that the semigroup $e^{t\widetilde{A}}$ is analytic in $\widetilde{X}$, then the restriction of $e^{t\widetilde{A}}$ to X maps X into itself for each $t \ge 0$. To prove this statement it is convenient to replace the path $\gamma_{r,\eta}$ by the path $\gamma = \{\lambda \in \mathbb{C} : \lambda = \omega' + \rho e^{\pm i\theta},\ \rho \ge 0\}$, with $\omega' > \omega$, in formula (1.10). For each $x \in X$ we get

$$e^{t\widetilde{A}}x = \frac{1}{2\pi i}\int_0^{+\infty} e^{\omega' t}\left(e^{i\theta+\rho te^{i\theta}}R(\omega' + \rho e^{i\theta}, \widetilde{A}) - e^{-i\theta+\rho te^{-i\theta}}R(\omega' + \rho e^{-i\theta}, \widetilde{A})\right)x\,d\rho,\ \ t > 0.$$

The real part of the function under the integral vanishes (why?), and then $e^{t\widetilde{A}}x$ belongs to X. So, we have a semigroup of linear operators in X which enjoys all the properties that we have seen up to now.

Exercises 1.3.18

1. Let X_k, $k = 1, \dots, n$ be Banach spaces, and let $A_k : D(A_k) \to X_k$ be sectorial operators. Set
$$X = \prod_{k=1}^{n} X_k, \quad D(A) = \prod_{k=1}^{n} D(A_k),$$
and $A(x_1, \dots, x_n) = (A_1 x_1, \dots, A_n x_n)$, and show that A is a sectorial operator in X. X is endowed with the product norm $\|(x_1, \dots, x_n)\| = \left(\sum_{k=1}^{n} \|x_k\|^2\right)^{1/2}$.

2. (a) Let A, B be sectorial operators in X. Prove that $e^{tA}e^{tB} = e^{tB}e^{tA}$ for any $t > 0$ if and only if $e^{tA}e^{sB} = e^{sB}e^{tA}$ for any $t, s > 0$.

   (b) Prove that if A and B are as above, then $e^{tA}e^{sB} = e^{sB}e^{tA}$ for any $t, s > 0$ if and only if $R(\lambda, A)R(\mu, B) = R(\mu, B)R(\lambda, A)$ for large $\operatorname{Re}\lambda$ and $\operatorname{Re}\mu$.

3. Let $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ be, respectively, a sectorial operator and a closed operator such that $D(A) \subset D(B)$.

   (i) Show that there exist two positive constants a and b such that
$$\|Bx\| \le a\|Ax\| + b\|x\|$$
for every $x \in D(A)$.

   [Hint: use the closed graph theorem to show that $BR(\lambda, A)$ is bounded for any $\lambda \in \rho(A)$].

   (ii) Prove that if $BR(\lambda_0, A) = R(\lambda_0, A)B$ in $D(B)$ for some $\lambda_0 \in \rho(A)$, then $BR(\lambda, A) = R(\lambda, A)B$ in $D(B)$ for any $\lambda \in S_{\theta,\omega}$.

   [Hint: use Proposition B.3].

   (iii) Show that if $BR(\lambda_0, A) = R(\lambda_0, A)B$ in $D(B)$, then $Be^{tA} = e^{tA}B$ in $D(B)$ for every $t > 0$.

4. Prove Corollary 1.3.14.

5. Let X be a real Banach space. Prove that the function $f : X \times X \to \mathbb{R}$ defined by $f(x, y) = \sqrt{\|x\|^2 + \|y\|^2}$ for any $x, y \in X$, may not satisfy, in general, the homogeneity property
$$f(\lambda(x, y)) = |\lambda| f(x, y), \quad \lambda \in \mathbb{C}.$$

Chapter 2

Examples of sectorial operators

In this chapter we show several examples of sectorial operators.

The leading example is the Laplace operator Δ in one or more variables, i.e., $\Delta u = u''$ if $N = 1$ and $\Delta u = \sum_{i=1}^N D_{ii}u$ if $N > 1$. We shall see some realizations of the Laplacian in different Banach spaces, and with different domains, that turn out to be sectorial operators.

The Banach spaces taken into consideration are the usual spaces of complex valued functions defined in $\mathbb{R}^N$ or in an open set Ω of $\mathbb{R}^N$, that we recall briefly below.

The Lebesgue spaces $L^p(\Omega)$, $1 \leq p \leq +\infty$, are endowed with the norms

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx\right)^{1/p}, \quad 1 \leq p < +\infty,$$

$$\|f\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|.$$

When no confusion may arise, we write $\|f\|_p$ for $\|f\|_{L^p(\Omega)}$.

The Sobolev spaces $W^{k,p}(\Omega)$, where k is any positive integer and $1 \leq p \leq +\infty$, consist of all the functions f in $L^p(\Omega)$ which admit weak derivatives $D^\alpha f$ for $|\alpha| \leq k$ belonging to $L^p(\Omega)$. They are endowed with the norm

$$\|f\|_{W^{k,p}(\Omega)} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_p.$$

If $p = 2$, we write $H^k(\Omega)$ for $W^{k,p}(\Omega)$.

$C_b(\overline{\Omega})$ (resp., $BUC(\overline{\Omega})$) is the space of all the bounded and continuous (resp., bounded and uniformly continuous) functions $f : \overline{\Omega} \to \mathbb{C}$. They are endowed with the L^∞ norm.

If $k \in \mathbb{N}$, $C^k_b(\overline{\Omega})$ (resp. $BUC^k(\overline{\Omega})$) is the space of all the functions f in $C_b(\overline{\Omega})$ (resp. in $BUC(\overline{\Omega})$) which are k times continuously differentiable in $\overline{\Omega}$, with all the derivatives up to the order k in $C_b(\overline{\Omega})$ (resp. in $BUC(\overline{\Omega})$). They are endowed with the norm

$$\|f\|_{C^k_b(\overline{\Omega})} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_\infty.$$

If Ω is bounded, we drop the subindex b and we write $C(\overline{\Omega})$, $C^k(\overline{\Omega})$.

2.1 The operator $\mathcal{A}u = u''$

2.1.1 The second order derivative in the real line

Throughout the section we shall use square roots of complex numbers, defined by $\sqrt{\lambda} = |\lambda|^{1/2}e^{i\theta/2}$ if $\arg\lambda = \theta \in (-\pi, \pi]$. Therefore, $\operatorname{Re}\sqrt{\lambda} > 0$ if $\lambda \in \mathbb{C} \setminus (-\infty, 0]$.

Let us define the realizations of the second order derivative in $L^p(\mathbb{R})$ $(1 \le p < +\infty)$, and in $C_b(\mathbb{R})$, endowed with the maximal domains

$$\begin{array}{lll} D(A_p) = W^{2,p}(\mathbb{R}) \subset L^p(\mathbb{R}), & A_p u = u'', & 1 \le p < +\infty, \\ D(A_\infty) = C_b^2(\mathbb{R}), & A_\infty u = u''. & \end{array}$$

Let us determine the spectrum of A_p and estimate the norm of its resolvent.

Proposition 2.1.1 *For all $1 \le p \le +\infty$ the spectrum of A_p is the halfline $(-\infty, 0]$. If $\lambda = |\lambda|e^{i\theta}$ with $|\theta| < \pi$ then*

$$\|R(\lambda, A)\|_{\mathcal{L}(L^p(\mathbb{R}))} \le \frac{1}{|\lambda|\cos(\theta/2)}.$$

Proof. First we show that $(-\infty, 0] \subset \sigma(A_p)$. Fix $\lambda \le 0$ and consider the function $u(x) = \exp(i\sqrt{-\lambda}x)$ which satisfies $u'' = \lambda u$. For $p = +\infty$, u is an eigenfunction of A_∞ with eigenvalue λ. For $p < +\infty$, u does not belong to $L^p(\mathbb{R})$. To overcome this difficulty, consider a cut-off function $\psi : \mathbb{R} \to \mathbb{R}$, supported in $[-2, 2]$ and identically equal to 1 in $[-1, 1]$ and set $\psi_n(x) = \psi(x/n)$, for any $n \in \mathbb{N}$.

If $u_n = \psi_n u$, then $u_n \in D(A_p)$ and $\|u_n\|_p \approx n^{1/p}$ as $n \to +\infty$. Moreover, $\|Au_n - \lambda u_n\|_p \le Cn^{1/p-1}$. Setting $v_n = u_n/\|u_n\|_p$, it follows that $\|(\lambda - A)v_n\|_p \to 0$ as $n \to +\infty$, and then $\lambda \in \sigma(A)$. See Exercise B.9.

Now let $\lambda \notin (-\infty, 0]$. If $p = +\infty$, the equation $\lambda u - u'' = 0$ has no nonzero bounded solution, hence $\lambda I - A_\infty$ is one to one. If $p < +\infty$, it is easy to see that all the nonzero solutions $u \in W^{2,p}_{loc}(\mathbb{R})$ to the equation $\lambda u - u'' = 0$ belong to $C^\infty(\mathbb{R})$ and they are classical solutions, but they do not belong to $L^p(\mathbb{R})$, so that the operator $\lambda I - A_p$ is one to one. We recall that $W^{2,p}_{loc}(\mathbb{R})$ denotes the set of all the functions $f : \mathbb{R} \to \mathbb{R}$ which belong to $W^{2,p}(I)$ for any bounded interval $I \subset \mathbb{R}$.

Let us show that $\lambda I - A_p$ is onto. We write $\sqrt{\lambda} = \mu$. If $f \in C_b(\mathbb{R})$ the variation of constants method gives the (unique) bounded solution to $\lambda u - u'' = f$, written as

$$u(x) = \frac{1}{2\mu}\left(\int_{-\infty}^{x} e^{-\mu(x-y)} f(y)dy + \int_{x}^{+\infty} e^{\mu(x-y)} f(y)dy\right) = (f \star h_\mu)(x), \tag{2.1}$$

where $h_\mu(x) = e^{-\mu|x|}/2\mu$. Since $\|h_\mu\|_{L^1(\mathbb{R})} = (|\mu| \operatorname{Re}\mu)^{-1}$, we get

$$\|u\|_\infty \le \|h_\mu\|_{L^1(\mathbb{R})}\|f\|_\infty = \frac{1}{|\lambda|\cos(\theta/2)}\|f\|_\infty,$$

where $\theta = \arg\lambda$. If $|\theta| \le \theta_0 < \pi$ we get $\|u\|_\infty \le (|\lambda|\cos(\theta_0/2))^{-1}\|f\|_\infty$, and therefore A_∞ is sectorial, with $\omega = 0$ and any $\theta \in (\pi/2, \pi)$.

If $p < +\infty$ and $f \in L^p(\mathbb{R})$, the natural candidate to be $R(\lambda, A_p)f$ is still the function u defined by (2.1). We have to check that $u \in D(A_p)$ and that $(\lambda I - A_p)u = f$. By Young's inequality (see e.g. [3, Th. IV.15]), $u \in L^p(\mathbb{R})$ and again

$$\|u\|_p \le \|f\|_p\|h_\mu\|_1 \le \frac{1}{|\lambda|\cos(\theta/2)}\|f\|_p.$$

That $u \in D(A_p)$ may be seen in several ways; all of them need some knowledge of elementary properties of Sobolev spaces. The following proof relies on the fact that smooth functions are dense in $W^{1,p}(\mathbb{R})$[1].

Approximate $f \in L^p(\mathbb{R})$ by a sequence $(f_n) \subset C_0^\infty(\mathbb{R})$. The corresponding solutions u_n to $\lambda u_n - u_n'' = f_n$ are smooth and they are given by formula (2.1) with f_n instead of f, therefore they converge to u by Young's inequality. Moreover,

$$u_n'(x) = -\frac{1}{2}\int_{-\infty}^{x} e^{-\mu(x-y)} f_n(y)dy + \frac{1}{2}\int_{x}^{+\infty} e^{\mu(x-y)} f_n(y)dy$$

converge to the function

$$g(x) = -\frac{1}{2}\int_{-\infty}^{x} e^{-\mu(x-y)} f(y)dy + \frac{1}{2}\int_{x}^{+\infty} e^{\mu(x-y)} f(y)dy$$

again by Young's inequality. Hence $g = u' \in L^p(\mathbb{R})$, and $u_n'' = \lambda u_n - f_n$ converge to $\lambda u - f$, hence $\lambda u - f = u'' \in L^p(\mathbb{R})$. Therefore $u \in W^{2,p}(\mathbb{R})$ and the statement follows. $\square$

Note that $D(A_\infty)$ is not dense in $C_b(\mathbb{R})$, and its closure is $BUC(\mathbb{R})$. Therefore, the associated semigroup e^{tA_∞} is not strongly continuous. But the part of A_∞ in $BUC(\mathbb{R})$, i.e. the operator

$$BUC^2(\mathbb{R}) \to BUC(\mathbb{R}), \ \ u \mapsto u''$$

has dense domain in $BUC(\mathbb{R})$ and it is sectorial, so that the restriction of e^{tA_∞} to $BUC(\mathbb{R})$ is strongly continuous. If $p < +\infty$, $D(A_p)$ is dense in $L^p(\mathbb{R})$, and e^{tA_p} is strongly continuous in $L^p(\mathbb{R})$.

This is one of the few situations in which we have a nice representation formula for e^{tA_p}, for $1 \leq p \leq +\infty$, and precisely

$$(e^{tA_p} f)(x) = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} f(y)dy, \ \ t > 0, \ x \in \mathbb{R}. \tag{2.2}$$

This formula will be discussed in Section 2.3, where we shall use a classical method, based on the Fourier transform, to obtain it. In principle, since we have an explicit representation formula for the resolvent, plugging it in (1.10) we should get (2.2). But the contour integral obtained in this way is not very easy to work out.

2.1.2 The operator $\mathcal{A}u = u''$ in a bounded interval, with Dirichlet boundary conditions

Without loss of generality, we fix $I = (0,1)$, and we consider the realizations of the second order derivative in $L^p(0,1)$, $1 \leq p < +\infty$,

$$D(A_p) = \{u \in W^{2,p}(0,1) : \ u(0) = u(1) = 0\} \subset L^p(0,1), \ \ A_p u = u'',$$

as well as its realization in $C([0,1])$,

$$D(A_\infty) = \{u \in C^2([0,1]) : \ u(0) = u(1) = 0\}, \ \ A_\infty u = u''.$$

[1]Precisely, a function $v \in L^p(\mathbb{R})$ belongs to $W^{1,p}(\mathbb{R})$ iff there is a sequence $(v_n) \subset C^\infty(\mathbb{R})$ with v_n, $v_n' \in L^p(\mathbb{R})$, such that $v_n \to v$ and $v_n' \to g$ in $L^p(\mathbb{R})$ as $n \to +\infty$. In this case, g is the weak derivative of v. See [3, Chapter 8].

We could follow the same approach of Subsection 2.1.1, by computing the resolvent operator $R(\lambda, A_\infty)$ for $\lambda \notin (-\infty, 0]$ and then showing that the same formula gives $R(\lambda, A_p)$. The formula turns out to be more complicated than before, but it leads to the same final estimate, see Exercise 3 in §2.1.3. Here we do not write it down explicitly, but we estimate separately its components, arriving at a less precise estimate for the norm of the resolvent, with simpler computations.

Proposition 2.1.2 *The operators $A_p : D(A_p) \to L^p(0,1)$, $1 \le p < +\infty$ and $A_\infty : D(A_\infty) \to C([0,1])$ are sectorial, with $\omega = 0$ and any $\theta \in (\pi/2, \pi)$.*

Proof. For $\lambda \notin (-\infty, 0]$ set $\mu = \sqrt{\lambda}$, so that $\operatorname{Re}\mu > 0$. For every $f \in X$, $X = L^p(0,1)$ or $X = C([0,1])$, extend f to a function $\widetilde{f} \in L^p(\mathbb{R})$ or $\widetilde{f} \in C_b(\mathbb{R})$, in such a way that $\|\widetilde{f}\| = \|f\|$. For instance we may define $\widetilde{f}(x) = 0$ for $x \notin (0,1)$ if $X = L^p(0,1)$, $\widetilde{f}(x) = f(1)$ for $x > 1$, $\widetilde{f}(x) = f(0)$ for $x < 0$ if $X = C([0,1])$. Let $\widetilde{u}$ be defined by (2.1) with $\widetilde{f}$ instead of f. We already know from Proposition 2.1.1 that $\widetilde{u}_{|[0,1]}$ is a solution of the equation $\lambda u - u'' = f$ satisfying $\|u\|_p \le \|f\|_p/(|\lambda|\cos(\theta/2))$, where $\theta = \arg\lambda$. However, it does not necessarily satisfy the boundary conditions. To find a solution that satisfies the boundary conditions we set

$$\gamma_0 = \frac{1}{2\mu}\int_{\mathbb{R}} e^{-\mu|s|}\widetilde{f}(s)\,ds = \widetilde{u}(0)$$

and

$$\gamma_1 = \frac{1}{2\mu}\int_{\mathbb{R}} e^{-\mu|1-s|}\widetilde{f}(s)\,ds = \widetilde{u}(1).$$

All the solutions of the equation $\lambda u - u'' = f$ belonging to $W^{2,p}(0,1)$ or to $C^2([0,1])$ are given by

$$u(x) = \widetilde{u}(x) + c_1 u_1(x) + c_2 u_2(x),$$

where $u_1(x) := e^{-\mu x}$ and $u_2(x) := e^{\mu x}$ are two independent solutions of the homogeneous equation $\lambda u - u'' = 0$. We determine uniquely c_1 and c_2 imposing $u(0) = u(1) = 0$ because the determinant

$$D(\mu) = e^{\mu} - e^{-\mu}$$

is nonzero since $\operatorname{Re}\mu > 0$. A straightforward computation yields

$$c_1 = \frac{1}{D(\mu)}\left[\gamma_1 - e^{\mu}\gamma_0\right], \qquad c_2 = \frac{1}{D(\mu)}\left[-\gamma_1 + e^{-\mu}\gamma_0\right],$$

so that for $1 \le p < +\infty$

$$\|u_1\|_p \le \frac{1}{(p\operatorname{Re}\mu)^{1/p}}; \qquad \|u_2\|_p \le \frac{e^{\operatorname{Re}\mu}}{(p\operatorname{Re}\mu)^{1/p}};$$

while $\|u_1\|_\infty = 1$, $\|u_2\|_\infty = e^{\operatorname{Re}\mu}$. For $1 < p < +\infty$ by the Hölder inequality we also obtain

$$|\gamma_0| \le \frac{1}{2|\mu|(p'\operatorname{Re}\mu)^{1/p'}}\|f\|_p, \qquad |\gamma_1| \le \frac{1}{2|\mu|(p'\operatorname{Re}\mu)^{1/p'}}\|f\|_p$$

and

$$|\gamma_j| \le \frac{1}{2|\mu|}\|f\|_1, \quad \text{if } f \in L^1(0,1),\ \ j = 0,\ 1$$

$$|\gamma_j| \le \frac{1}{|\mu|\operatorname{Re}\mu}\|f\|_\infty, \quad \text{if } f \in C([0,1]),\ \ j = 0,\ 1.$$

Moreover $|D(\mu)| \approx e^{\mathrm{Re}\,\mu}$ for $|\mu| \to +\infty$. If $\lambda = |\lambda|e^{i\theta}$ with $|\theta| \leq \theta_0 < \pi$ then $\mathrm{Re}\,\mu \geq |\mu|\cos(\theta_0/2)$ and we easily get

$$\|c_1 u_1\|_p \leq \frac{C}{|\lambda|}\|f\|_p \qquad \text{and} \qquad \|c_2 u_2\|_p \leq \frac{C}{|\lambda|}\|f\|_p$$

for a suitable $C > 0$ and λ as above, $|\lambda|$ large enough. Finally

$$\|u\|_p \leq \frac{C}{|\lambda|}\|f\|_p$$

for $|\lambda|$ large, say $|\lambda| \geq R$, and $|\arg\lambda| \leq \theta_0$.

For $|\lambda| \leq R$ we may argue as follows: one checks easily that the spectrum of A_p consists only of the eigenvalues $-n^2\pi^2$, $n \in \mathbb{N}$. Since $\lambda \mapsto R(\lambda, A_p)$ is holomorphic in the resolvent set, it is continuous, hence it is bounded on the compact set $\{|\lambda| \leq R,\ |\arg\lambda| \leq \theta_0\} \cup \{0\}$. $\square$

Exercises 2.1.3

1. Let A_∞ be the operator defined in Subsection 2.1.1.

   (a) Prove that the resolvent $R(\lambda, A_\infty)$ leaves invariant the subspaces

   $$C_0(\mathbb{R}) := \{u \in C(\mathbb{R}) : \lim_{|x|\to+\infty} u(x) = 0\}$$

   and

   $$C_T(\mathbb{R}) := \{u \in C(\mathbb{R}) :\ u(x) = u(x+T),\ x \in \mathbb{R}\},$$

   with $T > 0$.

   (b) Using the previous results show that the operators

   $$A_0 : D(A_0) := \{u \in C^2(\mathbb{R}) \cap C_0(\mathbb{R}) :\ u'' \in C_0(\mathbb{R})\} \to C_0(\mathbb{R}),\ \ A_0 u = u'',$$

   and

   $$A_T : D(A_T) := C^2(\mathbb{R}) \cap C_T(\mathbb{R}) \to C_T(\mathbb{R}),\ \ A_T u = u''$$

   are sectorial in $C_0(\mathbb{R})$ and in $C_T(\mathbb{R})$, respectively.

2. (a) Let $\lambda > 0$ and set

   $$\phi(x) = \int_0^{+\infty} \frac{1}{\sqrt{4\pi t}} e^{-\lambda t} e^{-x^2/4t}\, dt.$$

   Prove that $\phi'' = \lambda\phi$ and $\phi(0) = (2\sqrt{\pi\lambda})^{-1}\Gamma(1/2) = (2\sqrt{\lambda})^{-1}$, $\phi(x) \to 0$ as $|x| \to +\infty$, so that ϕ coincides with the function $h_{\sqrt{\lambda}}$ in (2.1). (b) Use (a) and Proposition 1.3.7 to prove formula (2.2).

3. Consider again the operator $u \mapsto u''$ in $(0,1)$ as in Subsection 2.1.2, with the domains $D(A_p)$ defined there, $1 \leq p \leq +\infty$. Solving explicitly the differential equation $\lambda u - u'' = f$ in $D(A_p)$, show that the eigenvalues are $-n^2\pi^2$, $n \in \mathbb{N}$, and express the resolvent as an integral operator.

4. Consider the operator $A_p u = u''$ in $L^p(0,1)$, $1 \le p < \infty$, with the domain
$$D(A_p) = \{u \in W^{2,p}(0,1) :\ u'(0) = u'(1) = 0\} \subset L^p(0,1),$$
or in $C([0,1])$, with the domain
$$D(A_\infty) = \{u \in C^2((0,1)) \cap C([0,1]) :\ u'(0) = u'(1) = 0\},$$
corresponding to the Neumann boundary condition. Use the same argument of Subsection 2.1.2 to show that A_p is sectorial.

5. Let A_∞ be the realization of the second order derivative in $C([0,1])$ with Dirichlet boundary condition, as in Subsection 2.1.2. Prove that for each $\alpha \in (0,1)$ the part of A_∞ in $C^\alpha([0,1])$, i.e. the operator
$$\{u \in C^{2+\alpha}([0,1]) :\ u(0) = u(1) = 0\} \to C^\alpha([0,1]),\quad u \mapsto u''$$
is not sectorial in $C^\alpha([0,1])$, although the function $(0,+\infty) \to \mathcal{L}(C^\alpha([0,1]))$, $t \mapsto e^{tA_\infty}{}_{|C^\alpha([0,1])}$ is analytic.
[Hint: take $f \equiv 1$, compute explicitly $u := R(\lambda, A_\infty)f$ for $\lambda > 0$, and show that $\limsup_{\lambda\to+\infty} \lambda^{1+\alpha/2} u(\lambda^{-1/2}) = +\infty$, so that $\lambda[R(\lambda, A_\infty)f]_{C^\alpha}$ is unbounded as $\lambda \to +\infty$.]
Taking into account the behavior of $R(\lambda, A)1$, deduce that $\|e^{tA}\|_{\mathcal{L}(C^\alpha([0,1]))}$ is unbounded for $t \in (0,1)$.

2.2 Some abstract examples

The realization of the second order derivative in $L^2(\mathbb{R})$ is a particular case of the following general situation. Recall that, if H is a Hilbert space, and $A : D(A) \subset H \to H$ is a linear operator with dense domain, the *adjoint* A^* of A is the operator $A^* : D(A^*) \subset X \to X$ defined as follows,
$$D(A^*) = \{x \in H : \exists y \in H \text{ such that } \langle Az, x\rangle = \langle z, y\rangle,\ \ \forall z \in D(A)\}, \qquad A^*x = y.$$
The operator A is said to be *self-adjoint* if $D(A) = D(A^*)$ and $A = A^*$. It is said to be *dissipative* if
$$\|(\lambda - A)x\| \ge \lambda\|x\|, \tag{2.3}$$
for all $x \in D(A)$ and $\lambda > 0$, or equivalently (see Exercises 2.2.4) if $\mathrm{Re}\langle Ax, x\rangle \le 0$ for every $x \in D(A)$.

The following proposition holds.

Proposition 2.2.1 *Let H be a Hilbert space, and let $A : D(A) \subset H \to H$ be a self-adjoint dissipative operator. Then A is sectorial, with an arbitrary $\theta < \pi$ and $\omega = 0$.*

Proof. Let us first show that $\sigma(A) \subset \mathbb{R}$. Let $\lambda = a + ib \in \mathbb{C}$. Since $\langle Ax, x\rangle \in \mathbb{R}$ for every $x \in D(A)$, we have
$$\|(\lambda I - A)x\|^2 = (a^2 + b^2)\|x\|^2 - 2a\langle x, Ax\rangle + \|Ax\|^2 \ge b^2\|x\|^2. \tag{2.4}$$

Hence, if $b \neq 0$ then $\lambda I - A$ is one to one. Let us check that the range is both closed and dense in H, so that A is onto. Take $x_n \in D(A)$ such that $\lambda x_n - Ax_n$ converges as $n \to +\infty$. From the inequality

$$\|(\lambda I - A)(x_n - x_m)\|^2 \geq b^2 \|x_n - x_m\|^2, \ \ n, m \in \mathbb{N},$$

it follows that (x_n) is a Cauchy sequence, and by difference (Ax_n) is a Cauchy sequence too. Hence there are $x, y \in H$ such that $x_n \to x$, $Ax_n \to y$. Since A is self-adjoint, it is closed, and then $x \in D(A)$, $Ax = y$, and $\lambda x_n - Ax_n$ converges to $\lambda x - Ax \in \text{Range}\,(\lambda I - A)$. Therefore, the range of $\lambda I - A$ is closed.

If y is orthogonal to the range of $\lambda I - A$, then for every $x \in D(A)$ we have $\langle y, \lambda x - Ax\rangle = 0$. Hence $y \in D(A^*) = D(A)$ and $\overline{\lambda} y - A^* y = \overline{\lambda} y - Ay = 0$. Since $\overline{\lambda} I - A$ is one to one, then $y = 0$, and the range of $\lambda I - A$ is dense.

Let us check that $\sigma(A) \subset (-\infty, 0]$. Indeed, if $\lambda > 0$ and $x \in D(A)$, we have

$$\|(\lambda I - A)x\|^2 = \lambda^2 \|x\|^2 - 2\lambda\langle x, Ax\rangle + \|Ax\|^2 \geq \lambda^2 \|x\|^2, \tag{2.5}$$

and arguing as above we get $\lambda \in \rho(A)$.

Let us now verify condition (1.9)(ii) for $\lambda = \rho e^{i\theta}$, with $\rho > 0$, $-\pi < \theta < \pi$. Take $x \in H$ and $u = R(\lambda, A)x$. From the equality $\lambda u - Au = x$, multiplying by $e^{-i\theta/2}$ and taking the inner product with u, we deduce

$$\rho e^{i\theta/2}\|u\|^2 - e^{-i\theta/2}\langle Au, u\rangle = e^{-i\theta/2}\langle x, u\rangle,$$

from which, taking the real part,

$$\rho \cos(\theta/2)\|u\|^2 - \cos(\theta/2)\langle Au, u\rangle = \text{Re}(e^{-i\theta/2}\langle x, u\rangle) \leq \|x\|\, \|u\|.$$

Therefore, taking into account that $\cos(\theta/2) > 0$ and $\langle Au, u\rangle \leq 0$, we get

$$\|u\| \leq \frac{\|x\|}{|\lambda| \cos(\theta/2)},$$

with $\theta = \arg \lambda$. □

Let us see another example, where X is a general Banach space.

Proposition 2.2.2 *Let A be a linear operator such that the resolvent set $\rho(A)$ contains $\mathbb{C} \setminus i\mathbb{R}$, and there exists $M > 0$ such that $\|R(\lambda, A)\| \leq M/|\text{Re}\,\lambda|$ for $\text{Re}\,\lambda \neq 0$. Then A^2 is sectorial, with $\omega = 0$ and any $\theta < \pi$.*

Proof. For every $\lambda \in \mathbb{C}\setminus(-\infty, 0]$ and for every $y \in X$, the resolvent equation $\lambda x - A^2 x = y$ is equivalent to

$$(\sqrt{\lambda} I - A)(\sqrt{\lambda} I + A)x = y.$$

Since $\text{Re}\,\sqrt{\lambda} > 0$, then $\sqrt{\lambda} \in \rho(A) \cap (\rho(-A))$, so that

$$x = R(\sqrt{\lambda}, A)R(\sqrt{\lambda}, -A)y = -R(\sqrt{\lambda}, A)R(-\sqrt{\lambda}, A)y \tag{2.6}$$

and, since $|\text{Re}\,\sqrt{\lambda}| = \sqrt{|\lambda|}\cos(\theta/2)$ if $\arg \lambda = \theta$, we get

$$\|x\| \leq \frac{M^2}{|\lambda|(\cos(\theta/2))^2}\|y\|,$$

for $\lambda \in S_{\theta,0}$, and the statement follows. □

Remark 2.2.3 The proof of Proposition 2.2.2 shows that $\lim_{\mathrm{Re}\,\lambda\to+\infty,\ \lambda\in S_{\theta,\omega}} \|AR(\lambda, A^2)\| = 0$. Therefore, Remark 1.3.15 implies that $A^2+\alpha A$ is the generator of an analytic semigroup for any $\alpha \in \mathbb{R}$.

Proposition 2.2.2 gives an alternative way to show that the realization of the second order derivative in $L^p(\mathbb{R})$, or in $C_b(\mathbb{R})$, is sectorial. But there are also other interesting applications. See next exercise 3.

Exercises 2.2.4

1. Let A be a sectorial operator with $\theta > 3\pi/4$. Show that $-A^2$ is sectorial.

2. Let H be a Hilbert space and $A : D(A) \subset H \to H$ be a linear operator. Show that the dissipativity condition (2.3) is equivalent to $\mathrm{Re}\langle Ax, x\rangle \le 0$ for any $x \in D(A)$.

3. (a) Show that the operator $A : D(A) = \{f \in C_b(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\}) : x \mapsto xf'(x) \in C_b(\mathbb{R}), \lim_{x\to 0} xf'(x) = 0\}$, $Af(x) = xf'(x)$ for $x \neq 0$, $Af(0) = 0$, satisfies the assumptions of Proposition 2.2.2, so that A^2 is sectorial in $C_b(\mathbb{R})$.

   (b) Prove that for each a, $b \in \mathbb{R}$ a suitable realization of the operator $\mathcal{A}$ defined by $(\mathcal{A}f)(x) = x^2 f''(x) + axf'(x) + bf(x)$ is sectorial.

   [Hint. First method: use (a), Exercise 1 and Remark 2.2.3. Second method: determine explicitly the resolvent operator using the changes of variables $x = e^t$ and $x = -e^t$].

2.3 The Laplacian in $\mathbb{R}^N$

Let us consider the heat equation

$$\begin{cases} u_t(t,x) = \Delta u(t,x), & t > 0, \quad x \in \mathbb{R}^N, \\ u(0,x) = f(x), & \qquad\quad\ x \in \mathbb{R}^N, \end{cases} \tag{2.7}$$

where f is a given function in X, $X = L^p(\mathbb{R}^N)$, $1 \le p < +\infty$, or $X = C_b(\mathbb{R}^N)$.

To get a representation formula for the solution, let us apply (just formally) the Fourier transform, denoting by $\hat{u}(t,\xi)$ the Fourier transform of u with respect to the space variable x. We get

$$\begin{cases} \hat{u}_t(t,\xi) = -|\xi|^2 \hat{u}(t,\xi), & t > 0, \quad \xi \in \mathbb{R}^N, \\ \hat{u}(0,\xi) = \hat{f}(\xi), & \qquad\quad\ \xi \in \mathbb{R}^N, \end{cases}$$

whose solution is $\hat{u}(t,\xi) = \hat{f}(\xi)e^{-|\xi|^2 t}$. Taking the inverse Fourier transform, we get $u = T(\cdot)f$, where the *heat semigroup* $\{T(t)\}_{t\ge 0}$ is defined by the Gauss-Weierstrass formula

$$(T(t)f)(x) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} f(y)dy, \ \ t > 0, \ x \in \mathbb{R}^N \tag{2.8}$$

(as usual, we define $(T(0)f)(x) = f(x)$). The verification that $(T(t))_{t\ge 0}$ is a semigroup is left as an exercise.

Now, we check that formula (2.8) gives in fact a solution to (2.7) and defines an analytic semigroup whose generator is a sectorial realization of the Laplacian in X. For clarity reason, we split the proof in several steps.

(a) Let us first notice that $T(t)f = G_t \star f$, where

$$G_t(x) = \frac{1}{(4\pi t)^{N/2}}\, e^{-\frac{|x|^2}{4t}}, \qquad \int_{\mathbb{R}^N} G_t(x)dx = 1, \quad t > 0,$$

and $\star$ denotes the convolution. By Young's inequality,

$$\|T(t)f\|_p \le \|f\|_p, \;\; t > 0, \; 1 \le p \le +\infty. \tag{2.9}$$

Since G_t and all its derivatives belong to $C^\infty(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, $1 \le p \le +\infty$, it readily follows that the function $u(t,x) := (T(t)f)(x)$ belongs to $C^\infty((0,+\infty)\times\mathbb{R}^N)$, because we can differentiate under the integral sign. Since $\partial G_t/\partial t = \Delta G_t$, then u solves the heat equation in $(0,+\infty)\times\mathbb{R}^N$.

Let us show that $T(t)f \to f$ in X as $t \to 0^+$ if $f \in L^p(\mathbb{R}^N)$ or $f \in BUC(\mathbb{R}^N)$. If $f \in L^p(\mathbb{R}^N)$ we have

$$\begin{aligned}
\|T(t)f - f\|_p^p &= \int_{\mathbb{R}^N} \Big| \int_{\mathbb{R}^N} G_t(y) f(x-y)dy - f(x)\Big|^p dx \\
&= \int_{\mathbb{R}^N} \Big| \int_{\mathbb{R}^N} G_t(y)[f(x-y) - f(x)]dy\Big|^p dx \\
&= \int_{\mathbb{R}^N} \Big| \int_{\mathbb{R}^N} G_1(v)[f(x-\sqrt{t}v) - f(x)]dv\Big|^p dx \\
&\le \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} G_1(v)|f(x-\sqrt{t}v) - f(x)|^p dv\, dx \\
&= \int_{\mathbb{R}^N} G_1(v) \int_{\mathbb{R}^N} |f(x-\sqrt{t}v) - f(x)|^p dx\, dv.
\end{aligned}$$

Here we used twice the property that the integral of G_t is 1; the first one to put $f(x)$ under the integral sign and the second one to get

$$\Big| \int_{\mathbb{R}^N} G_1(v)[f(x-\sqrt{t}v) - f(x)]dv\Big|^p \le \int_{\mathbb{R}^N} G_1(v)|f(x-\sqrt{t}v) - f(x)|^p dv$$

through Hölder inequality, if $p > 1$. Now, the function $\varphi(t,v) := \int_{\mathbb{R}^N} |f(x-\sqrt{t}v) - f(x)|^p dx$ goes to zero as $t \to 0^+$ for each v, by a well known property of the L^p functions, and it does not exceed $2^p\|f\|_p^p$. By dominated convergence, $\|T(t)f - f\|_p^p$ tends to 0 as $t \to 0^+$.

If $f \in BUC(\mathbb{R}^N)$ we have

$$\begin{aligned}
\sup_{x\in\mathbb{R}^N} |(T(t)f - f)(x)| &\le \sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} G_t(y)|f(x-y) - f(x)|dy \\
&= \sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} G_1(v)|f(x-\sqrt{t}v) - f(x)|dv \\
&\le \int_{\mathbb{R}^N} G_1(v) \sup_{x\in\mathbb{R}^N} |f(x-\sqrt{t}v) - f(x)|dv.
\end{aligned}$$

Again, the function $\varphi(t,v) := \sup_{x\in\mathbb{R}^N} |f(x-\sqrt{t}v) - f(x)|$ goes to zero as $t \to 0^+$ for each v by the uniform continuity of f, and it does not exceed $2\|f\|_\infty$. By dominated convergence, $T(t)f - f$ goes to 0 as $t \to 0^+$ in the supremum norm.

If $f \in C_b(\mathbb{R}^N)$ the same argument shows that $T(t)f \to f$, as $t \to 0^+$, uniformly on compact sets. In particular, the function $(t,x) \mapsto (T(t)f)(x)$ is continuous and bounded in $[0,+\infty) \times \mathbb{R}^N$.

(b) If $f \in X$, the function

$$R(\lambda)f = \int_0^{+\infty} e^{-\lambda t} T(t)f\,dt$$

is well defined and holomorphic in the halfplane $\Pi := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$. Observe that $t \mapsto T(t)f$ is continuous from $[0,+\infty)$ to X, if $X = L^p(\mathbb{R}^N)$ and bounded and continuous from $(0,+\infty)$ to X, if $X = C_b(\mathbb{R}^N)$ (the continuity in $(0,+\infty)$ follows from the fact that $T(s)f \in BUC(\mathbb{R}^N)$ for every $s > 0$, see Exercise 5 in §2.3.1 below). In both cases $R(\lambda)f$ is well defined.

It is easily seen that R verifies the resolvent identity in the halfplane Π: indeed, for $\lambda \neq \mu,\ \lambda,\mu \in \Pi$, we have

$$\begin{aligned}
R(\lambda)R(\mu)f &= \int_0^{+\infty} e^{-\lambda t} T(t) \int_0^{+\infty} e^{-\mu s} T(s) f\,ds\,dt = \int_0^{+\infty}\int_0^{+\infty} e^{-\lambda t - \mu s} T(t+s) f\,dt\,ds \\
&= \int_0^{+\infty} e^{-\mu\sigma} T(\sigma) f \int_0^{\sigma} e^{(\mu-\lambda)t} dt\,d\sigma = \int_0^{+\infty} e^{-\mu\sigma} T(\sigma) f \frac{e^{(\mu-\lambda)\sigma} - 1}{\mu - \lambda}\,d\sigma \\
&= \frac{1}{\mu-\lambda}(R(\lambda)f - R(\mu)f).
\end{aligned}$$

Let us prove that $R(\lambda)$ is one to one for $\lambda \in \Pi$. Suppose that there are $f \in X,\ \lambda_0 \in \Pi$ such that $R(\lambda_0)f = 0$. From the resolvent identity it follows that $R(\lambda)f = 0$ for all $\lambda \in \Pi$, hence, for all $g \in X'$

$$\langle R(\lambda)f, g\rangle \ = \int_0^{+\infty} e^{-\lambda t}\langle T(t)f, g\rangle dt = 0, \quad \lambda \in \Pi.$$

Since $\langle R(\lambda)f, g\rangle$ is the Laplace transform of the scalar function $t \mapsto \langle T(t)f, g\rangle$, we get $\langle T(t)f, g\rangle \equiv 0$ in $(0,+\infty)$, and then $T(t)f \equiv 0$ in $(0,+\infty)$, since g is arbitrary. Letting $t \to 0^+$ ge get $f = 0$. Thus, by Proposition B.2 there is a linear operator $A : D(A) \subset X \to X$ such that $\rho(A) \supset \Pi$ and $R(\lambda, A) = R(\lambda)$ for $\lambda \in \Pi$.

(c) Let us show that the operator A is sectorial in X and that $T(t) = e^{tA}$ for any $t > 0$. For $\operatorname{Re} z > 0$, $f \in X$, we define $T(z)f = G_z * f$ where

$$G_z(x) = \frac{1}{(4\pi z)^{N/2}} e^{-\frac{|x|^2}{4z}}, \qquad \int_{\mathbb{R}^N} |G_z(x)|dx = \left(\frac{|z|}{\operatorname{Re} z}\right)^{N/2}.$$

By Young's inequality $\|T(z)f\|_p \leq (\cos\theta_0)^{-N/2}\|f\|_p$ if $z \in S_{\theta_0,0}$ and $\theta_0 < \pi/2$. Moreover, since $G_z \to G_{z_0}$ in $L^1(\mathbb{R}^N)$ as $z \to z_0$ in Π (this is easily seen using dominated convergence), the map $z \mapsto T(z)f$ is continuous from Π to X. Writing for every $f \in L^p(\mathbb{R}^N)$, $g \in L^{p'}(\mathbb{R}^N)$ $(1/p + 1/p' = 1)$,

$$\langle T(z)f, g\rangle = \frac{1}{(4\pi z)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4z}} \langle f(\cdot - y), g\rangle dy$$

and using Theorem A.6 one sees that $z \mapsto T(z)f$ is holomorphic from Π to $L^p(\mathbb{R}^N)$. In the case $p = +\infty$, $X = C_b(\mathbb{R}^N)$, the function $z \mapsto T(z)f(x)$ is holomorphic in Π for every $x \in \mathbb{R}^N$.

Now we prove the resolvent estimate in the halfplane $\{\mathrm{Re}\, z > 0\}$. If $\lambda = a + ib$ with $a > 0$ and $b \geq 0$, by Cauchy integral theorem we have

$$R(\lambda, A)f = \int_0^{+\infty} e^{-\lambda t} T(t) f dt = \int_\gamma e^{-\lambda z} T(z) f dz$$

where $\gamma = \{z = x - ix, x \geq 0\}$. Therefore

$$\|R(\lambda, A)f\|_p \leq 2^{N/4} \|f\|_p \int_0^{+\infty} e^{-(a+b)x} dx \leq \frac{1}{a+b} (\sqrt{2})^{N/2} \|f\|_p \leq \frac{2^{N/4}}{|\lambda|} \|f\|_p.$$

If $b \leq 0$ one gets the same estimate considering $\tilde{\gamma} = \{z = x + ix,\ x \geq 0\}$.

By Proposition 1.3.12, A is sectorial in X.

Let e^{tA} be the analytic semigroup generated by A. By Proposition 1.3.7, for $\mathrm{Re}\, \lambda > 0$ we have

$$R(\lambda, A)f = \int_0^{+\infty} e^{-\lambda t} e^{tA} f dt = \int_0^{+\infty} e^{-\lambda t} T(t) f dt$$

hence for every $f \in X$, $g \in X'$,

$$\int_0^{+\infty} e^{-\lambda t} \langle e^{tA} f, g\rangle dt = \int_0^{+\infty} e^{-\lambda t} \langle T(t) f, g\rangle dt.$$

This shows that the Laplace transforms of the scalar-valued functions $t \mapsto \langle e^{tA} f, g\rangle$, $t \mapsto \langle T(t)f, g\rangle$ coincide, hence $\langle e^{tA} f, g\rangle = \langle T(t)f, g\rangle$. Since f, g are arbitrary, $e^{tA} = T(t)$.

(d) Let us now show that A is an extension of the Laplacian defined in $W^{2,p}(\mathbb{R}^N)$, if $X = L^p(\mathbb{R}^N)$, and in $C^2_b(\mathbb{R}^N)$ if $X = C_b(\mathbb{R}^N)$.

To begin with, we consider the case of $L^p(\mathbb{R}^N)$. The Schwartz space $\mathcal{S}(\mathbb{R}^N)$ is invariant under each $T(t)$ and it is dense in $L^p(\mathbb{R}^N)$ because it contains $C^\infty_0(\mathbb{R}^N)$[2]. For $f \in \mathcal{S}(\mathbb{R}^N)$, it is easily checked that $u := T(\cdot)f$ belongs to $C^2([0, +\infty) \times \mathbb{R}^N)$ (in fact, it belongs to $C^\infty([0, +\infty) \times \mathbb{R}^N)$) and that $u_t = \Delta u = T(t)\Delta f$. Therefore

$$\frac{u(t, x) - u(0, x)}{t} = \frac{1}{t} \int_0^t u_t(s, x) ds = \frac{1}{t} \int_0^t \Delta u(s, x) ds \to \Delta f(x) \text{ as } t \to 0^+ \tag{2.10}$$

pointwise and also in $L^p(\mathbb{R}^N)$, because

$$\frac{1}{t} \int_0^t \|\Delta u(s, \cdot) - \Delta f\|_p\, ds \leq \sup_{0<s<t} \|T(s)\Delta f - \Delta f\|_p.$$

Then, by Proposition 1.1.6(iii), $\mathcal{S}(\mathbb{R}^N)$ is contained in $D(A)$ and $Au = \Delta u$ for $u \in \mathcal{S}(\mathbb{R}^N)$. Moreover, by Theorem 1.3.16 it is a core for A. Let $u \in W^{2,p}(\mathbb{R}^N)$ and let $u_n \in \mathcal{S}(\mathbb{R}^N)$ be such that $u_n \to u$ in $W^{2,p}(\mathbb{R}^N)$. Then $Au_n = \Delta u_n \to \Delta u$ in $L^p(\mathbb{R}^N)$ and, since A is closed, $u \in D(A)$ and $Au = \Delta u$.

In the case of $C_b(\mathbb{R}^N)$ we argue differently because the Schwartz space is not dense in $C_b(\mathbb{R}^N)$ and in $C^2_b(\mathbb{R}^N)$. Instead, we use the identities $T(t)\Delta f = \Delta T(t) f = \frac{\partial}{\partial t} T(t) f$ which hold pointwise in $(0, +\infty) \times \mathbb{R}^N$. Setting $g = f - \Delta f$ we have

$$R(1, A)g = \int_0^{+\infty} e^{-t} T(t)(f - \Delta f) dt = \int_0^{+\infty} e^{-t} (I - \Delta) T(t) f dt$$

[2] We recall that $\mathcal{S}(\mathbb{R}^N)$ is the space of all the functions $f : \mathbb{R}^N \to \mathbb{R}$ such that $|x|^\alpha |D^\beta f(x)|$ tends to 0 as $|x|$ tends to $+\infty$ for any multiindices α and β; $C^\infty_0(\mathbb{R}^N)$ is the space of all compactly supported infinitely many times differentiable functions $f : \mathbb{R}^N \to \mathbb{R}$.

$$= \int_0^{+\infty} e^{-t}\left(I - \frac{\partial}{\partial t}\right) T(t)f dt = f,$$

by a simple integration by parts in the last identity and using the fact that $T(t)f \to f$ pointwise as $t \to 0^+$. This shows that $f \in D(A)$ and that $Af = \Delta f$.

(e) If $N = 1$ we already know that $D(A) = W^{2,p}(\mathbb{R})$ if $X = L^p(\mathbb{R})$, and $D(A) = C_b^2(\mathbb{R})$, if $X = C_b(\mathbb{R})$. The problem of giving an explicit characterization of $D(A)$ in terms of known functional spaces is more difficult if $N > 1$. The answer is nice, i.e. $D(A) = W^{2,p}(\mathbb{R}^N)$ if $X = L^p(\mathbb{R}^N)$ and $1 < p < +\infty$, but the proof is not easy for $p \neq 2$. For $p = 1$, $W^{2,1}(\mathbb{R}^N) \neq D(A)$ and for $p = +\infty$, $C_b^2(\mathbb{R}^N) \neq D(A)$ (see next Exercise 6 in §2.3.1).

Here we give an easy proof that the domain of A in $L^2(\mathbb{R}^N)$ is $H^2(\mathbb{R}^N)$.

The domain of A in L^2 is the closure of $\mathcal{S}(\mathbb{R}^N)$ with respect to the graph norm $u \mapsto \|u\|_{L^2(\mathbb{R}^N)} + \|\Delta u\|_{L^2(\mathbb{R}^N)}$, which is weaker than the H^2-norm. To conclude it suffices to show that the two norms are in fact equivalent on $\mathcal{S}(\mathbb{R}^N)$: indeed, in this case $D(A)$ is the closure of $\mathcal{S}(\mathbb{R}^N)$ in $H^2(\mathbb{R}^N)$, that is $H^2(\mathbb{R}^N)$. The main point to be proved is that $\|D_{ij}u\|_{L^2(\mathbb{R}^N)} \leq \|\Delta u\|_{L^2(\mathbb{R}^N)}$ for each $u \in \mathcal{S}(\mathbb{R}^N)$ and $i, j = 1, \ldots, N$. Integrating by parts twice we get

$$\begin{aligned} \|\,|D^2u|\,\|_2^2 &= \sum_{i,j=1}^N \int_{\mathbb{R}^N} D_{ij}u\overline{D_{ij}u}\,dx = -\sum_{i,j=1}^N \int_{\mathbb{R}^N} D_{ijj}u\overline{D_iu}\,dx \\ &= \sum_{i,j=1}^N \int_{\mathbb{R}^N} D_{ii}u\overline{D_{jj}u}\,dx = \|\Delta u\|_2^2. \end{aligned} \tag{2.11}$$

The L^2 norm of the first order derivatives of u may be estimated as follows. For $u \in H^2(\mathbb{R}^N)$, the identity

$$\int_{\mathbb{R}^N} \Delta u\,\overline{u}\,dx = -\int_{\mathbb{R}^N} |Du|^2 dx$$

yields $\|Du\|_2^2 \leq \|\Delta u\|_2\|u\|_2$, and this concludes the proof.

Exercises 2.3.1

1. (a) Using the Fourier transform show that $T(t)$ maps $\mathcal{S}(\mathbb{R}^N)$ into itself for each $t > 0$ and that

$$T(t)T(s)f = T(t+s)f, \quad t,\ s > 0,$$

for every $f \in \mathcal{S}(\mathbb{R}^N)$ and, hence, for every $f \in L^p(\mathbb{R}^N)$, $1 \leq p < +\infty$.

(b) Show that if $f_n, f \in C_b(\mathbb{R}^N)$, $f_n \to f$ pointwise and $\|f_n\|_\infty \leq C$, then $T(t)f_n \to T(t)f$ pointwise. Use this fact to prove the semigroup law in $C_b(\mathbb{R}^N)$.

2. Show that $BUC^2(\mathbb{R}^N)$ is a core of the Laplacian in $BUC(\mathbb{R}^N)$.

3. Use the Fourier transform to prove the resolvent estimate for the Laplacian in $L^2(\mathbb{R}^N)$, $\|u\|_{L^2(\mathbb{R}^N)} \leq \|f\|_2/\operatorname{Re}\lambda$, if $\operatorname{Re}\lambda > 0$ and $\|u\|_2 \leq \|f\|_2/|\operatorname{Im}\lambda|$ if $\operatorname{Im}\lambda \neq 0$, where $\lambda u - \Delta u = f$.

4. Prove that the Laplace operator is sectorial in $L^p(\mathbb{R}^N)$ and in $C_b(\mathbb{R}^N)$ with $\omega = 0$ and every $\theta < \pi$. [Hint: argue as in (c)].

5. (a) Using the representation formula (2.8), prove the following estimates for the heat semigroup $T(t)$ in $L^p(\mathbb{R}^N)$, $1 \le p \le +\infty$:

$$\|D^\alpha T(t)f\|_p \le \frac{c_\alpha}{t^{|\alpha|/2}}\|f\|_p \tag{2.12}$$

for every multiindex α, $1 \le p \le +\infty$ and suitable constants c_α.

(b) Let $0 < \theta < 1$, and let $C^\theta_b(\mathbb{R}^N)$ be the space of all functions f such that $[f]_{C^\theta_b(\mathbb{R}^N)} := \sup_{x\neq y} |f(x)-f(y)|/|x-y|^\theta < +\infty$. Use the fact that D_iG_t is odd with respect to x_i to prove that for each $f \in C^\theta_b(\mathbb{R}^N)$, and for each $i = 1,\dots,N$

$$\|D_iT(t)f\|_\infty \le \frac{C}{t^{1/2-\theta/2}}[f]_{C^\theta_b(\mathbb{R}^N)}, \quad t > 0.$$

(c) Use the estimates in (a) for $|\alpha| = 1$ to prove that

$$\|D_iu\|_X \le C_1t^{1/2}\|\Delta u\|_X + C_2t^{-1/2}\|u\|_X, \quad t > 0,$$

$$\|D_iu\|_X \le C_3\|\Delta u\|_X^{1/2}\|u\|_X^{1/2},$$

for $X = L^p(\mathbb{R}^N)$, $1 \le p < +\infty$, $X = C_b(\mathbb{R}^N)$, and u in the domain of the Laplacian in X.

6. (a) Let B be the unit ball of $\mathbb{R}^2$. Show that the function $u(x,y) = xy\log(x^2+y^2)$ belongs to $C^1(B)$ and that $u_{xx}, u_{yy} \in L^\infty(B)$ whereas $u_{xy} \notin L^\infty(B)$.

(b) Using the functions $u_\varepsilon(x,y) = xy\log(\varepsilon + x^2 + y^2)$, show that there exists no $C > 0$ such that $\|u\|_{C^2_b(\mathbb{R}^2)} \le C(\|u\|_\infty + \|\Delta u\|_\infty)$ for any $u \in C^\infty_0(\mathbb{R}^2)$. Deduce that the domain of the Laplacian in $C_b(\mathbb{R}^2)$ is not $C^2_b(\mathbb{R}^2)$.

2.4 The Dirichlet Laplacian in a bounded open set

Now we consider the realization of the Laplacian with Dirichlet boundary condition in $L^p(\Omega)$, $1 < p < +\infty$, where Ω is an open bounded set in $\mathbb{R}^N$ with C^2 boundary $\partial\Omega$. Even for $p = 2$ the theory is much more difficult than in the case $\Omega = \mathbb{R}^N$. In fact, the Fourier transform is useless, and estimates such as (2.11) are not available integrating by parts because boundary integrals appear.

In order to prove that the operator A_p defined by

$$D(A_p) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega), \qquad A_pu = \Delta u\ , \quad u \in D(A_p)$$

is sectorial, one shows that the resolvent set $\rho(A_p)$ contains a sector

$$S_\theta = \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg\lambda| < \theta\}$$

for some $\theta \in (\pi/2, \pi)$, and that the resolvent estimate

$$\|R(\lambda, A_p)\|_{\mathcal{L}(L^p(\Omega))} \le \frac{M}{|\lambda|}$$

holds for some $M > 0$ and for all $\lambda \in S_{\theta,\omega}$. The hard part is the proof of the existence of a solution $u \in D(A_p)$ to $\lambda u - \Delta u = f$, i.e. the following theorem that we state without any proof.

Theorem 2.4.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with C^2 boundary, and let $f \in L^p(\Omega)$, $\lambda \notin (-\infty, 0]$. Then, there is $u \in D(A_p)$ such that $\lambda u - \Delta u = f$, and the estimate*

$$\|u\|_{W^{2,p}} \le C\|f\|_p \tag{2.13}$$

holds, with C depending only upon Ω and λ.

The resolvent estimate is much easier. Its proof is quite simple for $p \ge 2$, and in fact we shall consider only this case. For $1 < p < 2$ the method still works, but some technical problems occur.

Proposition 2.4.2 *Let $2 \le p < +\infty$, let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge 0$ and let $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$, be such that $\lambda u - \Delta u = f \in L^p(\Omega)$. Then*

$$\|u\|_p \le \sqrt{1 + \frac{p^2}{4}}\,\frac{\|f\|_p}{|\lambda|}.$$

Proof. To simplify the notation, throughout the proof, we denote simply by $\|\cdot\|$ the usual L^p-norm.

If $u = 0$ the statement is obvious. If $u \ne 0$, we multiply the equation $\lambda u - \Delta u = f$ by $|u|^{p-2}\overline{u}$, which belongs to $W^{1,p'}(\Omega)$ (see Exercises 2.2.4), and we integrate over Ω. We have

$$\lambda\|u\|^p + \int_\Omega \sum_{k=1}^N \frac{\partial u}{\partial x_k}\,\frac{\partial}{\partial x_k}\Big(|u|^{p-2}\overline{u}\Big)dx = \int_\Omega f\,|u|^{p-2}\overline{u}\,dx.$$

Notice that

$$\frac{\partial}{\partial x_k}|u|^{p-2}\overline{u} = |u|^{p-2}\frac{\partial\overline{u}}{\partial x_k} + \frac{p-2}{2}\overline{u}|u|^{p-4}\left(\overline{u}\frac{\partial u}{\partial x_k} + u\frac{\partial\overline{u}}{\partial x_k}\right).$$

Setting

$$|u|^{(p-4)/2}\overline{u}\frac{\partial u}{\partial x_k} = a_k + ib_k, \quad k = 1, \dots, N,$$

with a_k, $b_k \in \mathbb{R}$, we have

$$\begin{aligned} &\int_\Omega \sum_{k=1}^N \frac{\partial u}{\partial x_k}\,\frac{\partial}{\partial x_k}\Big(|u|^{p-2}\overline{u}\Big)dx \\ = \;&\int_\Omega \sum_{k=1}^N \left((|u|^{(p-4)/2})^2 u\overline{u}\frac{\partial u}{\partial x_k}\frac{\partial\overline{u}}{\partial x_k} + \frac{p-2}{2}(|u|^{(p-4)/2})^2\overline{u}\frac{\partial u}{\partial x_k}\left(\overline{u}\frac{\partial u}{\partial x_k} + u\frac{\partial\overline{u}}{\partial x_k}\right)\right)dx \\ = \;&\int_\Omega \sum_{k=1}^N \Big(a_k^2 + b_k^2 + (p-2)a_k(a_k + ib_k)\Big)dx, \end{aligned}$$

whence

$$\lambda\|u\|^p + \int_\Omega \sum_{k=1}^N ((p-1)a_k^2 + b_k^2)dx + i(p-2)\int_\Omega \sum_{k=1}^N a_k b_k\,dx = \int_\Omega f|u|^{p-2}\overline{u}\,dx.$$

Taking the real part we get

$$\mathrm{Re}\,\lambda\|u\|^p + \int_\Omega \sum_{k=1}^N ((p-1)a_k^2 + b_k^2)dx = \mathrm{Re}\int_\Omega f|u|^{p-2}\overline{u}\,dx \le \|f\|\,\|u\|^{p-1},$$

and then

$$\begin{cases} (a) \quad \mathrm{Re}\,\lambda\,\|u\| \le \|f\|; \\ \\ (b) \quad \displaystyle\int_\Omega \sum_{k=1}^N ((p-1)a_k^2 + b_k^2)dx \le \|f\|\,\|u\|^{p-1}. \end{cases}$$

Taking the imaginary part we get

$$\mathrm{Im}\,\lambda\,\|u\|^p + (p-2)\int_\Omega \sum_{k=1}^N a_k b_k\,dx = \mathrm{Im}\int_\Omega f|u|^{p-2}\overline{u}\,dx$$

and then

$$|\mathrm{Im}\,\lambda|\,\|u\|^p \le \frac{p-2}{2}\int_\Omega \sum_{k=1}^N (a_k^2 + b_k^2)dx + \|f\|\,\|u\|^{p-1},$$

so that, using (b),

$$|\mathrm{Im}\,\lambda|\,\|u\|^p \le \left(\frac{p-2}{2} + 1\right)\|f\|\,\|u\|^{p-1},$$

i.e.,

$$|\mathrm{Im}\,\lambda|\,\|u\| \le \frac{p}{2}\|f\|.$$

From this inequality and from (a), squaring and summing, we obtain

$$|\lambda|^2\|u\|^2 \le \left(1 + \frac{p^2}{4}\right)\|f\|^2,$$

and the statement follows. □

2.5 More general operators

In this section we state without proofs some important theorems about generation of analytic semigroups by second order strongly elliptic operators. Roughly speaking, the realizations of elliptic operators with good coefficients and good boundary conditions are sectorial in the most common functional spaces. This is the reason why the general theory has a wide range of applications.

Let us consider general second order elliptic operators in an open set $\Omega \subset \mathbb{R}^N$. Ω is either the whole $\mathbb{R}^N$ or a bounded open set with C^2 boundary $\partial\Omega$. Let us denote by $n(x)$ the outer unit vector normal to $\partial\Omega$ at x.

Let $\mathcal{A}$ be the differential operator

$$(\mathcal{A}u)(x) = \sum_{i,j=1}^N a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^N b_i(x)D_iu(x) + c(x)u(x) \tag{2.14}$$

with real, bounded and continuous coefficients a_{ij}, b_i, c on $\overline{\Omega}$. We assume that for every $x \in \overline{\Omega}$ the matrix $[a_{ij}(x)]_{i,j=1,\dots,N}$ is symmetric and strictly positive definite, i.e.,

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^N, \tag{2.15}$$

for some $\nu > 0$. Moreover, if $\Omega = \mathbb{R}^N$ we need that the leading coefficients a_{ij} are uniformly continuous.

The following results hold.

Theorem 2.5.1 (S. Agmon, [1]) *Let $p \in (1, +\infty)$.*

(i) Let $A_p : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ be defined by $A_p u = \mathcal{A}u$. The operator A_p is sectorial in $L^p(\mathbb{R}^N)$ and $D(A_p)$ is dense in $L^p(\mathbb{R}^N)$.

(ii) Let Ω and $\mathcal{A}$ be as above, and let A_p be defined by

$$D(A_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \quad A_p u = \mathcal{A}u.$$

Then, the operator A_p is sectorial in $L^p(\Omega)$, and $D(A_p)$ is dense in $L^p(\Omega)$.

(iii) Let Ω and $\mathcal{A}$ be as above, and let A_p be defined by

$$D(A_p) = \{u \in W^{2,p}(\Omega) : \mathcal{B}u_{|\partial\Omega} = 0\}, \quad A_p u = \mathcal{A}u, \quad u \in D(A_p),$$

where

$$(\mathcal{B}u)(x) = b_0(x)u(x) + \sum_{i=1}^{N} b_i(x)D_i u(x), \tag{2.16}$$

the coefficients b_i, $i = 1, \dots, N$ are in $C^1(\overline{\Omega})$ and the transversality condition

$$\sum_{i=1}^{N} b_i(x)n_i(x) \neq 0, \quad x \in \partial\Omega \tag{2.17}$$

holds. Then, the operator A_p is sectorial in $L^p(\Omega)$, and $D(A_p)$ is dense in $L^p(\Omega)$.

We have also the following result.

Theorem 2.5.2 (H.B. Stewart, [16, 17]) *Let $\mathcal{A}$ be the differential operator in* (2.14).

(i) Consider the operator $A : D(A) \to X = C_b(\mathbb{R}^N)$ defined by

$$\begin{cases} D(A) = \{u \in C_b(\mathbb{R}^N) \bigcap_{1 \leq p < +\infty} W^{2,p}_{loc}(\mathbb{R}^N) : \mathcal{A}u \in C_b(\mathbb{R}^N)\}, \\ Au = \mathcal{A}u, \quad u \in D(A). \end{cases} \tag{2.18}$$

Then, A is sectorial in X, and $\overline{D(A)} = BUC(\mathbb{R}^N)$.

(ii) Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with C^2 boundary $\partial\Omega$, and consider the operator

$$\begin{cases} D(A) = \{u \in \bigcap_{1 \leq p < +\infty} W^{2,p}(\Omega) : u_{|\partial\Omega} = 0,\ \mathcal{A}u \in C(\overline{\Omega})\}, \\ Au = \mathcal{A}u, \quad u \in D(A). \end{cases} \tag{2.19}$$

Then, the operator A is sectorial in X, and $\overline{D(A)} = C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) :\ u =$ 0 at $\partial\Omega\}$.

(iii) Let Ω be as in (ii), and let $X = C(\overline{\Omega})$,

$$\begin{cases} D(A) = \{u \in \bigcap_{1\le p<+\infty} W^{2,p}(\Omega) : \mathcal{B}u_{|\partial\Omega} = 0,\ \mathcal{A}u \in C(\overline{\Omega})\}, \\ Au = \mathcal{A}u, \quad u \in D(A), \end{cases} \tag{2.20}$$

where $\mathcal{B}$ is defined in (2.16) and the coefficients b_i, $i = 1, \dots, N$ are in $C^1(\overline{\Omega})$ and satisfy (2.17). Then, the operator A is sectorial in X, and $D(A)$ is dense in X.

Moreover, in all the cases above there is $M > 0$ such that $\lambda \in S_{\theta,\omega}$ implies

$$\|D_i R(\lambda, A) f\|_\infty \le \frac{M}{|\lambda|^{1/2}} \|f\|_\infty, \quad f \in X,\ i = 1, \dots, N. \tag{2.21}$$

Exercises 2.5.3

1. Show that if $p \ge 2$ and $u \in W^{1,p}(\Omega)$ then the function $|u|^{p-2}u$ belongs to $W^{1,p'}(\Omega)$. Is this true for $1 < p < 2$?

2. Let A be the Laplacian in $L^2(\mathbb{R}^N)$ with domain $D(A) = H^2(\mathbb{R}^N)$. Prove that the operator $-A^2$ is sectorial in $L^2(\mathbb{R}^N)$ and characterize its domain.

Chapter 3

Intermediate spaces

3.1 The interpolation spaces $D_A(\alpha,\infty)$

Let $A : D(A) \subset X \to X$ be a sectorial operator, and set

$$M_0 = \sup_{0<t\leq 1} \|e^{tA}\|, \;\; M_1 = \sup_{0<t\leq 1} \|tAe^{tA}\|. \tag{3.1}$$

We have seen in Proposition 1.3.6 that for all $x \in \overline{D(A)}$ the function $t \mapsto u(t) = e^{tA}x$ belongs to $C([0,T];X)$, and for all $x \in D(A)$ such that $Ax \in \overline{D(A)}$, it belongs to $C^1([0,T];X)$. We also know that for $x \in X$ the function $t \mapsto v(t) = \|Ae^{tA}x\|$ has in general a singularity of order 1 as $t \to 0^+$, whereas for $x \in D(A)$ it is bounded near 0. It is then natural to raise the following related questions:

1. how can we characterize the class of initial data such that the function $u(t) = e^{tA}x$ has an intermediate regularity, e.g., it is α-Hölder continuous for some $0 < \alpha < 1$?

2. how can we characterize the class of initial data x such that the function $t \mapsto \|Ae^{tA}x\|$ has a singularity of order α, with $0 < \alpha < 1$?

To answer such questions, we introduce some intermediate Banach spaces between X and $D(A)$.

Definition 3.1.1 *Let $A : D(A) \subset X \to X$ be a sectorial operator, and fix $0 < \alpha < 1$. We set*

$$\begin{cases} D_A(\alpha,\infty) = \{x \in X : \; [x]_\alpha = \sup_{0<t\leq 1} \|t^{1-\alpha}Ae^{tA}x\| < +\infty\}, \\ \\ \|x\|_{D_A(\alpha,\infty)} = \|x\| + [x]_\alpha. \end{cases} \tag{3.2}$$

Note that what characterizes $D_A(\alpha,\infty)$ is the behavior of $\|t^{1-\alpha}Ae^{tA}x\|$ near $t = 0$. Indeed, for $0 < a < b < +\infty$ and for each $x \in X$, estimate (1.15) with $k = 1$ implies that $\sup_{a\leq t\leq b} \|t^{1-\alpha}Ae^{tA}x\| \leq C\|x\|$, with $C = C(a,b,\alpha)$. Therefore, the interval $(0,1]$ in the definition of $D_A(\alpha,\infty)$ could be replaced by any $(0,T]$ with $T > 0$, and for each $T > 0$ the norm $x \mapsto \|x\| + \sup_{0<t\leq T} \|t^{1-\alpha}Ae^{tA}x\|$ is equivalent to the norm in (3.2).

Once we have an estimate for $\|Ae^{tA}\|_{\mathcal{L}(D_A(\alpha,\infty);X)}$ we easily obtain estimates for $\|A^k e^{tA}\|_{\mathcal{L}(D_A(\alpha,\infty);X)}$ for every $k \in \mathbb{N}$, just using the semigroup law and (1.15). For instance for $k = 2$ and for each $x \in D_A(\alpha,\infty)$ we obtain

$$\sup_{0<t\leq T} \|t^{2-\alpha}A^2e^{tA}x\| \leq \sup_{0<t\leq T} \|tAe^{t/2\,A}\|_{\mathcal{L}(X)} \|t^{1-\alpha}Ae^{t/2\,A}x\| \leq C\|x\|_{D_A(\alpha,\infty)}.$$

It is clear that if $x \in D_A(\alpha,\infty)$ and $T>0$, then the function $s \mapsto \|Ae^{sA}x\|$ belongs to $L^1(0,T)$, so that, by Proposition 1.3.6(ii),

$$e^{tA}x - x = \int_0^t Ae^{sA}x\,ds, \quad t \ge 0, \quad x = \lim_{t\to 0} e^{tA}x.$$

In particular, all the spaces $D_A(\alpha,\infty)$ are contained in the closure of $D(A)$. The following inclusions follow, with continuous embeddings:

$$D(A) \subset D_A(\alpha,\infty) \subset D_A(\beta,\infty) \subset \overline{D(A)}, \;\; 0<\beta<\alpha<1.$$

Proposition 3.1.2 *For $0<\alpha<1$ the equality*

$$D_A(\alpha,\infty) = \{x \in X: \; [[x]]_{D_A(\alpha,\infty)} = \sup_{0<t\le 1} t^{-\alpha}\|e^{tA}x - x\| < +\infty\}$$

holds, and the norm

$$x \mapsto \|x\| + [[x]]_{D_A(\alpha,\infty)}$$

is equivalent to the norm of $D_A(\alpha,\infty)$.

Proof. Let $x \in D_A(\alpha,\infty)$ be given. For $0<t\le 1$ we have

$$t^{-\alpha}(e^{tA}x - x) = t^{-\alpha}\int_0^t s^{1-\alpha}Ae^{sA}x\frac{1}{s^{1-\alpha}}\,ds, \tag{3.3}$$

so that

$$[[x]]_{D_A(\alpha,\infty)} = \sup_{0<t\le 1}\|t^{-\alpha}(e^{tA}x - x)\| \le \alpha^{-1}[x]_{D_A(\alpha,\infty)}. \tag{3.4}$$

Conversely, let $[[x]]_{D_A(\alpha,\infty)} < +\infty$, and write

$$Ae^{tA}x = Ae^{tA}\frac{1}{t}\int_0^t (x - e^{sA}x)ds + e^{tA}\frac{1}{t}A\int_0^t e^{sA}xds.$$

It follows that

$$\|t^{1-\alpha}Ae^{tA}x\| \le t^{1-\alpha}\frac{M_1}{t^2}\int_0^t s^\alpha\frac{\|x - e^{sA}x\|}{s^\alpha}ds + M_0t^{-\alpha}\|e^{tA}x - x\|, \tag{3.5}$$

and the function $s \mapsto \|x - e^{sA}x\|/s^\alpha$ is bounded, so that $t \mapsto t^{1-\alpha}Ae^{tA}x$ is also bounded, and

$$[x]_{D_A(\alpha,\infty)} = \sup_{0<t\le 1}\|t^{1-\alpha}Ae^{tA}x\| \le (M_1(\alpha+1)^{-1} + M_0)[[x]]_{D_A(\alpha,\infty)}. \tag{3.6}$$

We can conclude that the seminorms $[\,\cdot\,]_{D_A(\alpha,\infty)}$ and $[[\,\cdot\,]]_{D_A(\alpha,\infty)}$ are equivalent. □

The next corollary follows from the semigroup law, and it gives an answer to the first question at the beginning of this section.

Corollary 3.1.3 *Given $x \in X$, the function $t \mapsto e^{tA}x$ belongs to $C^\alpha([0,1];X)$ if and only if x belongs to $D_A(\alpha,\infty)$. In this case, $t \mapsto e^{tA}x$ belongs to $C^\alpha([0,T];X)$ for every $T>0$.*

Proof. The proof follows from the equality

$$e^{tA}x - e^{sA}x = e^{sA}(e^{(t-s)A}x - x), \quad 0 \le s < t,$$

recalling that $\|e^{\xi A}\|_{\mathcal{L}(X)}$ is bounded by a constant independent of ξ if ξ runs in any bounded interval. □

It is easily seen that the spaces $D_A(\alpha,\infty)$ are Banach spaces. Moreover, it can be proved that they do not depend explicitly on the operator A, but only on its domain $D(A)$ and on the graph norm of A. More precisely, for every sectorial operator $B : D(B) \to X$ such that $D(B) = D(A)$, with equivalent graph norms, the equality $D_A(\alpha,\infty) = D_B(\alpha,\infty)$ holds, with equivalent norms.

Starting from $D_A(\alpha,\infty)$ we define other normed spaces, as follows.

Definition 3.1.4 *Let $A : D(A) \subset X \to X$ be a sectorial operator. For any $k \in \mathbb{N}$ and any $\alpha \in (0,1)$ we set*

$$\begin{cases} D_A(k+\alpha,\infty) = \{x \in D(A^k) : A^k x \in D_A(\alpha,\infty)\}, \\ \\ \|x\|_{D_A(k+\alpha,\infty)} = \|x\|_{D(A^k)} + [A^k x]_\alpha. \end{cases} \tag{3.7}$$

Corollary 3.1.3 yields that the function $t \mapsto u(t) := e^{tA}x$ belongs to $C^\alpha([0,1];D(A))$ (and then to $C^\alpha([0,T];D(A))$ for all $T > 0$) if and only if x belongs to $D_A(1+\alpha,\infty)$. Similarly, since $\frac{d}{dt}e^{tA}x = e^{tA}Ax$ for $x \in D(A)$, u belongs to $C^{1+\alpha}([0,1];X)$ (and then to $C^{1+\alpha}([0,T];X)$ for all $T > 0$) if and only if x belongs to $D_A(1+\alpha,\infty)$.

An important feature of spaces $D_A(\alpha,\infty)$ is that the part of A in $D_A(\alpha,\infty)$, i.e.

$$A_\alpha : D_A(1+\alpha,\infty) \to D_A(\alpha,\infty), \;\; A_\alpha x = Ax,$$

is a sectorial operator.

Proposition 3.1.5 *For $0 < \alpha < 1$ the resolvent set of A_α contains $\rho(A)$, the restriction of $R(\lambda,A)$ to $D_A(\alpha,\infty)$ is $R(\lambda,A_\alpha)$, and the inequality*

$$\|R(\lambda,A_\alpha)\|_{\mathcal{L}(D_A(\alpha,\infty))} \le \|R(\lambda,A)\|_{\mathcal{L}(X)}$$

holds for every $\lambda \in \rho(A)$. In particular, A_α is a sectorial operator in $D_A(\alpha,\infty)$ and e^{tA_α} is the restriction of e^{tA} to $D_A(\alpha,\infty)$.

Proof. Fix $\lambda \in \rho(A)$ and $x \in D_A(\alpha,\infty)$. The resolvent equation $\lambda y - Ay = x$ has a unique solution $y \in D(A)$, and since $D(A) \subset D_A(\alpha,\infty)$ then $Ay \in D_A(\alpha,\infty)$ and therefore $y = R(\lambda,A)x \in D_A(1+\alpha,\infty)$.

Moreover for $0 < t \le 1$ the inequality

$$\|t^{1-\alpha}Ae^{tA}R(\lambda,A)x\| = \|R(\lambda,A)t^{1-\alpha}Ae^{tA}x\| \le \|R(\lambda,A)\|_{\mathcal{L}(X)}\|t^{1-\alpha}Ae^{tA}x\|$$

holds. Therefore,

$$[R(\lambda,A)x]_{D_A(\alpha,\infty)} \le \|R(\lambda,A)\|_{\mathcal{L}(X)}[x]_{D_A(\alpha,\infty)},$$

and the claim is proved. □

Let us see an interpolation property of the spaces $D_A(\alpha,\infty)$.

Proposition 3.1.6 *Let M_0, M_1 be the constants in* (3.1)*. For every $x\in D(A)$ we have*

$$[x]_{D_A(\alpha,\infty)} \le M_0^\alpha M_1^{1-\alpha}\|Ax\|^\alpha\|x\|^{1-\alpha}.$$

Proof. For all $t\in(0,1)$ we have

$$\|t^{1-\alpha}Ae^{tA}x\| \le \begin{cases} M_0 t^{1-\alpha}\|Ax\|, \\ \\ M_1 t^{-\alpha}\|x\|. \end{cases}$$

It follows that

$$\|t^{1-\alpha}Ae^{tA}x\| \le (M_0t^{1-\alpha}\|Ax\|)^\alpha (M_1 t^{-\alpha}\|x\|)^{1-\alpha} = M_0^\alpha M_1^{1-\alpha}\|Ax\|^\alpha\|x\|^{1-\alpha}.$$

□

An immediate consequence of Proposition 3.1.6 are estimates for $\|e^{tA}\|_{\mathcal{L}(X,D_A(\alpha,\infty))}$ and more generally for $\|A^ne^{tA}\|_{\mathcal{L}(X,D_A(\alpha,\infty))}$, $n\in\mathbb{N}$: indeed, for each $x\in X$ and $t>0$, $e^{tA}x$ belongs to $D(A)$, so that

$$\|e^{tA}x\|_{D_A(\alpha,\infty)} \le M_0^\alpha M_1^{1-\alpha}\|Ae^{tA}x\|^\alpha\|e^{tA}x\|^{1-\alpha} \le \frac{C_T}{t^\alpha}\|x\|, \quad 0<t\le T, \tag{3.8}$$

and similarly, for each $n\in\mathbb{N}$,

$$\sup_{0<t\le T}\|t^{n+\alpha}A^ne^{tA}\|_{\mathcal{L}(X,D_A(\alpha,\infty))} < +\infty. \tag{3.9}$$

Let us discuss in detail a fundamental example. We recall that for any open set $\Omega\subset\mathbb{R}^N$ and any $\theta\in(0,1)$ the Hölder space $C^\theta_b(\overline{\Omega})$ consists of the bounded functions $f:\overline{\Omega}\to\mathbb{C}$ such that

$$[f]_{C^\theta_b(\Omega)} = \sup_{x,y\in\overline{\Omega},\, x\ne y}\frac{|f(x)-f(y)|}{|x-y|^\theta} < +\infty,$$

and it is a Banach space with the norm

$$\|f\|_{C^\theta_b(\overline{\Omega})} = \|f\|_\infty + [f]_{C^\theta_b(\overline{\Omega})}.$$

Moreover, for $k\in\mathbb{N}$, $C^{k+\theta}_b(\overline{\Omega})$ denotes the space of all the functions f which are differentiable up to the k-th order in $\overline{\Omega}$, with bounded derivatives, and such that $D^\alpha f\in C^\theta_b(\overline{\Omega})$ for any multiindex α with $|\alpha|=k$. It is a Banach space with the norm

$$\|f\|_{C^{k+\theta}_b(\overline{\Omega})} = \sum_{|\alpha|\le k}\|D^\alpha f\|_\infty + \sum_{|\alpha|=k}[D^\alpha f]_{C^\theta_b(\overline{\Omega})}.$$

We drop the index b when Ω is bounded.

Example 3.1.7 *Let us consider $X=C_b(\mathbb{R}^N)$, and let $A:D(A)\to X$ be the realization of the Laplacian in X. For $0<\alpha<1$, $\alpha\ne 1/2$, we have*

$$D_A(\alpha,\infty) = C^{2\alpha}_b(\mathbb{R}^N), \tag{3.10}$$

$$D_A(1+\alpha,\infty) = C^{2+2\alpha}_b(\mathbb{R}^N), \tag{3.11}$$

with equivalence of the respective norms.

Proof. We prove the statement for $\alpha < 1/2$. Let $T(t)$ be the heat semigroup, given by formula (2.8). We recall that for each $f \in C_b(\mathbb{R}^N)$ we have

$$(a)\ \| \, |DT(t)f| \, \|_\infty \leq \frac{c}{\sqrt{t}} \|f\|_\infty, \quad (b)\ \|AT(t)f\|_\infty \leq \frac{c}{t}\|f\|_\infty, \tag{3.12}$$

for some $c > 0$, by (2.12).

Let us first prove the inclusion $D_A(\alpha,\infty) \supset C^{2\alpha}_b(\mathbb{R}^N)$. If $f \in C^{2\alpha}_b(\mathbb{R}^N)$ we write

$$(T(t)f)(x) - f(x) = \frac{1}{(4\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4}} \big[f(x-\sqrt{t}y) - f(x)\big]\, dy,$$

and we get

$$\|T(t)f - f\|_\infty \leq \frac{1}{(4\pi)^{N/2}} [f]_{C^{2\alpha}_b} t^\alpha \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4}} |y|^{2\alpha} dy.$$

Therefore, $f \in D_A(\alpha,\infty)$ and $[[f]]_{D_A(\alpha,\infty)} \leq C[f]_{C^{2\alpha}_b(\mathbb{R}^N)}$.

Conversely, let $f \in D_A(\alpha,\infty)$. Then, for every $t > 0$ we have

$$\begin{aligned} |f(x) - f(y)| &\leq |T(t)f(x) - f(x)| + |T(t)f(x) - T(t)f(y)| + |T(t)f(y) - f(y)| \\ &\leq 2[[f]]_{D_A(\alpha,\infty)} t^\alpha + \| \, |DT(t)f| \, \|_\infty |x-y|. \end{aligned} \tag{3.13}$$

We want to choose $t = |x-y|^2$ to get the statement, but estimate (3.12)(a) is not sufficient for this purpose. To get a better estimate we use the equality

$$T(n)f - T(t)f = \int_t^n AT(s)f\, ds, \quad 0 < t < n,$$

that implies, for each $i = 1, \ldots, N$,

$$D_iT(n)f - D_iT(t)f = \int_t^n D_iAT(s)f\, ds, \quad 0 < t < n. \tag{3.14}$$

Note that $\|AT(t)f\|_\infty \leq t^{\alpha-1}[f]_\alpha$ for $0 < t \leq 1$ by definition, and $\|AT(t)f\|_\infty \leq Ct^{-1}\|f\|_\infty \leq Ct^{\alpha-1}\|f\|_\infty$ for $t \geq 1$ by (3.12)(b). Using this estimate and (3.12)(a) we get

$$\begin{aligned} \|D_iAT(s)f\|_\infty &= \|D_iT(s/2)AT(s/2)f\|_\infty \leq \|D_iT(s/2)\|_{\mathcal{L}(C_b(\mathbb{R}^N))} \|AT(s/2)f\|_\infty \\ &\leq \frac{C}{s^{3/2-\alpha}} \|f\|_{D_A(\alpha,\infty)} \end{aligned}$$

so that we may let $n \to +\infty$ in (3.14), to get

$$D_iT(t)f = -\int_t^{+\infty} D_iAT(s)f\, ds, \quad t > 0,$$

and

$$\|D_iT(t)f\|_\infty \leq \|f\|_{D_A(\alpha,\infty)} \int_t^{+\infty} \frac{C}{s^{3/2-\alpha}}\, ds = \frac{C(\alpha)}{t^{1/2-\alpha}} \|f\|_{D_A(\alpha,\infty)}. \tag{3.15}$$

This estimate is what we need for (3.13) to prove that f is 2α-Hölder continuous. For $|x-y| \leq 1$ choose $t = |x-y|^2$ to get

$$|f(x) - f(y)| \leq 2[[f]]_{D_A(\alpha,\infty)}|x-y|^{2\alpha} + C(\alpha)\|f\|_{D_A(\alpha,\infty)}|x-y|^{2\alpha}$$

$$\leq \quad C\|f\|_{D_A(\alpha,\infty)}|x-y|^{2\alpha}.$$

If $|x-y|\geq 1$ then $|f(x)-f(y)|\leq 2\|f\|_\infty \leq 2\|f\|_{D_A(\alpha,\infty)}|x-y|^{2\alpha}$.

Let us prove (3.11). The embedding $C_b^{2+2\alpha}(\mathbb{R}^N)\subset D_A(1+\alpha,\infty)$ is an obvious consequence of (3.10), since $C_b^{2+2\alpha}(\mathrm{Re}^N)\subset D(A)$. To prove the other embedding we have to show that the functions in $D_A(1+\alpha,\infty)$ have second order derivatives belonging to $C_b^{2\alpha}(\mathbb{R}^N)$.

Fix any $\lambda>0$ and any $f\in D_A(1+\alpha,\infty)$. Then $f=R(\lambda,A)g$ where $g:=\lambda f-\Delta f\in D_A(\alpha,\infty)=C_b^{2\alpha}(\mathbb{R}^N)$, and by (1.22) we have

$$f(x)=\int_0^{+\infty}e^{-\lambda t}(T(t)g)(x)dt, \quad x\in\mathbb{R}^N.$$

We can differentiate twice with respect to x, because for each $i,\ j=1,\ldots,N$, both $\|e^{-\lambda t}D_iT(t)g\|_\infty$ and $\|e^{-\lambda t}D_{ij}T(t)g\|_\infty$ are integrable in $(0,+\infty)$. Indeed, (3.15) implies $\|D_iT(t/2)g\|_\infty\leq C(\alpha)(t/2)^{\alpha-1/2}$ for every i, so that using once again (3.12)(a) we get

$$\begin{aligned}\|D_{ij}T(t)g\|_\infty &= \|D_jT(t/2)D_iT(t/2)g\|_\infty\\ &\leq \frac{c}{(t/2)^{1/2}}\frac{C(\alpha)}{(t/2)^{1/2-\alpha}}\|g\|_{D_A(\alpha,\infty)}\\ &= \frac{k}{t^{1-\alpha}}\|g\|_{D_A(\alpha,\infty)}.\end{aligned}\tag{3.16}$$

Therefore, the integral $\int_0^{+\infty}e^{-\lambda t}T(t)g\,dt$ is well defined as a $C_b^2(\mathbb{R}^N)$-valued integral, so that $f\in C_b^2(\mathbb{R}^N)$. We could go on estimating the seminorm $[D_{ij}T(t)g]_{C_b^{2\alpha}(\mathbb{R}^N)}$, but we get $[D_{ij}T(t)g]_{C_b^{2\alpha}(\mathbb{R}^N)}\leq C\|g\|_{D_A(\alpha,\infty)}/t$, and it is not obvious that the integral is well defined as a $C_b^{2+2\alpha}(\mathbb{R}^N)$-valued integral. So, we have to choose another approach. Since we already know that $D_A(\alpha,\infty)=C_b^{2\alpha}(\mathbb{R}^N)$, it is sufficient to prove that $D_{ij}f\in D_A(\alpha,\infty)$, i.e. that

$$\sup_{0<\xi\leq 1}\|\xi^{1-\alpha}AT(\xi)D_{ij}f\|_\infty<+\infty, \quad i,j=1,\ldots,N.$$

Let k be the constant in formula (3.16). Using (3.16) and (3.12)(b), for each $\xi\in(0,1)$ we get

$$\begin{aligned}\|\xi^{1-\alpha}AT(\xi)D_{ij}f\|_\infty &= \left\|\int_0^{+\infty}\xi^{1-\alpha}e^{-\lambda t}AT(\xi+t/2)D_{ij}T(t/2)g\,dt\right\|_\infty\\ &\leq \|g\|_{D_A(\alpha,\infty)}\int_0^{+\infty}\xi^{1-\alpha}\frac{ck}{(\xi+t/2)(t/2)^{1-\alpha}}\,dt\\ &= \|g\|_{D_A(\alpha,\infty)}\int_0^{+\infty}\frac{2ck}{(1+s)s^{1-\alpha}}\,ds.\end{aligned}\tag{3.17}$$

Therefore, all the second order derivatives of f are in $D_A(\alpha,\infty)=C_b^{2\alpha}(\mathbb{R}^N)$, their $C_b^{2\alpha}$ norm is bounded by $C\|g\|_\alpha\leq C(\lambda\|f\|_\alpha+\|\Delta f\|_\alpha)\leq\max\{\lambda C,C\}\|f\|_{D_A(1+\alpha,\infty)}$, and the statement follows. $\square$

Remark 3.1.8 The case $\alpha=1/2$ is more delicate. In fact, the inclusion $\mathrm{Lip}(\mathbb{R}^N)\subset D_A(1/2,\infty)$ follows as in the first part of the proof, but it is strict. Indeed, it is possible to prove that

$$D_A(1/2,\infty)=\left\{u\in C_b(\mathbb{R}^N):\sup_{x\neq y}\frac{|u(x)+u(y)-2u((x+y)/2)|}{|x-y|}<+\infty\right\},$$

and this space is strictly larger than $\mathrm{Lip}(\mathbb{R}^N)$ (see [19]).

Example 3.1.7 and Corollary 3.1.3 imply that the solution $u(t,x) = (T(t)u_0)(x)$ of the Cauchy problem for the heat equation in $\mathbb{R}^N$,

$$\begin{cases} u_t(t,x) = \Delta u(t,x), & t > 0, \quad x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), & \qquad\quad\;\; x \in \mathbb{R}^N, \end{cases}$$

is α-Hölder continuous with respect to t on $[0,T]\times\mathbb{R}^N$, with Hölder constant independent of x, if and only if the initial datum u_0 belongs to $C^{2\alpha}_b(\mathbb{R}^N)$. In this case, Proposition 3.1.5 implies that $\|u(t,\cdot)\|_{D_A(\alpha,\infty)} \le C\|u_0\|_{D_A(\alpha,\infty)}$ for $0 \le t \le T$, so that u is 2α-Hölder continuous with respect to x as well, with Hölder constant independent of t. We say that u belongs to the parabolic Hölder space $C^{\alpha,2\alpha}([0,T]\times\mathbb{R}^N)$, for all $T > 0$.

This is a first example of a typical feature of second order parabolic partial differential equations: time regularity implies space regularity, and the degree of regularity with respect to time is one half of the regularity with respect to the space variables.

Moreover, Example 3.1.7 gives an alternative proof of the classical Schauder Theorem for the Laplacian (see e.g. [7, ch. 6]).

Theorem 3.1.9 *If $u \in C^2_b(\mathbb{R}^N)$ and $\Delta u \in C^\alpha_b(\mathbb{R}^N)$ for some $\alpha \in (0,1)$, then $u \in C^{2+\alpha}_b(\mathbb{R}^N)$.*

Proof. In fact such a u belongs to $D_A(1+\alpha/2,\infty) = C^{2+\alpha}_b(\mathbb{R}^N)$. □

As a consequence of Proposition 3.1.5 and of Example 3.1.7 we also obtain that the Laplacian with domain $C^{2+\alpha}_b(\mathbb{R}^N)$ is sectorial in $C^\alpha_b(\mathbb{R}^N)$ for every $\alpha \in (0,1)$. The proof follows immediately from the equalities

$$D_\Delta(1+\alpha/2,\infty) = C^{2+\alpha}_b(\mathbb{R}^N), \;\; D_\Delta(\alpha/2,\infty) = C^\alpha_b(\mathbb{R}^N).$$

A characterization of the spaces $D_A(\alpha,\infty)$ for general second order elliptic operators is similar to the above one, but the proof is less elementary since it relies on the deep results of Theorem 2.5.2 and on general interpolation techniques.

Theorem 3.1.10 *Let $\alpha \in (0,1)$, $\alpha \ne 1/2$. The following statements hold.*

(i) Let $X = C_b(\mathbb{R}^N)$, and let A be defined by (2.18). Then, $D_A(\alpha,\infty) = C^{2\alpha}_b(\mathbb{R}^n)$, with equivalence of the norms.

(ii) Let Ω be an open bounded set of $\mathbb{R}^N$ with C^2 boundary, let $X = C(\overline{\Omega})$, and let A be defined by (2.19). Then,

$$D_A(\alpha,\infty) = C^{2\alpha}_0(\overline{\Omega}) := \{f \in C^{2\alpha}(\overline{\Omega}) : \, f_{|\partial\Omega} = 0\},$$

with equivalence of the norms.

(iii) Let Ω be an open bounded set of $\mathbb{R}^N$ with C^2 boundary, let $X = C(\overline{\Omega})$, and let A be defined by (2.20). Then

$$D_A(\alpha,\infty) = \begin{cases} C^{2\alpha}(\overline{\Omega}), & \text{if } 0 < \alpha < 1/2, \\ \{f \in C^{2\alpha}(\overline{\Omega}) : \, \mathcal{B}f_{|\partial\Omega} = 0\}, & \text{if } 1/2 < \alpha < 1, \end{cases}$$

with equivalence of the norms.

Remark 3.1.11 Proposition 3.1.5 and Theorem 3.1.10(ii) show that, for any $\alpha \in (0,1)$, the operator $A : \{u \in C^{2+2\alpha}([0,1]) : u(0) = u''(0) = u(1) = u''(1) = 0\} \to C^{2\alpha}_0([0,1])$, $Au = u''$ is sectorial. This result should be compared with Exercise 2.1.3(5) which states that the realization of the second order derivative with Dirichlet boundary condition in $C^{2\alpha}([0,1])$ is not sectorial.

Exercises 3.1.12

1. Show that if $\omega < 0$ in Definition 1.3.1 then $D_A(\alpha,\infty) = \{x \in X : \ |x|_\alpha = \sup_{t>0} \|t^{1-\alpha}Ae^{tA}x\| < +\infty\}$, and that $x \mapsto |x|_\alpha$ is an equivalent norm in $D_A(\alpha,\infty)$ for each $\alpha \in (0,1)$. What about $\omega = 0$?
2. Show that $D_A(\alpha,\infty) = D_{A+\lambda I}(\alpha,\infty)$ for each $\lambda \in \mathbb{R}$ and $\alpha \in (0,1)$, with equivalence of the norms.
3. Show that $D_A(\alpha,\infty)$ is a Banach space.
4. Show that
$$D_A(\alpha,\infty) = D_{A_0}(\alpha,\infty),$$
where A_0 is the part of A in $X_0 := \overline{D(A)}$ (see Definition 1.3.11).
5. Show that the closure of $D(A)$ in $D_A(\alpha,\infty)$ is the subspace of all $x \in X$ such that $\lim_{t\to 0} t^{1-\alpha}Ae^{tA}x = 0$. This implies that, even if $D(A)$ is dense in X, it is not necessarily dense in $D_A(\alpha,\infty)$.

   [Hint: to prove that $e^{tA}x - x$ tends to zero in $D_A(\alpha,\infty)$ provided $t^{1-\alpha}Ae^{tA}x$ tends to zero as $t \to 0$, split the supremum over $(0,1]$ in the definition of $[\,\cdot\,]_\alpha$ into the supremum over $(0,\varepsilon]$ and over $[\varepsilon,1]$, ε small].

3.2 Spaces of class J_α

Definition 3.2.1 *Given three Banach spaces $Z \subset Y \subset X$ (with continuous embeddings), and given $\alpha \in (0,1)$, we say that Y is of class J_α between X and Z if there is $C > 0$ such that*
$$\|y\|_Y \le C\|y\|_Z^\alpha \|y\|_X^{1-\alpha}, \ \ y \in Z.$$

From Proposition 3.1.6 it follows that for all $\alpha \in (0,1)$ the space $D_A(\alpha,\infty)$ is of class J_α between X and the domain of A. From Exercise 5(c) in §2.3.1 we obtain that $W^{1,p}(\mathbb{R}^N)$ is in the class $J_{1/2}$ between $L^p(\mathbb{R}^N)$ and $W^{2,p}(\mathbb{R}^N)$ for each $p \in [1,+\infty)$, and that $C^1_b(\mathbb{R}^N)$ is in the class $J_{1/2}$ between $C_b(\mathbb{R}^N)$ and the domain of the Laplacian in $C_b(\mathbb{R}^N)$.

Other examples of spaces of class J_α between a Banach space X and the domain of a sectorial operator A are the real interpolation spaces $D_A(\alpha,p)$ with $1 \le p < +\infty$, the complex interpolation spaces $[X,D(A)]_\alpha$, the domains of the fractional powers $D(-A^\alpha)$, ... but the treatment of such spaces goes beyond the aims of this introductory course. The main reference on the subject is the book [18], a simplified treatment may be found in the lecture notes [11].

Several properties of the spaces $D_A(\alpha,\infty)$ are shared by any space of class J_α.

Proposition 3.2.2 *Let $A : D(A) \to X$ be a sectorial operator, and let X_α be any space of class J_α between X and $D(A)$, $0 < \alpha < 1$. Then the following statements hold:*

(i) For $\varepsilon \in (0, 1-\alpha)$ we have

$$D_A(\alpha+\varepsilon, \infty) \subset X_\alpha,$$

with continuous embedding.

(ii) For $k \in \mathbb{N} \cup \{0\}$ there are constants $M_{k,\alpha} > 0$ such that

$$\|A^k e^{tA}\|_{\mathcal{L}(X, X_\alpha)} \leq \frac{M_{k,\alpha}}{t^{k+\alpha}}, \quad 0 < t \leq 1.$$

(iii) If $B \in \mathcal{L}(X_\alpha, X)$ then $A + B : D(A+B) := D(A) \to X$ is sectorial.

Proof. *Proof of (i).* Let $x \in D_A(\alpha+\varepsilon, \infty)$. From formula (1.19) with $t = 1$ we obtain

$$x = e^A x - \int_0^1 A e^{sA} x \, ds.$$

The function $s \mapsto A e^{sA} x$ is integrable over $[0, 1]$ with values in X_α, because

$$\begin{aligned} &\|A e^{sA} x\|_{X_\alpha} \leq C(\|A e^{sA} x\|_{D(A)})^\alpha \|A e^{sA} x\|_X)^{1-\alpha} \\ &\leq C_\varepsilon (s^{-2+\alpha+\varepsilon} \|x\|_{D_A(\alpha+\varepsilon,\infty)})^\alpha (s^{-1+\alpha+\varepsilon} \|x\|_{D_A(\alpha+\varepsilon,\infty)})^{1-\alpha} = C_\varepsilon s^{-1+\varepsilon} \|x\|_{D_A(\alpha+\varepsilon,\infty)}. \end{aligned}$$

Therefore, $x \in X_\alpha$, and the statement follows.

Proof of (ii). For each $x \in X$ we have $\|A^k e^{tA} x\|_{X_\alpha} \leq C(\|A^k e^{tA} x\|_{D(A)})^\alpha (\|A^k e^{tA} x\|_X)^{1-\alpha}$, and the statement follows using (1.15).

Proof of (iii). It is an immediate consequence of corollary 1.3.14. □

Note that in general a space X_α of class J_α between X and $D(A)$ may not be contained in any $D_A(\beta, \infty)$. For instance, if $X = C([0, 1])$, A is the realization of the second order derivative with Dirichlet boundary condition X, i.e. $D(A) = \{u \in C^2([0, 1]) : \ u(0) = u(1) = 0\}$ and $Au = u''$, then $C^1([0, 1])$ is of class $J_{1/2}$ between X and $D(A)$ but it is not contained in $\overline{D(A)}$ (and hence, in any $D_A(\beta, \infty)$) because the functions in $\overline{D(A)}$ vanish at $x = 0$ and at $x = 1$.

Similarly, the part A_α of A in X_α could not be sectorial. Note that the embeddings $D(A) \subset X_\alpha \subset X$ imply that $t \mapsto e^{tA}$ is analytic in $(0, +\infty)$ with values in $\mathcal{L}(X_\alpha)$, hence $\|e^{tA}\|_{\mathcal{L}(X_\alpha)}$ is bounded by a constant independent of t if t runs in any interval $[a, b] \subset (0, +\infty)$, but it could blow up as $t \to 0$.

Exercises 3.2.3

1. Let $A : D(A) \to X$ be a sectorial operator. Prove that $D(A)$ is of class $J_{1/2}$ between X and $D(A^2)$.

   [Hint: If $\omega = 0$, use formula (1.19) to get $\|Ax\| \leq M_1 \|x\|/t + M_0 t \|A^2 x\|$ for each $t > 0$ and then take the minimum for $t \in (0, +\infty)$. If $\omega > 0$, replace A by $A - \omega I \ldots$].

2. Let $A : D(A) \to X$ be a linear operator satisfying the assumptions of Proposition 2.2.2. Prove that $D(A)$ is of class $J_{1/2}$ between X and $D(A^2)$.

   [Hint: Setting $\lambda x - A^2 x = y$ for $x \in D(A^2)$ and $\lambda > 0$, use formula (2.6) to estimate $\|Ax\|$ and then take the minimum for $\lambda \in (0, +\infty)$].

3. Prove that $C^1_b(\mathbb{R})$ is of class $J_{1/4}$ between $C_b(\mathbb{R})$ and $C^4_b(\mathbb{R})$.

4. (a) Following the proof of Proposition 3.1.6, show that $D_A(\alpha,\infty)$ is of class $J_{\alpha/\theta}$ between X and $D_A(\theta,\infty)$, for every $\theta\in(\alpha,1)$.

   (b) Show that any space of class J_α between X and $D(A)$ is of class $J_{\alpha/\theta}$ between X and $D_A(\theta,\infty)$, for every $\theta\in(\alpha,1)$.

   (c) Using (a), prove that any function which is continuous with values in X and bounded with values in $D_A(\theta,\infty)$ in an interval $[a,b]$, is also continuous with values in $D_A(\alpha,\infty)$ in $[a,b]$, for $\alpha<\theta$.

5. Prove that for every $\theta\in(0,1)$ there is $C=C(\theta)>0$ such that

$$\|D_i\varphi\|_\infty\leq C(\|\varphi\|_{C^{2+\theta}_b(\mathbb{R}^N)})^{(1-\theta)/2}(\|\varphi\|_{C^\theta_b(\mathbb{R}^N)})^{(1+\theta)/2},$$

$$\|D_{ij}\varphi\|_\infty\leq C(\|\varphi\|_{C^{2+\theta}_b(\mathbb{R}^N)})^{1-\theta/2}(\|\varphi\|_{C^\theta_b(\mathbb{R}^N)})^{\theta/2},$$

   for every $\varphi\in C^{2+\theta}_b(\mathbb{R}^N)$, $i,\ j=1,\dots,N$. Deduce that $C^1_b(\mathbb{R}^N)$ and $C^2_b(\mathbb{R}^N)$ are of class $J_{(1-\theta)/2}$ and $J_{1-\theta/2}$, respectively, between $C^\theta_b(\mathbb{R}^N)$ and $C^{2+\theta}_b(\mathbb{R}^N)$.

   [Hint: write $\varphi=\varphi-T(t)\varphi+T(t)\varphi=-\int_0^t T(s)\Delta\varphi\,ds+T(t)\varphi$, $T(t)=$ heat semigroup, and use the estimates $\|D_iT(t)f\|_\infty\leq Ct^{-1/2+\theta/2}\|f\|_{C^\theta_b}$, $\|D_{ij}T(t)f\|_\infty\leq Ct^{-1+\theta/2}\|f\|_{C^\theta_b}$].

6. Let b_i, $i=1,\dots,N$, $c:\mathbb{R}^N\to\mathbb{C}$ be given functions, and let $\mathcal{A}$ be the differential operator $(\mathcal{A}u)(x)=\Delta u(x)+\sum_{i=1}^N b_i(x)D_iu(x)+c(x)u(x)$. Following the notation of Section 2.3, let $D(A_p)$ be the domain of the Laplacian in $L^p(\mathbb{R}^N)$ for $1\leq p<+\infty$, in $C_b(\mathbb{R}^N)$ for $p=+\infty$.

   Show that if b_i, $c\in L^\infty(\mathbb{R}^N)$ then the operator $D(A_p)\to L^p(\mathbb{R}^N)$, $u\mapsto\mathcal{A}u$ is sectorial in $L^p(\mathbb{R}^N)$ for $1\leq p<+\infty$, and if b_i, $c\in C_b(\mathbb{R}^N)$ then the operator $D(A_\infty)\to C_b(\mathbb{R}^N)$, $u\mapsto\mathcal{A}u$ is sectorial in $C_b(\mathbb{R}^N)$.

Chapter 4

Non homogeneous problems

Let $A : D(A) \subset X \to X$ be a sectorial operator and let $T > 0$. In this chapter we study the nonhomogeneous Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t), & 0 < t \le T, \\ \\ u(0) = x, \end{cases} \tag{4.1}$$

where $f : [0,T] \to X$.

Throughout the chapter we use standard notation. We recall that if Y is any Banach space and $a < b \in \mathbb{R}$, $B([a,b];Y)$ and $C([a,b];Y)$ are the Banach spaces of all bounded (respectively, continuous) functions from $[a,b]$ to Y, endowed with the sup norm $\|f\|_\infty = \sup_{a\le s\le b} \|f(s)\|_Y$. $C^\alpha([a,b];Y)$ is the Banach space of all α-Hölder continuous functions from $[a,b]$ to Y, endowed with the norm $\|f\|_{C^\alpha([a,b];Y)} = \|f\|_\infty + [f]_{C^\alpha([a,b];Y)}$, where $[f]_{C^\alpha([a,b];Y)} = \sup_{a\le s,t\le b} \|f(t) - f(s)\|_Y/(t-s)^\alpha$.

4.1 Strict, classical, and mild solutions

Definition 4.1.1 *Let $f : [0,T] \to X$ be a continuous function, and let $x \in X$. Then:*

(i) $u \in C^1([0,T];X) \cap C([0,T];D(A))$ *is a* strict solution *of* (4.1) *in* $[0,T]$ *if* $u'(t) = Au(t) + f(t)$ *for every* $t \in [0,T]$, *and* $u(0) = x$.

(ii) $u \in C^1((0,T];X) \cap C((0,T];D(A)) \cap C([0,T];X)$ *is a* classical solution *of* (4.1) *in* $[0,T]$ *if* $u'(t) = Au(t) + f(t)$ *for every* $t \in (0,T]$, *and* $u(0) = x$.

From Definition 4.1.1 it is easily seen that if (4.1) has a strict solution, then

$$x \in D(A), \;\; Ax + f(0) = u'(0) \in \overline{D(A)}, \tag{4.2}$$

and if (4.1) has a classical solution, then

$$x \in \overline{D(A)}. \tag{4.3}$$

We will see that if (4.1) has a classical (or a strict) solution, then it is given, as in the case of a bounded A, by the *variation of constants formula* (see Proposition 1.2.3)

$$u(t) = e^{tA}x + \int_0^t e^{(t-s)A} f(s)ds, \;\; 0 \le t \le T. \tag{4.4}$$

Whenever the integral in (4.4) does make sense, the function u defined by (4.4) is said to be a *mild solution* of (4.1).

Proposition 4.1.2 *Let $f \in C((0,T],X)$ be such that $t \mapsto \|f(t)\| \in L^1(0,T)$, and let $x \in \overline{D(A)}$ be given. If u is a classical solution of (4.1), then it is given by formula (4.4).*

Proof. Let u be a classical solution, and fix $t \in (0,T]$. Since $u \in C^1((0,T];X) \cap C((0,T];D(A)) \cap C([0,T];X)$, the function

$$v(s) = e^{(t-s)A}u(s), \;\; 0 \le s \le t,$$

belongs to $C([0,t];X) \cap C^1((0,t),X)$, and

$$\begin{array}{rcl} v(0) & = & e^{tA}x, \;\; v(t) = u(t), \\ v'(s) & = & -Ae^{(t-s)A}u(s) + e^{(t-s)A}(Au(s)+f(s)) = e^{(t-s)A}f(s), \;\; 0 < s < t. \end{array}$$

As a consequence, for $0 < 2\varepsilon < t$ we have

$$v(t-\varepsilon) - v(\varepsilon) = \int_\varepsilon^{t-\varepsilon} e^{(t-s)A}f(s)ds,$$

so that letting $\varepsilon \to 0^+$ we get

$$v(t) - v(0) = \int_0^t e^{(t-s)A}f(s)ds,$$

and the statement follows. □

Remark 4.1.3 Under the assumptions of Proposition 4.1.2, the classical solution of (4.1) is unique. In particular, for $f \equiv 0$ and $x \in \overline{D(A)}$, the function

$$t \mapsto u(t) = e^{tA}x, \;\; t \ge 0,$$

is the unique solution of the homogeneous problem (4.1). Of course, Proposition 4.1.2 also implies uniqueness of the strict solution.

Therefore, existence of a classical or strict solution of (1.1) is reduced to the problem of regularity of the mild solution. In general, even for $x = 0$ the continuity of f is not sufficient to guarantee that the mild solution is classical. Trying to show that $u(t) \in D(A)$ by estimating $\|Ae^{(t-s)A}f(s)\|$ is useless, because we have $\|Ae^{(t-s)A}f(s)\| \le C\|f\|_\infty(t-s)^{-1}$ and this is not sufficient to make the integral convergent. More sophisticated arguments, such as in the proof of Proposition 1.3.6(ii), do not work. We refer to Exercise 3 in §4.1.13 for a rigorous counterexample.

The mild solution satisfies an integrated version of (4.1), as the next lemma shows.

Proposition 4.1.4 *Let $f \in C_b((0,T);X)$, and let $x \in X$. If u is defined by (4.4), then for every $t \in [0,T]$ the integral $\int_0^t u(s)ds$ belongs to $D(A)$, and*

$$u(t) = x + A\int_0^t u(s)ds + \int_0^t f(s)ds, \;\; 0 \le t \le T. \tag{4.5}$$

Proof. For every $t \in [0,T]$ we have

$$\begin{aligned}\int_0^t u(s)ds &= \int_0^t e^{sA}xds + \int_0^t ds \int_0^s e^{(s-\sigma)A} f(\sigma)d\sigma \\ &= \int_0^t e^{sA}xds + \int_0^t d\sigma \int_\sigma^t e^{(s-\sigma)A} f(\sigma)ds.\end{aligned}$$

The integral $\int_\sigma^t e^{(s-\sigma)A}f(\sigma)ds = \int_0^{t-\sigma} e^{\tau A}f(\sigma)d\tau$ belongs to $D(A)$ by Proposition 1.3.6(ii) and $A\int_\sigma^t e^{(s-\sigma)A}f(\sigma)ds = (e^{(t-\sigma)A} - I)f(\sigma)$. Lemma A.4 yields

$$\int_0^t d\sigma \int_\sigma^t e^{(s-\sigma)A}f(\sigma)ds \in D(A)$$

and

$$A\int_0^t d\sigma \int_\sigma^t e^{(s-\sigma)A}f(\sigma)ds = \int_0^t \left(e^{(t-\sigma)A} - I\right) f(\sigma)d\sigma.$$

Hence, using once again Proposition 1.3.6(ii), the integral $\int_0^t u(s)ds$ belongs to $D(A)$ and

$$A\int_0^t u(s)ds = e^{tA}x - x + \int_0^t \left(e^{(t-\sigma)A} - I\right) f(\sigma)d\sigma, \quad 0 \le t \le T,$$

so that (4.5) holds. □

In the next proposition we show that the mild solution with $x = 0$ is Hölder continuous in all intervals $[0,T]$. For the proof we define

$$M_k := \sup_{0<t\le T+1} \|t^k A^k e^{tA}\|, \quad k = 0,1,2, \tag{4.6}$$

and

$$v(t) = (e^{tA} * f)(t) := \int_0^t e^{(t-s)A}f(s)ds, \quad 0 \le t \le T, \tag{4.7}$$

Proposition 4.1.5 *Let $f \in C_b((0,T);X)$. Then the function v defined above belongs to $C^\alpha([0,T];X)$ for every $\alpha \in (0,1)$, and there is $C = C(\alpha,T)$ such that*

$$\|v\|_{C^\alpha([0,T];X)} \le C \sup_{0<s<T} \|f(s)\|. \tag{4.8}$$

Proof. For $0 \le t \le T$ we have

$$\|v(t)\| \le M_0 t \|f\|_\infty, \tag{4.9}$$

whereas for $0 \le s \le t \le T$ we have

$$\begin{aligned} v(t) - v(s) &= \int_0^s \left(e^{(t-\sigma)A} - e^{(s-\sigma)A}\right) f(\sigma)d\sigma + \int_s^t e^{(t-\sigma)A}f(\sigma)d\sigma \\ &= \int_0^s d\sigma \int_{s-\sigma}^{t-\sigma} Ae^{\tau A}f(\sigma)d\tau + \int_s^t e^{(t-\sigma)A}f(\sigma)d\sigma. \end{aligned} \tag{4.10}$$

Since $\tau \geq s-\sigma$, this implies that

$$\begin{aligned}
\|v(t)-v(s)\| &\leq M_1\|f\|_\infty \int_0^s d\sigma \int_{s-\sigma}^{t-\sigma} \frac{d\tau}{\tau} + M_0\|f\|_\infty (t-s) \\
&\leq M_1\|f\|_\infty \int_0^s \frac{d\sigma}{(s-\sigma)^\alpha} \int_{s-\sigma}^{t-\sigma} \frac{1}{\tau^{1-\alpha}}\, d\tau + M_0\|f\|_\infty (t-s) \\
&\leq M_1\|f\|_\infty \int_0^s \frac{d\sigma}{(s-\sigma)^\alpha} \int_0^{t-s} \frac{1}{\tau^{1-\alpha}}\, d\tau + M_0\|f\|_\infty (t-s) \\
&\leq \left(\frac{M_1 T^{1-\alpha}}{\alpha(1-\alpha)}(t-s)^\alpha + M_0(t-s)\right)\|f\|_\infty,
\end{aligned} \tag{4.11}$$

so that v is α-Hölder continuous. Estimate (4.8) follows immediately from (4.9) and (4.11). □

The result of Proposition 4.1.4 is used in the next lemma, where we give sufficient conditions in order that a mild solution be classical or strict.

Lemma 4.1.6 *Let $f \in C_b((0,T];X)$, let $x \in \overline{D(A)}$, and let u be the mild solution of (4.1). The following conditions are equivalent.*

(a) $u \in C((0,T];D(A))$,

(b) $u \in C^1((0,T];X)$,

(c) u is a classical solution of (4.1).

If in addition $f \in C([0,T];X)$, then the following conditions are equivalent.

(a′) $u \in C([0,T];D(A))$,

(b′) $u \in C^1([0,T];X)$,

(c′) u is a strict solution of (4.1).

Proof. Of course, (c) implies both (a) and (b). Let us show that if either (a) or (b) holds, then u is a classical solution. We already know that u belongs to $C([0,T];X)$ and that it satisfies (4.5). Therefore, for every t, h such that t, $t+h \in (0,T]$,

$$\frac{u(t+h)-u(t)}{h} = \frac{1}{h}A\int_t^{t+h} u(s)ds + \frac{1}{h}\int_t^{t+h} f(s)ds. \tag{4.12}$$

Since f is continuous at t, then

$$\lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} f(s)ds = f(t). \tag{4.13}$$

Let (a) hold. Then Au is continuous at t, so that

$$\lim_{h\to 0^+} \frac{1}{h}A\int_t^{t+h} u(s)ds = \lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} Au(s)ds = Au(t).$$

By (4.12) and (4.13) we obtain that u is differentiable at the point t, with $u'(t) = Au(t) + f(t)$. Since both Au and f are continuous in $(0, T]$, then u' is continuous, and u is a classical solution.

Now let (b) hold. Since u is continuous at t, then

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} u(s)ds = u(t).$$

On the other hand, (4.12) and (4.13) imply the existence of the limit

$$\lim_{h\to 0^+} A\left(\frac{1}{h}\int_t^{t+h} u(s)ds\right) = u'(t) - f(t).$$

Since A is a closed operator, then $u(t)$ belongs to $D(A)$, and $Au(t) = u'(t) - f(t)$. Since both u' and f are continuous in $(0, T]$, then Au is also continuous in $(0, T]$, so that u is a classical solution.

The equivalence of (a'), (b'), (c') may be proved in the same way. □

In the following two theorems we prove that, under some regularity conditions on f, the mild solution is strict or classical. In the theorem below we assume time regularity whereas in the next one we assume "space" regularity on f.

Theorem 4.1.7 *Let $0 < \alpha < 1$, $f \in C^\alpha([0, T], X)$, $x \in X$, an let u be the function defined in* (4.4). *Then u belongs to $C^\alpha([\varepsilon, T], D(A)) \cap C^{1+\alpha}([\varepsilon, T], X)$ for every $\varepsilon \in (0, T)$, and the following statements hold:*

(i) if $x \in \overline{D(A)}$, then u is a classical solution of (4.1)*;*

(ii) if $x \in D(A)$ and $Ax + f(0) \in \overline{D(A)}$, then u is a strict solution of (4.1), *and there is $C > 0$ such that*

$$\|u\|_{C^1([0,T],X)} + \|u\|_{C([0,T],D(A))} \le C(\|f\|_{C^\alpha([0,T],X)} + \|x\|_{D(A)}). \tag{4.14}$$

(iii) if $x \in D(A)$ and $Ax + f(0) \in D_A(\alpha, \infty)$, then u' and Au belong to $C^\alpha([0, T], X)$, u' belongs to $B([0, T]; D_A(\alpha, \infty))$, and there is C such that

$$\begin{aligned} &\|u\|_{C^{1+\alpha}([0,T];X)} + \|Au\|_{C^\alpha([0,T];X)} + \|u'\|_{B([0,T];D_A(\alpha,\infty))} \\ &\le C(\|f\|_{C^\alpha([0,T];X)} + \|x\|_{D(A)} + \|Ax + f(0)\|_{D_A(\alpha,\infty)}). \end{aligned} \tag{4.15}$$

Proof. We are going to show that if $x \in \overline{D(A)}$ then $u \in C((0, T]; D(A))$, and that if $x \in D(A)$ and $Ax + f(0) \in \overline{D(A)}$ then $u \in C([0, T]; D(A))$. In both cases statements (i) and (ii) will follow from Lemma 4.1.6.

Set

$$\begin{cases} u_1(t) = \displaystyle\int_0^t e^{(t-s)A}(f(s) - f(t))ds, \ \ 0 \le t \le T, \\ \\ u_2(t) = e^{tA}x + \displaystyle\int_0^t e^{(t-s)A} f(t)ds, \ \ 0 \le t \le T, \end{cases} \tag{4.16}$$

so that $u = u_1 + u_2$. Notice that both $u_1(t)$ and $u_2(t)$ belong to $D(A)$ for $t > 0$. Concerning $u_1(t)$, the estimate

$$\|Ae^{(t-s)A}(f(s) - f(t))\| \leq \frac{M_1}{t-s}(t-s)^\alpha [f]_{C^\alpha}$$

implies that the function $s \mapsto e^{(t-s)A}(f(s) - f(t))$ is integrable with values in $D(A)$, whence $u_1(t) \in D(A)$ for every $t \in (0, T]$ (the same holds, of course, for $t = 0$ as well). Concerning $u_2(t)$, we know that $e^{tA}x$ belongs to $D(A)$ for $t > 0$, and that $\int_0^t e^{(t-s)A} f(t) ds$ belongs to $D(A)$ by Proposition 1.3.6(ii). Moreover, we have

$$\begin{cases} (i) \quad Au_1(t) = \displaystyle\int_0^t Ae^{(t-s)A}(f(s) - f(t))ds, \quad 0 \leq t \leq T, \\ \\ (ii) \quad Au_2(t) = Ae^{tA}x + (e^{tA} - I)f(t), \quad 0 < t \leq T. \end{cases} \tag{4.17}$$

If $x \in D(A)$, then equality (4.17)(ii) holds for $t = 0$, too. Let us show that Au_1 is Hölder continuous in $[0, T]$. For $0 \leq s < t \leq T$ we have

$$\begin{aligned} Au_1(t) - Au_1(s) &= \int_0^s \left(Ae^{(t-\sigma)A}(f(\sigma) - f(t)) - Ae^{(s-\sigma)A}(f(\sigma) - f(s)) \right) d\sigma \\ &\quad + \int_s^t Ae^{(t-\sigma)A}(f(\sigma) - f(t))d\sigma \\ &= \int_0^s \left(Ae^{(t-\sigma)A} - Ae^{(s-\sigma)A} \right) (f(\sigma) - f(s))d\sigma \\ &\quad + \int_0^s Ae^{(t-\sigma)A}(f(s) - f(t))d\sigma + \int_s^t Ae^{(t-\sigma)A}(f(\sigma) - f(t))d\sigma \\ &= \int_0^s \int_{s-\sigma}^{t-\sigma} A^2 e^{\tau A} d\tau (f(\sigma) - f(s))d\sigma \\ &\quad + (e^{tA} - e^{(t-s)A})(f(s) - f(t)) + \int_s^t Ae^{(t-\sigma)A}(f(\sigma) - f(t))d\sigma, \end{aligned} \tag{4.18}$$

so that

$$\begin{aligned} \|Au_1(t) - Au_1(s)\| &\leq M_2 [f]_{C^\alpha} \int_0^s (s-\sigma)^\alpha \int_{s-\sigma}^{t-\sigma} \tau^{-2} d\tau \, d\sigma \\ &\quad + 2M_0 [f]_{C^\alpha} (t-s)^\alpha + M_1 [f]_{C^\alpha} \int_s^t (t-\sigma)^{\alpha-1} d\sigma \\ &\leq M_2 [f]_{C^\alpha} \int_0^s d\sigma \int_{s-\sigma}^{t-\sigma} \tau^{\alpha-2} d\tau + (2M_0 + M_1 \alpha^{-1})[f]_{C^\alpha} (t-s)^\alpha \\ &\leq \left(\frac{M_2}{\alpha(1-\alpha)} + 2M_0 + \frac{M_1}{\alpha} \right) [f]_{C^\alpha} (t-s)^\alpha, \end{aligned} \tag{4.19}$$

where M_k, $k = 0, 1, 2$, are the constants in (4.6). Hence, Au_1 is α-Hölder continuous in $[0, T]$. Moreover, it is easily checked that Au_2 is α-Hölder continuous in $[\varepsilon, T]$ for every $\varepsilon \in (0, T)$, and therefore $Au \in C^\alpha([\varepsilon, T]; X)$. Since $u \in C^\alpha([\varepsilon, T]; X)$ (because

$t \mapsto e^{tA}x \in C^\infty((0,T];X)$ and $t \mapsto \int_0^t e^{(t-s)A}f(s)ds \in C^\alpha([0,T];X)$ by Proposition 4.1.5), it follows that $u \in C^\alpha([\varepsilon,T];D(A))$. Since ε is arbitrary, then $u \in C((0,T];D(A))$.

Concerning the behavior as $t \to 0^+$, if $x \in \overline{D(A)}$, then $t \mapsto e^{tA}x \in C([0,T],X)$ and then $u \in C([0,T],X)$, see Proposition 4.1.5. This concludes the proof of (i).

If $x \in D(A)$, we may write $Au_2(t)$ in the form

$$Au_2(t) = e^{tA}(Ax + f(0)) + e^{tA}(f(t) - f(0)) - f(t), \quad 0 \le t \le T. \tag{4.20}$$

If $Ax + f(0) \in \overline{D(A)}$, then $\lim_{t\to 0^+} Au_2(t) = Ax$, hence Au_2 is continuous at $t = 0$, $u = u_1 + u_2$ belongs to $C([0,T];D(A))$ and it is a strict solution of (4.1). Estimate (4.14) easily follows since $u' = Au + f$ and

$$\|Au_1(t)\| \le M_1[f]_{C^\alpha}\int_0^t (t-s)^{\alpha-1}ds = \frac{M_1}{\alpha}[f]_{C^\alpha}t^\alpha,$$

$$\|Au_2(t)\| \le M_0\|Ax\| + (M_0+1)\|f\|_\infty.$$

This concludes the proof of (ii).

If $Ax + f(0) \in D_A(\alpha,\infty)$, we already know that $t \mapsto e^{tA}(Ax + f(0)) \in C^\alpha([0,T],X)$, with C^α norm estimated by C$\|Ax+f(0)\|_{D_A(\alpha,\infty)}$, for some positive constant C. Moreover $f \in C^\alpha([0,T],X)$ by assumption, so we have only to show that $t \mapsto e^{tA}(f(t) - f(0))$ is α-Hölder continuous.

For $0 \le s \le t \le T$ we have

$$\begin{aligned}
\|e^{tA}(f(t)-f(0)) - e^{sA}(f(s)-f(0))\| &\le \|(e^{tA}-e^{sA})(f(s)-f(0))\| + \|e^{tA}(f(t)-f(s))\| \\
&\le s^\alpha[f]_{C^\alpha}\left\|A\int_s^t e^{\sigma A}d\sigma\right\|_{\mathcal{L}(X)} + M_0(t-s)^\alpha[f]_{C^\alpha} \\
&\le M_1[f]_{C^\alpha}s^\alpha\int_s^t \frac{d\sigma}{\sigma} + M_0[f]_{C^\alpha}(t-s)^\alpha \qquad (4.21)\\
&\le M_1[f]_{C^\alpha}\int_s^t \sigma^{\alpha-1}d\sigma + M_0[f]_{C^\alpha}(t-s)^\alpha \\
&\le \left(\frac{M_1}{\alpha} + M_0\right)(t-s)^\alpha[f]_{C^\alpha}.
\end{aligned}$$

Hence Au_2 is α-Hölder continuous as well, and the estimate

$$\|u\|_{C^{1+\alpha}([0,T];X)} + \|Au\|_{C^\alpha([0,T];X)} \le c(\|f\|_{C^\alpha([0,T],X)} + \|x\|_X + \|Ax + f(0)\|_{D_A(\alpha,\infty)})$$

follows, since $u' = Au + f$ and $u = u_1 + u_2$.

Let us now estimate $[u'(t)]_{D_A(\alpha,\infty)}$. For $0 \le t \le T$ we have

$$u'(t) = \int_0^t Ae^{(t-s)A}(f(s)-f(t))ds + e^{tA}(Ax + f(0)) + e^{tA}(f(t)-f(0)),$$

so that for $0 < \xi \le 1$ we deduce

$$\begin{aligned}
\|\xi^{1-\alpha}Ae^{\xi A}u'(t)\| \le{}& \left\|\xi^{1-\alpha}\int_0^t A^2e^{(t+\xi-s)A}(f(s)-f(t))ds\right\| \\
&+ \|\xi^{1-\alpha}Ae^{(t+\xi)A}(Ax+f(0))\| + \|\xi^{1-\alpha}Ae^{(t+\xi)A}(f(t)-f(0))\|
\end{aligned}$$

$$\leq M_2[f]_{C^\alpha}\xi^{1-\alpha}\int_0^t (t-s)^\alpha (t+\xi-s)^{-2}ds \tag{4.22}$$
$$+ M_0[Ax+f(0)]_{D_A(\alpha,\infty)} + M_1[f]_{C^\alpha}\xi^{1-\alpha}(t+\xi)^{-1}t^\alpha$$
$$\leq M_2[f]_{C^\alpha}\int_0^\infty \sigma^\alpha(\sigma+1)^{-2}d\sigma + M_0[Ax+f(0)]_{D_A(\alpha,\infty)} + M_1[f]_{C^\alpha}.$$

Then, $[u'(t)]_{D_A(\alpha,\infty)}$ is bounded in $[0,T]$, and the proof is complete. □

Remark 4.1.8 The proof of Theorem 4.1.7 implies that the condition $Ax + f(0) \in D_A(\alpha,\infty)$ is necessary in order that $Au \in C^\alpha([0,T];X)$. Once this condition is satisfied, it is preserved through the whole interval $[0,T]$, in the sense that $Au(t)+f(t) = u'(t)$ belongs to $D_A(\alpha,\infty)$ for each $t \in [0,T]$.

In the proof of the next theorem we use the constants

$$M_{k,\alpha} := \sup_{0<t\leq T+1} \|t^{k-\alpha}A^k e^{tA}\|_{\mathcal{L}(D_A(\alpha,\infty),X)} < +\infty, \quad k=1,2. \tag{4.23}$$

Theorem 4.1.9 *Let $0<\alpha<1$, and let $f \in C([0,T];X)\cap B([0,T];D_A(\alpha,\infty))$. Then the function $v = (e^{tA} * f)$ belongs to $C([0,T];D(A)) \cap C^1([0,T];X)$, and it is the strict solution of*

$$\begin{cases} v'(t) = Av(t) + f(t), \quad 0<t\leq T, \\ v(0) = 0. \end{cases} \tag{4.24}$$

Moreover, v' and Av belong to $B([0,T];D_A(\alpha,\infty))$, Av belongs to $C^\alpha([0,T];\,X)$, and there is C such that

$$\|v'\|_{B([0,T];D_A(\alpha,\infty))} + \|Av\|_{B([0,T];D_A(\alpha,\infty))} + \|Av\|_{C^\alpha([0,T];X)} \leq C\|f\|_{B([0,T];D_A(\alpha,\infty))}. \tag{4.25}$$

Proof. Let us prove that v is a strict solution of (4.24), and that (4.25) holds. For $0\leq t\leq T$, $v(t)$ belongs to $D(A)$, and, denoting by $|f|$ the norm of f in $B([0,T];D_A(\alpha,\infty))$

$$\|Av(t)\| \leq M_{1,\alpha}|f|\int_0^t (t-s)^{\alpha-1}ds \leq \frac{T^\alpha M_{1,\alpha}}{\alpha}\,|f|. \tag{4.26}$$

Moreover, for $0<\xi\leq 1$ we have

$$\|\xi^{1-\alpha}Ae^{\xi A}Av(t)\| = \xi^{1-\alpha}\left\|\int_0^t A^2 e^{(t+\xi-s)A}f(s)ds\right\|$$
$$\leq M_{2,\alpha}\xi^{1-\alpha}\int_0^t (t+\xi-s)^{\alpha-2}ds|f| \leq \frac{M_{2,\alpha}}{1-\alpha}|f|, \tag{4.27}$$

so that Av is bounded with values in $D_A(\alpha,\infty)$. Let us prove that Av is Hölder continuous with values in X: for $0\leq s\leq t\leq T$ we have

$$\|Av(t)-Av(s)\| \leq \left\|A\int_0^s \left(e^{(t-\sigma)A} - e^{(s-\sigma)A}\right) f(\sigma)d\sigma\right\| + \left\|A\int_s^t e^{(t-\sigma)A}f(\sigma)d\sigma\right\|$$
$$\leq M_{2,\alpha}|f|\int_0^s d\sigma\int_{s-\sigma}^{t-\sigma}\tau^{\alpha-2}d\tau + M_{1,\alpha}|f|\int_s^t (t-\sigma)^{\alpha-1}d\sigma$$

$$\leq \left(\frac{M_{2,\alpha}}{\alpha(1-\alpha)} + \frac{M_{1,\alpha}}{\alpha} \right) (t-s)^\alpha |f|, \tag{4.28}$$

hence Av is α-Hölder continuous in $[0,T]$. Estimate (4.25) follows from (4.26), (4.27), (4.28).

The differentiability of v and the equality $v'(t) = Av(t) + f(t)$ follow from Lemma 4.1.6. □

Corollary 4.1.10 *Let $0 < \alpha < 1$, $x \in X$, $f \in C([0,T];X) \cap B([0,T];D_A(\alpha,\infty))$ be given, and let u be given by (4.4). Then, $u \in C^1((0,T];X) \cap C((0,T];D(A))$, and $u \in B([\varepsilon,T];D_A(\alpha+1,\infty))$ for every $\varepsilon \in (0,T)$. Moreover, the following statements hold:*

(i) If $x \in \overline{D(A)}$, then u is the classical solution of (4.1);

(ii) If $x \in D(A)$, $Ax \in \overline{D(A)}$, then u is the strict solution of (4.1);

(iii) If $x \in D_A(\alpha+1,\infty)$, then u' and Au belong to $B([0,T];D_A(\alpha,\infty)) \cap C([0,T];X)$, Au belongs to $C^\alpha([0,T];X)$, and there is $C > 0$ such that

$$\begin{aligned} &\|u'\|_{B([0,T];D_A(\alpha,\infty))} + \|Au\|_{B([0,T];D_A(\alpha,\infty))} + \|Au\|_{C^\alpha([0,T];X)} \\ &\leq C(\|f\|_{B([0,T];D_A(\alpha,\infty))} + \|x\|_{D_A(\alpha,\infty)}). \end{aligned} \tag{4.29}$$

Proof. Let us write $u(t) = e^{tA}x + (e^{tA} * f)(t)$. If $x \in \overline{D(A)}$, the function $t \mapsto e^{tA}x$ is the classical solution of $w' = Aw$, $t > 0$, $w(0) = x$. If $x \in D(A)$ and $Ax \in \overline{D(A)}$ it is in fact a strict solution; if $x \in D_A(\alpha+1,\infty)$ then it is a strict solution and it also belongs to $C^1([0,T];X) \cap B([0,T];D_A(\alpha+1,\infty))$. The claim then follows from Theorem 4.1.9. □

As a consequence of Theorem 4.1.7 and of Corollary 4.1.10 we get a classical theorem of the theory of PDE's. We need some notation.

We recall that for $0 < \theta < 1$ the parabolic Hölder space $C^{\theta/2,\theta}([0,T] \times \mathbb{R}^N)$ is the space of the continuous functions $f : \mathbb{R}^N \to \mathbb{C}$ such that

$$\|f\|_{C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)} := \|f\|_\infty + \sup_{x \in \mathbb{R}^N} [f(\cdot,x)]_{C^{\theta/2}([0,T])} + \sup_{t \in [0,T]} [f(t,\cdot)]_{C_b^\theta(\mathbb{R}^N)} < +\infty,$$

and $C^{1+\theta/2,2+\theta}([0,T] \times \mathbb{R}^N)$ is the space of the bounded functions u such that u_t, $D_{ij}u$ exist for all i, $j = 1,\ldots,N$ and belong to $C^{\theta/2,\theta}([0,T] \times \mathbb{R}^N)$. The norm is

$$\begin{aligned} \|u\|_{C^{1+\theta/2,2+\theta}([0,T]\times\mathbb{R}^N)} :=& \|u\|_\infty + \sum_{i=1}^N \|D_i u\|_\infty \\ &+ \|u_t\|_{C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)} + \sum_{i,j=1}^N \|D_{ij}u\|_{C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)}. \end{aligned}$$

Note that $f \in C^{\theta/2,\theta}([0,T] \times \mathbb{R}^N)$ if and only if $t \mapsto f(t,\cdot)$ belongs to $C^{\theta/2}([0,T];C_b(\mathbb{R}^N))$ $\cap\, B([0,T];C_b^\theta(\mathbb{R}^N))$.

Corollary 4.1.11 (Ladyzhenskaja – Solonnikov – Ural'ceva) *Let $0 < \theta < 1$, $T > 0$ and let $u_0 \in C^{2+\theta}_b(\mathbb{R}^N)$, $f \in C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)$. Then the initial value problem*

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + f(t,x), & 0 \le t \le T, \quad x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{4.30}$$

has a unique solution $u \in C^{1+\theta/2,2+\theta}([0,T]\times\mathbb{R}^N)$, and there is $C > 0$, independent of u_0 and f, such that

$$\|u\|_{C^{1+\theta/2,2+\theta}([0,T]\times\mathbb{R}^N)} \le C(\|u_0\|_{C^{2+\theta}_b(\mathbb{R}^N)} + \|f\|_{C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)}).$$

Proof. Set $X = C_b(\mathbb{R}^N)$, $A : D(A) \to X$, $A\varphi = \Delta\varphi$, $T(t) =$ heat semigroup. The function $t \mapsto f(t,\cdot)$ belongs to $C^{\theta/2}([0,T];X) \cap B([0,T];D_A(\theta/2,\infty))$, thanks to the characterization of example 3.1.7. The initial datum u_0 is in $D(A)$, and both Au_0 and $f(0,\cdot)$ are in $D_A(\theta/2,\infty)$. Then we may apply both Theorem 4.1.7 and Corollary 4.1.10 with $\alpha = \theta/2$. They imply that the function u given by the variation of constants formula (4.4) is the unique strict solution to problem (4.1), with initial datum u_0 and with $f(t) = f(t,\cdot)$. Therefore, the function

$$u(t,x) := u(t)(x) = (T(t)u_0)(x) + \int_0^t (T(t-s)f(s,\cdot))(x)ds,$$

is the unique bounded classical solution to (4.30) with bounded u_t. Moreover, Theorem 4.1.7 implies that $u' \in C^{\theta/2}([0,T];C_b(\mathbb{R}^N)) \cap B([0,T];C^\theta_b(\mathbb{R}^N))$, so that $u_t \in C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)$, with norm bounded by $C(\|u_0\|_{C^{2+\theta}_b(\mathbb{R}^N)} + \|f\|_{C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)})$ for some $C > 0$. Corollary 4.1.10 implies that u is bounded with values in $D_A(\theta/2+1,\infty)$, so that $u(t,\cdot) \in C^{2+\theta}_b(\mathbb{R}^N)$ for each t, and

$$\sup_{0\le t\le T} \|u(t,\cdot)\|_{C^{2+\theta}_b(\mathbb{R}^N)} \le C(\|u_0\|_{C^{2+\theta}_b(\mathbb{R}^N)} + \|f\|_{C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)}),$$

for some $C > 0$, by estimate (4.29).

To finish the proof it remains to show that each second order space derivative $D_{ij}u$ is $\theta/2$-Hölder continuous with respect to t. To this aim we use the interpolatory inequality

$$\|D_{ij}\varphi\|_\infty \le C(\|\varphi\|_{C^{2+\theta}_b(\mathbb{R}^N)})^{1-\theta/2}(\|\varphi\|_{C^\theta_b(\mathbb{R}^N)})^{\theta/2},$$

that holds for every $\varphi \in C^{2+\theta}_b(\mathbb{R}^N)$, i, $j = 1,\dots,N$. See Exercise 5 in §3.2.3. Applying it to the function $\varphi = u(t,\cdot) - u(s,\cdot)$ we get

$$\begin{aligned} &\|D_{ij}u(t,\cdot) - D_{ij}u(s,\cdot)\|_\infty \\ &\le C(\|u(t,\cdot) - u(s,\cdot)\|_{C^{2+\theta}_b(\mathbb{R}^N)})^{1-\theta/2}(\|u(t,\cdot) - u(s,\cdot)\|_{C^\theta_b(\mathbb{R}^N)})^{\theta/2} \\ &\le C(2\sup_{0\le t\le T} \|u(t,\cdot)\|_{C^{2+\theta}_b(\mathbb{R}^N)})^{1-\theta/2}(|t-s|\sup_{0\le t\le T} \|u_t(t,\cdot)\|_{C^\theta_b(\mathbb{R}^N)})^{\theta/2} \\ &\le C'|t-s|^{\theta/2}(\|u_0\|_{C^{2+\theta}_b(\mathbb{R}^N)} + \|f\|_{C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)}), \end{aligned}$$

and the statement follows. □

Remark 4.1.12 If we have a Cauchy problem in an interval $[a,b] \neq [0,T]$,

$$\begin{cases} v'(t) = Av(t) + g(t), \quad a < t \le b, \\ \\ v(a) = y, \end{cases} \tag{4.31}$$

we obtain results similar to the case $[a,b] = [0,T]$, by the change of time variable $\tau = T(t-a)/(b-a)$. The details are left as (very easy) exercises. We just write down the variation of constants formula for v,

$$v(t) = e^{(t-a)A}y + \int_a^t e^{(t-s)A}g(s)ds, \;\; a \le t \le b. \tag{4.32}$$

Exercises 4.1.13

1. Let $f \in C_b((0,T);X)$ and set $v = (e^{tA} * f)$. Let X_α be a space of class J_α between X and $D(A)$ ($\alpha \in (0,1)$). Using the technique of Proposition 4.1.5 prove that

   (a) $v \in B([0,T];X_\alpha)$ and $\|v\|_{B([0,T];X_\alpha)} \le C_1 \sup_{0<t<T} \|f(t)\|$;

   (b) $v \in C^{1-\alpha}([0,T];X_\alpha)$ and $\|v\|_{C^\alpha([0,T];X_\alpha)} \le C_2 \sup_{0<t<T} \|f(t)\|$.

2. Let $A : D(A) \to X$ be a sectorial operator, and let $0 < \alpha < 1$, $a < b \in \mathbb{R}$. Prove that if a function u belongs to $C^{1+\alpha}([a,b];X) \cap C^\alpha([a,b];D(A))$ then u' is bounded in $[a,b]$ with values in $D_A(\alpha,\infty)$.

   [Hint: set $u_0 = u(a)$, $f(t) = u'(t) - Au(t)$, and use Theorem 4.1.7(iii) and Remark 4.1.8].

3. Consider the sectorial operators A_p in the sequence spaces ℓ^p, $1 \le p < \infty$ given by

   $$D(A_p) = \{(x_n) \in \ell^p : (nx_n) \in \ell^p\}, \qquad A_p(x_n) = -(nx_n) \text{ for } (x_n) \in D(A_p)$$

   and assume that for every $f \in C([0,T];\ell^p)$ the mild solution of (4.1) with initial value $x = 0$ is a strict one.

   (i) Use the closed graph theorem to show that the linear operator

   $$S : C([0,1];\ell^p) \to C([0,1];D(A_p)), \qquad Sf = e^{tA} * f$$

   is bounded.

   (ii) Let (e_n) be the canonical basis of ℓ^p and consider a nonzero continuous function $g : [0,+\infty) \to [0,1]$ with support contained in $[1/2,1]$. Let $f_n(t) = g(2^n(1-t))e_{2^n}$; then $f_n \in C([0,1];\ell^p)$, $\|f_n\|_\infty \le 1$. Moreover, setting $h_N = f_1 + \cdots + f_N$, we have also $h_N \in C([0,1];\ell_p)$, $\|h_N\|_\infty \le 1$, since the functions f_n have disjoint supports. Show that $(e^{tA} * f_n)(1) = c2^{-n}e_{2^n}$ where $c = \int_0^\infty e^{-s}g(s)ds$, hence $\|(e^{tA} * h_N)(1)\|_{D(A_p)} \ge cN^{1/p}$. This implies that S is unbounded, contradicting (i).

Chapter 5

Asymptotic behavior in linear problems

5.1 Behavior of e^{tA}

One of the most useful properties of analytic semigroups is the so called *spectrum determining condition*: roughly speaking, the asymptotic behavior (as $t \to +\infty$) of e^{tA}, and, more generally, of $A^n e^{tA}$, is determined by the spectral properties of A. This is an analogy with the finite dimensional case where the asymptotic behavior of the solutions of the differential equation $u' = Au$ depends on the eigenvalues of the matrix A.

Define the *spectral bound* of any sectorial operator A by

$$s(A) = \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\}. \tag{5.1}$$

Clearly $s(A) \leq \omega$ for any real number ω satisfying (1.9).

Proposition 5.1.1 *For every $n \in \mathbb{N} \cup \{0\}$ and $\varepsilon > 0$ there exists $M_{n,\varepsilon} > 0$ such that*

$$\|t^n A^n e^{tA}\|_{\mathcal{L}(X)} \leq M_{n,\varepsilon} e^{(s(A)+\varepsilon)t}, \;\; t > 0. \tag{5.2}$$

Proof. Let $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$ satisfy (1.9), and fix $\eta \in (\pi/2, \theta)$.

For $0 < t \leq 1$, estimates (5.2) are an easy consequence of (1.15). If $t \geq 1$ and $s(A) + \varepsilon \geq \omega$, (5.2) is still a consequence of (1.15). Let us consider the case in which $t \geq 1$ and $s(A) + \varepsilon < \omega$. Since $\rho(A) \supset S_{\theta,\omega} \cup \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > s(A)\}$, setting $a = (\omega - s(A) - \varepsilon)|\cos\eta|^{-1}$, $b = (\omega - s(A) - \varepsilon)|\tan\eta|$, the path

$$\begin{aligned} \Gamma_\varepsilon = \;\; & \{\lambda \in \mathbb{C} \,:\, \lambda = \xi e^{-i\eta} + \omega, \; \xi \geq a\} \cup \{\lambda \in \mathbb{C} \,:\, \lambda = \xi e^{i\eta} + \omega, \; \xi \geq a\} \\ & \cup \{\lambda \in \mathbb{C} \,:\, \mathrm{Re}\,\lambda = s(A) + \varepsilon, \; |\mathrm{Im}\,\lambda| \leq b\} \end{aligned}$$

(see Figure 5.1) is contained in $\rho(A)$, and $\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq M_\varepsilon$ on Γ_ε, for some $M_\varepsilon > 0$.

Since for every t the function $\lambda \mapsto e^{\lambda t} R(\lambda, A)$ is holomorphic in $\rho(A)$, the path $\omega + \gamma_{r,\eta}$ in the definition of e^{tA} may be replaced by Γ_ε, obtaining for each $t \geq 1$,

$$\|e^{tA}\| = \left\| \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} e^{t\lambda} R(\lambda, A) d\lambda \right\|$$

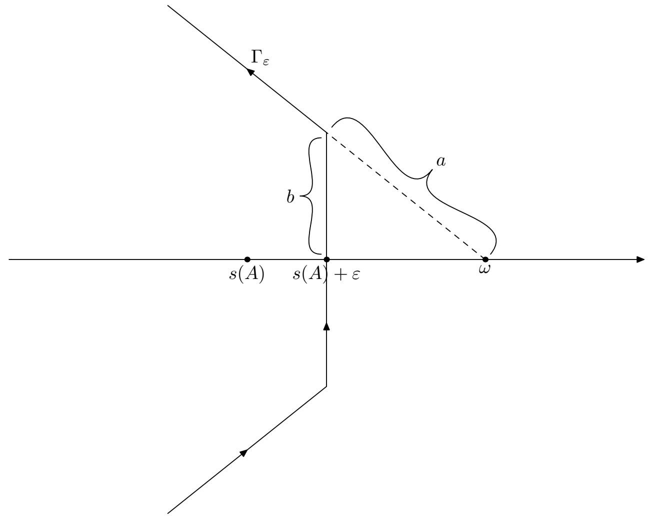

Figure 5.1: the curve Γ_ε.

$$
\begin{aligned}
&\le \frac{M_\varepsilon}{\pi}\int_a^{+\infty} e^{(\omega+\xi\cos\eta)t}d\xi + \frac{M_\varepsilon}{2\pi}\int_{-b}^{b} e^{(s(A)+\varepsilon)t}dy \\
&\le \frac{M_\varepsilon}{\pi}\left(\frac{1}{|\cos\eta|}+b\right)e^{(s(A)+\varepsilon)t}.
\end{aligned}
$$

Estimate (5.2) follows for $n = 0$. Arguing in the same way, for $t \ge 1$ we get

$$
\begin{aligned}
\|Ae^{tA}\| &= \left\|\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\lambda e^{t\lambda}R(\lambda,A)d\lambda\right\| \\
&\le \frac{M_\varepsilon}{2\pi}\left(2\int_a^{+\infty} e^{(\omega+\xi\cos\eta)t}d\xi + \int_{-b}^{b} e^{(s(A)+\varepsilon)t}dy\right) \\
&\le \frac{M_\varepsilon}{\pi}(|\cos\eta|^{-1}+b)e^{(s(A)+\varepsilon)t} \le \frac{\widetilde{M}_\varepsilon}{t}e^{(s(A)+2\varepsilon)t}.
\end{aligned}
$$

Since ε is arbitrary, (5.2) follows also for $n = 1$.

From the equality $A^n e^{tA} = (Ae^{tA/n})^n$ we get, for $n \ge 2$,

$$
\|A^n e^{tA}\|_{\mathcal{L}(X)} \le (M_{1,\varepsilon}nt^{-1}e^{t(s(A)+\varepsilon)/n})^n = (M_{1,\varepsilon}n)^n t^{-n}e^{(s(A)+\varepsilon)t},
$$

and (5.2) is proved. □

We remark that in the case $s(A) = \omega = 0$, estimates (1.14) are sharper than (5.2) for t large.

From Proposition 5.1.1 it follows that if $s(A) < 0$, then $t \mapsto e^{tA}x$ is bounded in $[0, +\infty)$ for every $x \in X$. In the case $s(A) \ge 0$, it is interesting to characterize the elements x such that $t \mapsto e^{tA}x$ is bounded in $[0, +\infty)$. We shall see that this is possible in the case where the spectrum of A does not intersect the imaginary axis.

5.2 Behavior of e^{tA} for a hyperbolic A

In this section we assume that

$$
\sigma(A) \cap i\mathbb{R} = \varnothing. \tag{5.3}
$$

In this case A is said to be *hyperbolic*. Set $\sigma(A) = \sigma_-(A) \cup \sigma_+(A)$, where

$$\sigma_-(A) = \sigma(A) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}, \quad \sigma_+(A) = \sigma(A) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}. \tag{5.4}$$

We write σ_+ and σ_-, respectively for $\sigma_+(A)$ and $\sigma_-(A)$ when there is no danger of confusion. Note that σ_+ is bounded. On the contrary, σ_- may be bounded or unbounded. For instance using Proposition 2.1.1 and Exercise 1, §1.3.5, we easily see that the spectrum of the realization of $u'' - u$ in $C_b(\mathbb{R})$ is the unbounded set $(-\infty, -1]$. On the other hand if $A \in \mathcal{L}(X)$ then A is sectorial and σ_- is bounded.

Since both σ_-, σ_+ are closed we have

$$-\omega_- := \sup\{\operatorname{Re}\lambda : \ \lambda \in \sigma_-\} < 0, \quad \omega_+ := \inf\{\operatorname{Re}\lambda : \ \lambda \in \sigma_+\} > 0. \tag{5.5}$$

σ_- and σ_+ may also be empty: in this case we set $\omega_- = +\infty$, $\omega_+ = +\infty$. Let P be the operator defined by

$$P = \frac{1}{2\pi i}\int_{\gamma_+} R(\lambda, A)d\lambda, \tag{5.6}$$

where γ_+ is a closed regular curve contained in $\rho(A)$, surrounding σ_+, oriented counterclockwise, with index 1 with respect to each point of σ_+, and with index 0 with respect to each point of σ_-. P is called *spectral projection* relative to σ_+.

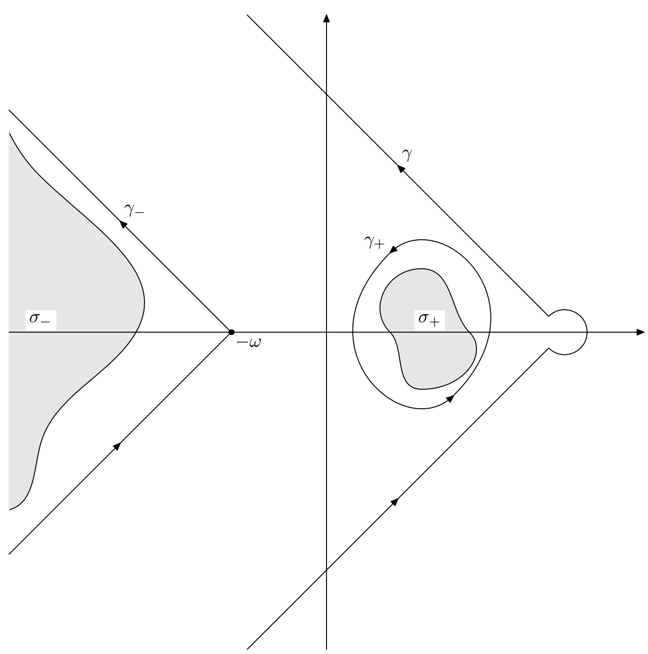

Figure 5.2: the curves γ_+, γ_-.

Proposition 5.2.1 *The following statements hold.*

(i) P *is a projection, that is* $P^2 = P$.

(ii) *For each* $t \geq 0$ *we have*

$$e^{tA}P = Pe^{tA} = \frac{1}{2\pi i}\int_{\gamma_+} e^{\lambda t}R(\lambda, A)d\lambda. \tag{5.7}$$

Consequently, $e^{tA}(P(X)) \subset P(X)$, $e^{tA}((I-P)(X)) \subset (I-P)(X)$.

(iii) $P \in \mathcal{L}(X, D(A^n))$ *for every* $n \in \mathbb{N}$. *Therefore,* $P(X) \subset D(A)$ *and the operator* $A_{|P(X)} : P(X) \to P(X)$ *is bounded.*

(iv) *For every* $\omega \in [0, \omega_+)$ *there exists* $N_\omega > 0$ *such that for every* $x \in P(X)$ *we have*[1]

$$\|e^{tA}x\| \le N_\omega e^{\omega t}\|x\|, \;\; t \le 0. \tag{5.8}$$

(v) *For each* $\omega \in [0, \omega_-)$ *there exists* $M_\omega > 0$ *such that for every* $x \in (I - P)(X)$ *we have*

$$\|e^{tA}x\| \le M_\omega e^{-\omega t}\|x\|, \;\; t \ge 0. \tag{5.9}$$

Proof. *Proof of (i).* Let γ_+, γ_+' be regular curves contained in $\rho(A)$ surrounding σ_+, with index 1 with respect to each point of σ_+, and such that γ_+ is contained in the bounded connected component of $\mathbb{C} \setminus \gamma_+'$. By the resolvent identity we have

$$\begin{aligned} P^2 &= \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_+'} R(\xi, A) d\xi \int_{\gamma_+} R(\lambda, A) d\lambda \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_+' \times \gamma_+} [R(\lambda, A) - R(\xi, A)](\xi - \lambda)^{-1} d\xi d\lambda \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_+} R(\lambda, A) d\lambda \int_{\gamma_+'} (\xi - \lambda)^{-1} d\xi - \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_+'} R(\xi, A) d\xi \int_{\gamma_+} (\xi - \lambda)^{-1} d\lambda \\ &= P. \end{aligned}$$

The proof of (ii) is similar and it is left as an exercise.

Proof of (iii). Since the path γ_+ is bounded and $\lambda \mapsto R(\lambda, A)$ is continuous with values in $\mathcal{L}(X, D(A))$, then $P \in \mathcal{L}(X, D(A))$, and

$$AP = \frac{1}{2\pi i} \int_{\gamma_+} AR(\lambda, A) d\lambda = \frac{1}{2\pi i} \int_{\gamma_+} \lambda R(\lambda, A) d\lambda.$$

Therefore, $AP \in \mathcal{L}(X, D(A))$ too. Moreover, if $x \in D(A)$ then $PAx = APx$. By recurrence, $P \in \mathcal{L}(X, D(A^n))$ for every $n \in \mathbb{N}$.

Proof of (iv). Since the part of A in $P(X)$ is bounded and its spectrum is σ_+ (see Exercise 3, the restriction of e^{tA} to $P(X)$ may be analytically continued to $(-\infty, 0)$, using formula (5.7). See Proposition 1.2.2.

For $\omega \in [0, \omega_+)$, we choose γ_+ such that $\inf_{\lambda \in \gamma_+} \operatorname{Re} \lambda = \omega$. Then for each $t \le 0$ and $x \in P(X)$ we have

$$\|e^{tA}x\| = \frac{1}{2\pi} \left\| \int_{\gamma_+} e^{\lambda t} R(\lambda, A)x \, d\lambda \right\| \le c \sup_{\lambda \in \gamma_+} |e^{\lambda t}| \, \|x\| = ce^{\omega t}\|x\|,$$

with $c = (2\pi)^{-1}|\gamma_+| \sup\{\|R(\lambda, A)\| : \; \lambda \in \gamma_+\}$, $|\gamma_+|$ = lenght of γ_+.

[1]For obvious notational reasons for each $x \in P(X)$ and $t < 0$ we write $e^{tA}x$ instead of $e^{tA_{|P(X)}}x$.

Proof of (v). For t small, say $t < 1$, estimate (5.9) is a consequence of (1.15). For $t \geq 1$ we write $e^{tA}(I-P)$ as

$$e^{tA}(I-P) = \frac{1}{2\pi i}\left(\int_{\gamma} - \int_{\gamma_+}\right) e^{\lambda t} R(\lambda, A) d\lambda = \frac{1}{2\pi i}\int_{\gamma_-} e^{\lambda t} R(\lambda, A) d\lambda,$$

where γ is the curve used in the definition of e^{tA} (see (1.10)), $\gamma_- = \{\lambda \in \mathbb{C} : \lambda = -\omega + re^{\pm i\eta},\ r \geq 0\}$ is oriented as usual and $\eta > \pi/2$. See Figure 5.2. The estimate is obtained as in the proof of Proposition 5.1.1, and it is left as an exercise. □

Corollary 5.2.2 *Let $x \in X$. Then*
(i) We have

$$\sup_{t\geq 0} \|e^{tA}x\| < +\infty \iff Px = 0.$$

In this case, $\|e^{tA}x\|$ decays exponentially to 0 as $t \to +\infty$.
(ii) For $x \in X$, the backward Cauchy problem

$$\begin{cases} v'(t) = Av(t), & t \leq 0, \\ v(0) = x, \end{cases} \tag{5.10}$$

has a bounded solution in $(-\infty, 0]$ if and only if $x \in P(X)$. In this case, the bounded solution is unique, it is given by $v(t) = e^{tA}x$, and it decays exponentially to 0 as $t \to -\infty$.

Proof. (i) Split every $x \in X$ as $x = Px + (I-P)x$, so that $e^{tA}x = e^{tA}Px + e^{tA}(I-P)x$. The norm of the second addendum decays exponentially to 0 as $t \to +\infty$. The norm of the first one is unbounded if $Px \neq 0$. Indeed, $Px = e^{-tA}e^{tA}Px$, so that $\|Px\| \leq \|e^{-tA}\|_{\mathcal{L}(P(X))}\|e^{tA}Px\| \leq N_\omega e^{-\omega t}\|e^{tA}Px\|$ with $\omega > 0$, which implies that $\|e^{tA}Px\| \geq e^{\omega t}\|Px\|/N_\omega$. Therefore $t \mapsto e^{tA}x$ is bounded in $\mathbb{R}_+$ if and only if $Px = 0$.

(ii) If $x \in P(X)$, the function $t \mapsto e^{tA}x$ is a strict solution of the backward Cauchy problem, and it decays exponentially as $t \to -\infty$. Conversely, if a backward bounded solution v does exist, then for $a < t \leq 0$ we have

$$v(t) = e^{(t-a)A}v(a) = e^{(t-a)A}(I-P)v(a) + e^{(t-a)A}Pv(a),$$

where $e^{(t-a)A}(I-P)v(a) = (I-P)v(t)$, $e^{(t-a)A}Pv(a) = Pv(t)$. Since $\|e^{(t-a)A}(I-P)\| \leq M_\omega e^{-\omega(t-a)}$, letting $a \to -\infty$ we get $(I-P)v(t) = 0$ for each $t \leq 0$, so that v is a solution to the backward problem in $P(X)$, $v(0) = x \in P(X)$ and hence $v(t) = e^{tA}x$. □

Note that problem (5.10) is ill posed in general. Changing t to $-t$, it is equivalent to a forward Cauchy problem with A replaced by $-A$, and $-A$ may have very bad properties. If A is sectorial, $-A$ is sectorial if and only if it is bounded (see Exercise 4, §1.3.5).

The subspaces $(I-P)(X)$ and $P(X)$ are often called the stable subspace and the unstable subspace, respectively.

Example 5.2.3 Let us consider again the operator $A_\infty : C_b^2(\mathbb{R}) \to C_b(\mathbb{R})$ studied in Subsection 2.1.1. We have $\rho(A_\infty) = \mathbb{C} \setminus (-\infty, 0]$, $\|\lambda R(\lambda, A_\infty)\| \leq (\cos\theta/2)^{-1}$, with $\theta =$

arg λ. In this case $\omega = s(A_\infty) = 0$, and estimates (5.2) are worse than (1.14) for large t. It is convenient to use (1.14), which gives

$$\|e^{tA_\infty}\| \leq M_0, \quad \|t^k A_\infty^k e^{tA_\infty}\| \leq M_k, \;\; k \in \mathbb{N}, \; t > 0.$$

Therefore $e^{tA}u_0$ is bounded for every initial datum u_0, and the k-th derivative with respect to time, the $2k$-th derivative with respect to x decay at least like t^{-k}, as $t \to +\infty$, in the sup norm.

Example 5.2.4 Let us now consider the problem

$$\begin{cases} u_t(t,x) = u_{xx}(t,x) + \alpha u(t,x), & t > 0, \;\; 0 \leq x \leq 1, \\ u(t,0) = u(t,1) = 0, & t \geq 0, \\ u(0,x) = u_0(x), & 0 \leq x \leq 1, \end{cases} \tag{5.11}$$

with $\alpha \in \mathbb{R}$. Choose $X = C([0,1])$, $A : D(A) = \{f \in C^2([0,1]) : f(0) = f(1) = 0\} \to X$, $Au = u'' + \alpha u$. Then the spectrum of A consists of the sequence of eigenvalues

$$\lambda_n = -\pi^2 n^2 + \alpha, \;\; n \in \mathbb{N}.$$

In particular, if $\alpha < \pi^2$ the spectrum is contained in the halfplane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$, and by Proposition 5.1.1 the solution $u(t,\cdot) = e^{tA}u_0$ of (5.11) and all its derivatives decay exponentially as $t \to +\infty$, for any initial datum u_0.

If $\alpha = \pi^2$, assumption (1.9) holds with $\omega = 0$. This is not immediate. A possible way to show it is to study the explicit expression of $R(\lambda, A)$ (which coincides with $R(\lambda - \pi^2, B)$ where $B : D(A) \to X$, $Bf = f''$) near $\lambda = 0$, see Example 2.1.2). Here we follow another approach. We observe that the operator $A_2 u := u'' + \pi^2 u$ with domain $D(A_2) = \{u \in H^2(0,1) : u(0) = u(1) = 0\}$ is sectorial in $L^2(0,1)$ and e^{tA_2} coincides with e^{tA} on $C([0,1])$. Indeed, if $f \in C([0,1])$ any solution $u \in D(A_2)$ of $\lambda u - A_2 u = f$ actually belongs to $C^2([0,1])$, so that $R(\lambda, A) = R(\lambda, A_2)$ in $C([0,1])$ for any $\lambda \in \rho(A_2) = \rho(A)$. Since the functions $u_k(x) = \sin(k\pi x)$ are eigenfunctions of A_2 with eigenvalue $(-k^2+1)\pi^2$ for any $k \in \mathbb{N}$, then (see Exercise 3, §1.3.5) $e^{tA_2}u_k = e^{-(k^2-1)\pi^2 t}u_k$ for any $t \geq 0$.

If $f \in C([0,1]) \subset L^2(0,1)$, we expand it in a sine series in $L^2(0,1)$,

$$f = \sum_{k=1}^{+\infty} c_k u_k, \qquad c_k = 2\int_0^1 f(x)u_k(x)dx. \tag{5.12}$$

To justify the expansion, it suffices to observe that (5.12) is the Fourier series of the function $\overline{f} : [-1,1] \to \mathbb{R}$ which is the odd extension of f. Hence,

$$e^{tA}f = e^{tA_2}f = \sum_{k=1}^{+\infty} c_k e^{-(k^2-1)\pi^2 t}u_k, \quad t \geq 0,$$

yields

$$\|e^{tA}f\|_\infty \leq 2\|f\|_\infty \sum_{k=1}^{+\infty} e^{-(k^2-1)\pi^2 t}, \quad t > 0.$$

which is bounded in $[1, +\infty)$. Since e^{tA} is an analytic semigroup, then $\|e^{tA}\|$ is bounded in $[0,1]$.

If $\alpha > \pi^2$, there are elements of the spectrum of A with positive real part. In the case where $\alpha \neq n^2\pi^2$ for every $n \in \mathbb{N}$, assumption (5.3) is satisfied. Let $m \in \mathbb{N}$ be such that $\pi^2 m^2 < \alpha < \pi^2(m+1)^2$. By Corollary 5.2.2, the initial data u_0 such that the solution is bounded are those which satisfy $Pu_0 = 0$. The projection P may be written as

$$P = \sum_{k=1}^{m} P_k, \tag{5.13}$$

where $P_k = \int_{|\lambda-\lambda_k|<\varepsilon} R(\lambda, A)d\lambda/(2\pi i)$, and the numbers $\lambda_k = -\pi^2 k^2 + \alpha$, $k = 1, \dots, m$, are the eigenvalues of A with positive real part, ε small. Let us show that

$$(P_k f)(x) = 2\sin(k\pi x) \int_0^1 \sin(k\pi y)\, f(y)dy, \quad x \in [0,1]. \tag{5.14}$$

For any $\lambda \neq \lambda_k$ expand $f \in C([0,1])$ as in (5.12). Using Exercise 3 in §1.3.5 we get

$$R(\lambda, A)f = R(\lambda, A_2)f = \sum_{n=1}^{+\infty} \frac{c_n}{\lambda - \lambda_n} u_n.$$

Hence

$$P_k f = \frac{1}{2\pi i} \int_{|\lambda-\lambda_k|\leq\varepsilon} R(\lambda, A)f\, d\lambda = c_k u_k.$$

Consequently, from (5.13) and (5.14) it follows that the solution of (5.11) is bounded in $[0, +\infty)$ if and only if

$$\int_0^1 \sin(k\pi y)\, u_0(y)dy = 0, \quad k = 1, \dots, m.$$

Exercises 5.2.5

1. Prove statement (ii) of Proposition 5.2.1 and complete the proof of statement (v).

2. Let A be a sectorial operator in X. Define the *growth bound*

$$\omega_A = \inf\{\gamma \in \mathbb{R} : \exists M > 0 \text{ s.t. } \|e^{tA}\| \leq Me^{\gamma t},\ t \geq 0\}.$$

Show that $s(A) = \omega_A$.

[Hint: show that if $\mathrm{Re}\,\lambda > \omega_A$ then

$$R(\lambda) = \int_0^{+\infty} e^{-\lambda t} e^{tA} dt$$

is the inverse of $\lambda I - A$].

3. Prove that the spectrum of the restrictions A_+ and A_- of A to $P(X)$ and to $(I - P)(X)$ are, respectively, σ_+ and σ_-.

[Hint: Prove that

$$R(\lambda, A_+) = \frac{1}{2\pi i} \int_{\gamma_+} \frac{R(\xi, A)}{\lambda - \xi} d\xi,$$

if $\lambda \notin \sigma_+$ and γ_+ is suitably chosen, and that

$$R(\lambda, A_-) = -\frac{1}{2\pi i}\int_{\gamma_+} \frac{R(\xi, A)}{\lambda - \xi} d\xi,$$

if $\lambda \notin \sigma_-$ and γ_+ is suitably chosen.]

4. Let α, $\beta \in \mathbb{R}$, and let A be the realization of the second order derivative in $C([0,1])$, with domain $\{f \in C^2([0,1]) : \alpha f(i) + \beta f'(i) = 0,\ i = 0, 1\}$. Find $s(A)$.

5. Let A satisfy (5.3), and let $T > 0$, $f : [-T, 0] \to P(X)$ be a continuous function. Prove that for every $x \in P(X)$ the backward problem

$$\begin{cases} u'(t) = Au(t) + f(t), & -T \le t \le 0, \\ u(0) = x, \end{cases}$$

has a unique strict solution in the interval $[-T, 0]$ with values in $P(X)$, given by the variation of constants formula

$$u(t) = e^{tA}x + \int_0^t e^{(t-s)A} f(s)ds, \quad -T \le t \le 0.$$

Prove that for each $\omega \in [0, \omega_+)$ we have

$$\|u(t)\| \le N_\omega\left(\|x\| + \frac{1}{\omega}\sup_{-T<t<0}\|f(t)\|\right).$$

6. *(A generalization of Proposition 5.2.1).* Let A be a sectorial operator such that $\sigma(A) = \sigma_1 \cup \sigma_2$, where σ_1 is compact, σ_2 is closed, and $\sigma_1 \cap \sigma_2 = \varnothing$. Define Q by

$$Q = \frac{1}{2\pi i}\int_\gamma R(\lambda, A)d\lambda,$$

where γ is any regular closed curve in $\rho(A)$, around σ_1, with index 1 with respect to each point in σ_1 and with index 0 with respect to each point in σ_2.

Prove that Q is a projection, that the part A_1 of A in $Q(X)$ is a bounded operator, and that the group generated by A_1 in $Q(X)$ may be expressed as

$$e^{tA_1} = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}R(\lambda, A)d\lambda.$$

5.3 Bounded solutions of nonhomogeneous problems in unbounded intervals

In this section we consider nonhomogeneous Cauchy problems in halflines. We start with

$$\begin{cases} u'(t) = Au(t) + f(t), & t > 0, \\ u(0) = u_0, \end{cases} \tag{5.15}$$

where $f : [0, +\infty) \to X$ is a continuous function and $u_0 \in X$. We assume throughout that A is hyperbolic, i.e. (5.3) holds, and we define σ_-, σ_+ and ω_-, ω_+ as in Section 5.2.

Let P be the projection defined by (5.6). Fix once and for all a positive number ω such that

$$-\omega_- < -\omega < \omega < \omega_+,$$

and let M_ω, N_ω be the constants given by Proposition 5.2.1(iv)(v).

Proposition 5.3.1 *Let $f \in C_b([0,+\infty);X)$, $u_0 \in X$. Then the mild solution u of* (5.15) *is bounded in $[0,+\infty)$ with values in X if and only if*

$$Pu_0 = -\int_0^{+\infty} e^{-sA}Pf(s)ds. \tag{5.16}$$

If (5.16) *holds we have*

$$u(t) = e^{tA}(I-P)u_0 + \int_0^t e^{(t-s)A}(I-P)f(s)ds - \int_t^{+\infty} e^{(t-s)A}Pf(s)ds, \;\; t \geq 0. \tag{5.17}$$

Proof. For every $t \geq 0$ we have $u(t) = (I-P)u(t) + Pu(t)$, where

$$(I-P)u(t) = e^{tA}(I-P)u_0 + \int_0^t e^{(t-s)A}(I-P)f(s)ds,$$

and

$$\begin{aligned} Pu(t) &= e^{tA}Pu_0 + \int_0^t e^{(t-s)A}Pf(s)ds \\ &= e^{tA}Pu_0 + \left(\int_0^{+\infty} - \int_t^{+\infty}\right) e^{(t-s)A}Pf(s)ds \\ &= e^{tA}\left(Pu_0 + \int_0^{+\infty} e^{-sA}Pf(s)ds\right) - \int_t^{+\infty} e^{(t-s)A}Pf(s)ds. \end{aligned}$$

For every $t \geq 0$ we have

$$\begin{aligned} \|(I-P)u(t)\| &\leq M_\omega e^{-\omega t}\|(I-P)u_0\| + \int_0^t M_\omega e^{-\omega(t-s)}ds \sup_{0\leq s\leq t}\|(I-P)f(s)\| \\ &\leq M_\omega\|(I-P)\|\left(\|u_0\| + \frac{1}{\omega}\|f\|_\infty\right), \end{aligned}$$

so that $(I-P)u$ is bounded in $[0,+\infty)$ with values in X. The integral $\int_t^{+\infty} e^{(t-s)A}Pf(s)ds$ is bounded too, and its norm does not exceed

$$N_\omega\int_t^{\infty} e^{\omega(t-s)}ds \sup_{s\geq 0}\|Pf(s)\| = \frac{N_\omega}{\omega}\|P\|\,\|f\|_\infty.$$

Hence u is bounded if and only if $t \mapsto e^{tA}\left(Pu_0 + \int_0^{+\infty} e^{-sA}Pf(s)ds\right)$ is bounded. On the other hand $y := Pu_0 + \int_0^{+\infty} e^{-sA}Pf(s)ds$ is an element of $P(X)$. By Corollary 5.2.2, $e^{tA}y$ is bounded if and only if $y = 0$, namely (5.16) holds. In this case, u is given by (5.17). □

Now we consider a backward problem,

$$\begin{cases} v'(t) = Av(t) + g(t), \;\; t \leq 0, \\ v(0) = v_0, \end{cases} \tag{5.18}$$

where $g : (-\infty, 0] \to X$ is a bounded and continuous function, and $v_0 \in X$.

Problem (5.18) is in general ill posed, and to find a solution we will have to assume rather restrictive conditions on the data. On the other hand, such conditions will ensure nice regularity properties of the solutions.

Note that the variation of constants formula (4.4) is well defined only for forward problems. Therefore, we have to make precise the concept of mild solution. A function $v \in C((-\infty, 0]; X)$ is said to be a mild solution of (5.18) in $(-\infty, 0]$ if $v(0) = v_0$ and for each $a < 0$ we have

$$v(t) = e^{(t-a)A}v(a) + \int_a^t e^{(t-s)A}g(s)ds, \quad a \le t \le 0. \tag{5.19}$$

In other words, v is a mild solution of (5.18) if and only if for every $a < 0$, setting $y = v(a)$, v is a mild solution of the problem

$$\begin{cases} v'(t) = Av(t) + g(t), & a < t \le 0, \\ v(a) = y, \end{cases} \tag{5.20}$$

and moreover $v(0) = v_0$.

Proposition 5.3.2 *Let $g \in C_b((-\infty, 0]; X)$, $v_0 \in X$. Then problem* (5.18) *has a mild solution $v \in C_b((-\infty, 0]; X)$ if and only if*

$$(I - P)v_0 = \int_{-\infty}^0 e^{-sA}(I - P)g(s)ds. \tag{5.21}$$

If (5.21) *holds, the bounded mild solution is unique and it is given by*

$$v(t) = e^{tA}Pv_0 + \int_0^t e^{(t-s)A}Pg(s)ds + \int_{-\infty}^t e^{(t-s)A}(I - P)g(s)ds, \quad t \le 0. \tag{5.22}$$

Proof. Assume that (5.18) has a bounded mild solution v. Then for every $a < 0$ and for every $t \in [a, 0]$ we have $v(t) = (I - P)v(t) + Pv(t)$, where

$$\begin{aligned}
(I - P)v(t) &= e^{(t-a)A}(I - P)v(a) + \int_a^t e^{(t-s)A}(I - P)g(s)ds \\
&= e^{(t-a)A}(I - P)v(a) + \left(\int_{-\infty}^t - \int_{-\infty}^a\right) e^{(t-s)A}(I - P)g(s)ds \\
&= e^{(t-a)A}\left((I - P)v(a) - \int_{-\infty}^a e^{(a-s)A}(I - P)g(s)ds\right) + v_1(t) \\
&= e^{(t-a)A}((I - P)v(a) - v_1(a)) + v_1(t).
\end{aligned}$$

The function

$$v_1(t) := \int_{-\infty}^t e^{(t-s)A}(I - P)g(s)ds, \quad t \le 0,$$

is bounded in $(-\infty, 0]$. Indeed,

$$\|v_1(t)\| \le M_\omega \sup_{s \le 0} \|(I - P)g(s)\| \int_{-\infty}^t e^{-\omega(t-s)}ds \le \frac{M_\omega}{\omega}\|I - P\| \, \|g\|_\infty. \tag{5.23}$$

Moreover v is bounded by assumption, hence $\sup_{a\le 0}\|(I-P)v(a)\| < +\infty$. Letting $a \to -\infty$ and using estimate (5.9) we get

$$(I-P)v(t) = v_1(t), \;\; t \le 0.$$

Taking $t=0$, we get (5.21). On the other hand, Pv is a mild (in fact, strict) solution to $w'(t) = Aw(t) + Pg(t)$, and since $Pv(0) = Pv_0$, by Exercise 5 in §5.2.5, we have for $t \le 0$,

$$Pv(t) = e^{tA}Pv_0 + \int_0^t e^{(t-s)A}Pg(s)ds.$$

Summing up, v is given by (5.22).

Conversely, assume that (5.21) holds, and define the function $v(t) := v_1(t) + v_2(t)$, where v_1 is defined above and $v_2(t) := e^{tA}Pv_0 + \int_0^t e^{(t-s)A}Pg(s)ds$. Then v_1 is bounded by estimate (5.23), and v_2 is bounded by Exercise 5 in §5.2.5 again, so that v is bounded.

One checks easily that v is a mild solution of (5.20) for every $a<0$, and, since (5.21) holds, we have $v(0) = Pv_0 + \int_{-\infty}^0 e^{-sA}(I-P)g(s)ds = Pv_0 + (I-P)v_0 = v_0$. Then v is a bounded mild solution to (5.18). □

5.4 Solutions with exponential growth and exponential decay

We now replace assumption (5.3) by

$$\sigma(A) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = \omega\} = \varnothing, \tag{5.24}$$

for some $\omega \in \mathbb{R}$. Note that (5.24) is satisfied by every $\omega > s(A)$. If I is any (unbounded) interval and $\omega \in \mathbb{R}$ we set

$$C_\omega(I;X) := \{f : I \to X \text{ continuous}, \;\|f\|_{C_\omega} := \sup_{t\in I}\|e^{-\omega t}f(t)\| < +\infty\}.$$

Let $f \in C_\omega((0,+\infty);X)$, $g \in C_\omega((-\infty,0);X)$. Since $e^{t(A-\omega I)} = e^{-\omega t}e^{tA}$, one checks easily that problems (5.15) and (5.18) have mild solutions $u \in C_\omega((0,+\infty);X)$, $v \in C_\omega((-\infty,0];X)$ if and only if the problems

$$\begin{cases} \tilde u'(t) = (A-\omega I)\tilde u(t) + e^{-\omega t}f(t), \;\; t>0, \\ u(0) = u_0, \end{cases} \tag{5.25}$$

$$\begin{cases} \tilde v'(t) = (A-\omega I)\tilde v(t) + e^{-\omega t}g(t), \;\; t\le 0, \\ v(0) = v_0, \end{cases} \tag{5.26}$$

have mild solutions $\tilde u \in C_b((0,+\infty);X)$, $\tilde v \in C_b((-\infty,0];X)$, and in this case we have $u(t) = e^{\omega t}\tilde u(t)$, $v(t) = e^{\omega t}\tilde v(t)$. On the other hand, the operator $\tilde A = A - \omega I : D(A) \to X$ is sectorial and hyperbolic, hence all the results of the previous section may be applied to problems (5.25) and (5.26). Note that such results involve the spectral projection relative to $\sigma_+(\tilde A)$, i.e. the operator

$$\frac{1}{2\pi i}\int_{\gamma_+} R(\lambda, A-\omega I)d\lambda = \frac{1}{2\pi i}\int_{\gamma_+ + \omega} R(z,A)dz := P_\omega, \tag{5.27}$$

where the path $\gamma_+ + \omega$ surrounds $\sigma_+^\omega := \{\lambda \in \sigma(A) : \ \mathrm{Re}\,\lambda > \omega\}$ and is contained in the halfplane $\{\mathrm{Re}\,\lambda > \omega\}$. Set moreover $\sigma_-^\omega := \{\lambda \in \sigma(A) : \ \mathrm{Re}\,\lambda < \omega\}$. Note that if $\omega > s(A)$ then $P_\omega = 0$.

Applying the results of Propositions 5.3.1 and 5.3.2 we get the following theorem.

Theorem 5.4.1 *Under assumption* (5.24) *let* P_ω *be defined by* (5.27). *The following statements hold:*

(i) If $f \in C_\omega((0,+\infty);X)$ *and* $u_0 \in X$*, the mild solution* u *of problem* (5.15) *belongs to* $C_\omega((0,+\infty);X)$ *if and only if*

$$P_\omega u_0 = -\int_0^{+\infty} e^{-s(A-\omega I)} e^{-\omega s} P_\omega f(s)ds,$$

that is [2]

$$P_\omega u_0 = -\int_0^{+\infty} e^{-sA} P_\omega f(s)ds.$$

In this case u *is given by* (5.17)*, and there exists* $C_1 = C_1(\omega)$ *such that*

$$\sup_{t\geq 0} \|e^{-\omega t}u(t)\| \leq C_1(\|u_0\| + \sup_{t\geq 0}\|e^{-\omega t} f(t)\|).$$

(ii) If $g \in C_\omega((-\infty,0);X)$ *and* $v_0 \in X$*, problem* (5.18) *has a mild solution* $v \in C_\omega((-\infty, 0];X)$ *if and only if* (5.21) *holds. In this case the solution is unique in* $C_\omega((-\infty,0];X)$ *and it is given by* (5.22). *There is* $C_2 = C_2(\omega)$ *such that*

$$\sup_{t\leq 0} \|e^{-\omega t}v(t)\| \leq C_2(\|v_0\| + \sup_{t\leq 0}\|e^{-\omega t} g(t)\|).$$

Remark 5.4.2 The definition 5.3 of a hyperbolic operator requires that X be a complex Banach space, and the proofs of the properties of P, Pe^{tA} etc., rely on properties of Banach space valued holomorphic functions.

If X is a real Banach space, we have to use the complexification of X as in Remark 1.3.17. If $A : D(A) \to X$ is a linear operator such that the complexification $\widetilde{A}$ is sectorial in $\widetilde{X}$, the projection P maps X into itself. To prove this claim, it is convenient to choose as γ_+ a circumference $C = \{\omega' + re^{i\eta} : \eta \in [0,2\pi]\}$ with centre ω' on the real axis. For each $x \in X$ we have

$$\begin{aligned} Px &= \frac{1}{2\pi}\int_0^{2\pi} re^{i\eta}R(\omega' + re^{i\eta}, A)x\,d\eta \\ &= \frac{r}{2\pi}\int_0^{\pi} \left(e^{i\eta}R(\omega' + re^{i\eta}, A) - e^{-i\eta}R(\omega' + re^{-i\eta}, A)\right) x\,d\eta, \end{aligned}$$

and the imaginary part of the function in the integral is zero. Therefore, $P(X) \subset X$, and consequently $(I-P)(X) \subset X$. Thus, the results of the last two sections remain true even if X is a real Banach space.

[2] Note that since σ_+^ω is bounded, $e^{tA}P_\omega$ is well defined also for $t < 0$, and the results of Proposition 5.2.1 hold, with obvious modifications.

Example 5.4.3 Consider the nonhomogeneous heat equation

$$\begin{cases} u_t(t,x) = u_{xx}(t,x) + f(t,x), & t > 0, \quad 0 \le x \le 1, \\ u(t,0) = u(t,1) = 0, & t \ge 0, \\ u(0,x) = u_0(x), & 0 \le x \le 1, \end{cases} \tag{5.28}$$

where $f : [0,+\infty) \times [0,1] \to \mathbb{R}$ is continuous, u_0 is continuous and vanishes at $x = 0$, $x = 1$. We choose as usual $X = C([0,1])$, $A : D(A) = \{u \in C^2([0,1]) : u(0) = u(1) = 0\} \to X$, $Au = u''$. Since $s(A) = -\pi^2$, then A is hyperbolic, and in this case the projection P defined in (5.6) vanishes. Proposition 5.3.1 implies that for every bounded and continuous f and for every $u_0 \in C([0,1])$ such that $u_0(0) = u_0(1) = 0$, the solution of (5.28) is bounded. Note that $u_0(0) = u_0(1) = 0$ is a compatibility condition (i.e. a necessary condition) for the solution of problem (5.28) to be continuous up to $t = 0$ and to satisfy $u(0,\cdot) = u_0$.

As far as exponentially decaying solutions are concerned, we use Theorem 5.4.1(i). Fixed $\omega \ne \pi^2 n^2$ for each $n \in \mathbb{N}$, f continuous and such that

$$\sup_{t \ge 0,\, 0 \le x \le 1} |e^{\omega t} f(t,x)| < +\infty$$

the solution u of (5.28) satisfies

$$\sup_{t \ge 0,\, 0 \le x \le 1} |e^{\omega t} u(t,x)| < +\infty$$

if and only if (5.16) holds. This is equivalent to (see Example 5.2.4)

$$\int_0^1 u_0(x) \sin(k\pi x)\, dx = -\int_0^{+\infty} e^{k^2\pi^2 s} \int_0^1 f(s,x) \sin(k\pi x)\, dx\, ds,$$

for every natural number k such that $\pi^2 k^2 < \omega$. (We remark that since $A \sin(k\pi x) = -k^2\pi^2 \sin(k\pi x)$ we have $e^{tA} \sin(k\pi x) = e^{-t\pi^2 k^2} \sin(k\pi x)$, for every $t \in \mathbb{R}$).

Let us now consider the backward problem

$$\begin{cases} v_t(t,x) = v_{xx}(t,x) + g(t,x), & t < 0, \quad 0 \le x \le 1, \\ v(t,0) = v(t,1) = 0, & t \le 0, \\ v(0,x) = v_0(x), & \quad\quad\ \ 0 \le x \le 1, \end{cases} \tag{5.29}$$

to which we apply Proposition 5.3.2. Since $P = 0$, if $g : (-\infty, 0] \times [0,1] \to \mathbb{R}$ is bounded and continuous, there is only a final datum v_0 such that the solution is bounded, and it is given by (see formula (5.21))

$$v_0(x) = \left(\int_{-\infty}^0 e^{-sA} g(s,\cdot) ds \right)(x), \quad 0 \le x \le 1.$$

By Theorem 5.4.1(i), a similar conclusion holds if g is continuous and it decays exponentially,

$$\sup_{t \le 0,\, 0 \le x \le 1} |e^{-\omega t} g(t,x)| < +\infty$$

with $\omega > 0$.

Exercises 5.4.4

1. Let A be a hyperbolic sectorial operator. Using Propositions 5.3.1 and 5.3.2, prove that for every $h \in C_b(\mathbb{R}; X)$ the problem

$$z'(t) = Az(t) + h(t), \quad t \in \mathbb{R}, \tag{5.30}$$

has a unique mild solution $z \in C_b(\mathbb{R}; X)$, given by

$$z(t) = \int_{-\infty}^{t} e^{(t-s)A}(I-P)h(s)ds - \int_{t}^{\infty} e^{(t-s)A}Ph(s)ds, \quad t \in \mathbb{R}.$$

(The definition of a mild solution of (5.30) is like the definition of a mild solution to (5.18)). Prove that

(i) if h is constant, then z is constant;

(ii) if $\lim_{t\to+\infty} h(t) = h_\infty$ (respectively, $\lim_{t\to-\infty} h(t) = h_{-\infty}$) then

$$\lim_{t\to+\infty} z(t) = \int_{0}^{+\infty} e^{sA}(I-P)h_\infty ds - \int_{-\infty}^{0} e^{sA}Ph_\infty ds$$

(respectively, the same with $+\infty$ replaced by $-\infty$);

(iii) if h is T-periodic, then z is T-periodic.

2. Prove that the spectrum of the realization of the Laplacian in $C_b(\mathbb{R}^N)$ and in $L^p(\mathbb{R}^N)$ $(1 \leq p < +\infty)$ is $(-\infty, 0]$.

   [Hint: To prove that $\lambda \leq 0$ belongs to $\sigma(\Delta)$, use or approximate the functions $f(x_1, \ldots, x_N) = e^{i\sqrt{-\lambda}x_1}$].

3. Let Ω be a bounded open set with a boundary of class C^2. Let moreover

$$\begin{aligned} D(A_1) &= \Big\{u \in \bigcap_{1\leq p<+\infty} W^{2,p}(\Omega) : \Delta u \in C(\overline{\Omega}),\ u = 0 \text{ on } \partial\Omega\Big\}, \\ D(A_2) &= \Big\{u \in \bigcap_{1\leq p<+\infty} W^{2,p}(\Omega) : \Delta u \in C(\overline{\Omega}),\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\Big\} \end{aligned}$$

and $A_i u = \Delta u$ for any $u \in D(A_i)$, $i = 1, 2$.

Show that A_1 and A_2 have compact resolvent and that $s(A_1) < 0$ and $s(A_2) = 0$.

Chapter 6

Nonlinear problems

6.1 Nonlinearities defined in X

Consider the initial value problem

$$\begin{cases} u'(t) = Au(t) + F(t, u(t)), \quad t > 0, \\ \\ u(0) = u_0, \end{cases} \tag{6.1}$$

where $A : D(A) \subset X \to X$ is a sectorial operator and $F : [0, T] \times X \to X$. Throughout this section we shall assume that F is continuous, and that for every $R > 0$ there is $L > 0$ such that

$$\|F(t, x) - F(t, y)\| \leq L\|x - y\|, \quad t \in [0, T], \quad x, y \in B(0, R). \tag{6.2}$$

This means that F is Lipschitz continuous with respect to x on any bounded subset of X, with Lipschitz constant independent of t.

As in the case of linear problems, we say that a function u defined in an interval $I = [0, \tau)$ or $I = [0, \tau]$, with $\tau \leq T$, is a *strict solution* of problem (6.1) in I if it is continuous with values in $D(A)$ and differentiable with values in X in the interval I, and it satisfies (6.1). We say that it is a *classical solution* if it is continuous with values in $D(A)$ and differentiable with values in X in the interval $I \setminus \{0\}$, it is continuous in I with values in X, and it satisfies (6.1). We say that it is a *mild solution* if it is continuous with values in X in $I \setminus \{0\}$ and it satisfies

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}F(s, u(s))ds, \quad t \in I. \tag{6.3}$$

By Proposition 4.1.2 every strict or classical solution satisfies (6.3).

For notational convenience, throughout this section we set

$$M_0 = \sup_{0 \leq t \leq T} \|e^{tA}\|_{\mathcal{L}(X)}. \tag{6.4}$$

6.1.1 Local existence, uniqueness, regularity

It is natural to solve (6.3) using a fixed point theorem to find a mild solution, and then to show that, under appropriate assumptions, the mild solution is classical or strict.

Theorem 6.1.1 *The following statements hold.*

(a) *If* $u, v \in C_b((0,a];X)$ *are mild solutions for some* $a \in (0,T]$, *then* $u \equiv v$.

(b) *For every* $\overline{u} \in X$ *there exist* r, $\delta > 0$, $K > 0$ *such that for* $\|u_0 - \overline{u}\| \le r$ *problem* (6.1) *has a mild solution* $u = u(\cdot; u_0) \in C_b((0,\delta];X)$. *The function* u *belongs to* $C([0,\delta];X)$ *if and only if* $u_0 \in \overline{D(A)}$.

Moreover for every u_0, $u_1 \in B(\overline{u}, r)$ *we have*

$$\|u(t;u_0) - u(t;u_1)\| \le K\|u_0 - u_1\|, \ \ 0 \le t \le \delta. \tag{6.5}$$

Proof. *Proof of (a).* Let $u, v \in C_b((0,a];X)$ be mild solutions to (6.1) and set $w = v - u$. By (6.3), the function w satisfies

$$w(t) = \int_0^t e^{(t-s)A}\left(F(s,v(s)) - F(s,u(s))\right)ds, \quad 0 \le t < a.$$

Using (6.2) with $R = \max\{\sup_{0<t\le a}\|u(t)\|, \sup_{0<t\le a}\|v(t)\|\}$ we see that

$$\|w(t)\| \le LM_0 \int_0^t \|w(s)\| ds.$$

The Gronwall lemma (see Exercise 3 in §1.2.4) implies that $w = 0$ in $[0,a]$.

Proof of (b). Fix $R > 0$ such that $R \ge 8M_0\|\overline{u}\|$, so that if $\|u_0 - \overline{u}\| \le r = R/(8M_0)$ we have

$$\sup_{0\le t\le T} \|e^{tA}u_0\| \le R/4.$$

Here M_0 is given by (6.4). Moreover, let $L > 0$ be such that

$$\|F(t,v) - F(t,w)\| \le L\|v - w\|, \quad 0 \le t \le T, \ v, w \in B(0,R).$$

We look for a mild solution belonging to the metric space

$$Y = \{u \in C_b((0,\delta];X) : \ \|u(t)\| \le R \ \ \forall t \in (0,\delta]\},$$

where $\delta \in (0,T]$ has to be chosen properly. Y is the closed ball with centre at 0 and radius R in the space $C_b((0,\delta];X)$, and for every $v \in Y$ the function $t \mapsto F(t, v(t))$ belongs to $C_b((0,\delta];X)$. We define the operator Γ in Y, by means of

$$\Gamma(v)(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}F(s,v(s))ds, \ \ 0 \le t \le \delta. \tag{6.6}$$

Clearly, a function $v \in Y$ is a mild solution of (6.1) in $[0,\delta]$ if and only if it is a fixed point of Γ.

We shall show that Γ is a contraction and maps Y into itself provided that δ is sufficiently small.

Let v_1, $v_2 \in Y$. We have

$$\begin{aligned} \|\Gamma(v_1) - \Gamma(v_2)\|_{C_b((0,\delta];X)} &\le \delta M_0 \|F(\cdot, v_1(\cdot)) - F(\cdot, v_2(\cdot))\|_{C_b((0,\delta];X)} \\ &\le \delta M_0 L \|v_1 - v_2\|_{C_b((0,\delta];X)}. \end{aligned} \tag{6.7}$$

Therefore, if

$$\delta \le \delta_0 = (2M_0L)^{-1},$$

Γ is a contraction with constant $1/2$ in Y. Moreover if $\delta \le \delta_0$, for every $v \in Y$ we have

$$\begin{aligned}\|\Gamma(v)\|_{C_b((0,\delta];X)} &\le \|\Gamma(v) - \Gamma(0)\|_{C_b((0,\delta];X)} + \|\Gamma(0)\|_{C((0,\delta];X)} \\ &\le R/2 + \|e^{\cdot A}u_0\|_{C_b((0,\delta];X)} + M_0\delta\|F(\cdot,0)\|_{C_b((0,\delta];X)} \\ &\le R/2 + R/4 + M_0\delta\|F(\cdot,0)\|_{C_b((0,\delta];X)}.\end{aligned} \tag{6.8}$$

Therefore if $\delta \le \delta_0$ is such that

$$M_0\delta\|F(\cdot,0)\|_{C_b((0,\delta];X)} \le R/4,$$

then Γ maps Y into itself, so that it has a unique fixed point in Y.

Concerning the continuity of u up to $t = 0$, we remark that the function $t \mapsto u(t) - e^{tA}u_0$ belongs to $C([0,\delta];X)$, whereas by Proposition 1.3.6(i) $t \mapsto e^{tA}u_0$ belongs to $C([0,\delta];X)$ if and only if $u_0 \in \overline{D(A)}$. Therefore, $u \in C([0,\delta];X)$ if and only if $u_0 \in \overline{D(A)}$.

Let us prove the statement about the dependence on the initial data. Let u_0, u_1 belong to $B(\overline{u}, r)$. Since Γ is a contraction with constant $1/2$ in Y and both $u(\cdot;u_0)$, $u(\cdot;u_1)$ belong to Y, we have

$$\|u(\cdot;u_0) - u(\cdot;u_1)\|_{C_b((0,\delta];X)} \le 2\|e^{\cdot A}(u_0 - u_1)\|_{C_b((0,\delta];X)} \le 2M_0\|u_0 - u_1\|,$$

so that (6.5) holds, with $K = 2M_0$. □

6.1.2 The maximally defined solution

Now we can construct a maximally defined solution as follows. Set

$$\begin{cases} \tau(u_0) = \sup\{a > 0 : \text{problem (6.1) has a mild solution } u_a \text{ in } [0,a]\} \\ \\ u(t) = u_a(t), \quad \text{if } t \le a. \end{cases}$$

Recalling Theorem 6.1.1(a), u is well defined in the interval

$$I(u_0) := \cup\{[0,a] : \text{ problem (6.1) has a mild solution } u_a \text{ in } [0,a]\},$$

and we have $\tau(u_0) = \sup I(u_0)$.

Let us now prove results concerning regularity and existence in the large of the solution.

Proposition 6.1.2 *Assume that there is $\theta \in (0,1)$ such that for every $R > 0$ we have*

$$\|F(t,x) - F(s,x)\| \le C(R)(t-s)^\theta, \quad 0 \le s \le t \le T, \ \|x\| \le R. \tag{6.9}$$

Then, for every $u_0 \in X$, $u \in C^\theta([\varepsilon,\tau(u_0)-\varepsilon];D(A)) \cap C^{1+\theta}([\varepsilon,\tau(u_0)-\varepsilon];X)$ and $u' \in B([\varepsilon,\tau(u_0)-\varepsilon];D_A(\theta,\infty))$ for every $\varepsilon \in (0,\tau(u_0)/2)$. Moreover the following statements hold.

(i) If $u_0 \in \overline{D(A)}$ then u is a classical solution of (6.1).

(ii) If $u_0 \in D(A)$ and $Au_0 + F(0,u_0) \in \overline{D(A)}$ then u is a strict solution of (6.1).

Proof. Let $a < \tau(u_0)$ and $0 < \varepsilon < a$. Since $t \mapsto F(t, u(t))$ belongs to $C_b((0, a]; X)$, Proposition 4.1.5 implies that the function $v(t) := \int_0^t e^{(t-s)A} F(s, u(s))ds$ belongs to $C^\alpha([0, a]; X)$. Moreover, $t \mapsto e^{tA}u_0$ belongs to $C^\infty([\varepsilon, a]; X)$. Summing up, we find that u belongs to $C^\theta([\varepsilon, a]; X)$. Assumptions (6.2) and (6.9) imply that the function $t \mapsto F(t, u(t))$ belongs to $C^\theta([\varepsilon, a]; X)$. Since u satisfies

$$u(t) = e^{(t-\varepsilon)A}u(\varepsilon) + \int_\varepsilon^t e^{(t-s)A} F(s, u(s))ds, \;\; \varepsilon \le t \le a, \tag{6.10}$$

we may apply Theorem 4.1.7 in the interval $[\varepsilon, a]$ (see Remark 4.1.12), and we get $u \in C^\theta([2\varepsilon, a];\, D(A)) \cap C^{1+\theta}([2\varepsilon, a]; X)$ for each $\varepsilon \in (0, a/2)$, and

$$u'(t) = Au(t) + F(t, u(t)), \;\; \varepsilon < t \le a.$$

Exercise 2 in §4.1.13 implies that u' is bounded with values in $D_A(\theta, \infty)$ in $[2\varepsilon, a]$. Since a and ε are arbitrary, then $u \in C^\theta([\varepsilon, \tau(u_0) - \varepsilon]; D(A)) \cap C^{1+\theta}([\varepsilon, \tau(u_0) - \varepsilon]; X)$ for each $\varepsilon \in (0, \tau(u_0)/2)$. If $u_0 \in \overline{D(A)}$, then $t \mapsto e^{tA}u_0$ is continuous up to 0, and statement (i) follows.

Let us prove (ii). By Proposition 4.1.5, we already know that the function v defined above is θ-Hölder continuous up to $t = 0$ with values in X. Since $u_0 \in D(A) \subset D_A(\theta, \infty)$, then the function $t \mapsto e^{tA}u_0$ is θ-Hölder continuous up to $t = 0$, too. Therefore u is θ-Hölder continuous up to $t = 0$ with values in X, so that $t \mapsto F(t, u(t))$ is θ-Hölder continuous in $[0, a]$ with values in X. Statement (ii) follows now from Theorem 4.1.7(ii). □

Proposition 6.1.3 *Let u_0 be such that $I(u_0) \ne [0, T]$. Then $t \mapsto \|u(t)\|$ is unbounded in $I(u_0)$.*

Proof. Assume by contradiction that u is bounded in $I(u_0)$ and set $\tau = \tau(u_0)$. Then $t \mapsto F(t, u(t; u_0))$ is bounded and continuous with values in X in the interval $(0, \tau)$. Since u satisfies the variation of constants formula (6.3), it may be continuously extended to $t = \tau$, in such a way that the extension is Hölder continuous in every interval $[\varepsilon, \tau]$, with $0 < \varepsilon < \tau$. Indeed, $t \mapsto e^{tA}u_0$ is well defined and analytic in the whole halfline $(0, +\infty)$, and $u - e^{tA}u_0$ belongs to $C^\alpha([0, \tau]; X)$ for each $\alpha \in (0, 1)$ by Proposition 4.1.5.

By Theorem 6.1.1, the problem

$$v'(t) = Av(t) + F(t, v(t)), \;\; t \ge \tau, \quad v(\tau) = u(\tau),$$

has a unique mild solution $v \in C([\tau, \tau + \delta]; X)$ for some $\delta > 0$. Note that v is continuous up to $t = \tau$ because $u(\tau) \in \overline{D(A)}$ (why? See Exercise 6, §6.1.5, for a related stronger statement).

The function w defined by $w(t) = u(t)$ for $0 \le t < \tau$, and $w(t) = v(t)$ for $\tau \le t \le \tau + \delta$, is a mild solution of (6.1) in $[0, \tau + \delta]$. See Exercise 2 in §6.1.5. This is in contradiction with the definition of τ. Therefore, u cannot be bounded. □

Note that the proof of proposition 6.1.3 shows also that if $I(u_0) \ne [0, T]$ then $\tau(u_0) = \sup I(u_0) \notin I(u_0)$.

The result of Proposition 6.1.3 is used to prove existence in the large when we have an *a priori* estimate on the norm of $u(t)$. Such *a priori* estimate is easily available for each u_0 if f does not grow more than linearly as $\|x\| \to +\infty$. Note that Proposition 6.1.3 and next Proposition 6.1.4 are quite similar to the case of ordinary differential equations.

Proposition 6.1.4 *Assume that there is $C > 0$ such that*

$$\|F(t,x)\| \leq C(1 + \|x\|) \;\; x \in X, \; t \in [0,T]. \tag{6.11}$$

Let $u : I(u_0) \to X$ be the mild solution to (6.1). Then u is bounded in $I(u_0)$ with values in X. Consequently, $I(u_0) = [0,T]$.

Proof. For each $t \in I(u_0)$ we have

$$\|u(t)\| \;\leq M_0\|u_0\| + M_0 C\int_0^t (1 + \|u(s)\|)ds = M_0\|u_0\| + M_0 C\left(T + \int_0^t \|u(s)\|ds\right).$$

Applying the Gronwall lemma to the real-valued function $t \mapsto \|u(t)\|$ we get

$$\|u(t)\| \leq (M_0\|u_0\| + M_0 CT)e^{M_0 Ct}, \;\; t \in I(u_0),$$

and the statement follows. □

We remark that (6.11) is satisfied if F is globally Lipschitz continuous with respect to x, with Lipschitz constant independent of t.

Exercises 6.1.5

1. Let $F : [0,T] \times X \to X$ be a continuous function. Prove that

   (a) if F satisfies (6.2) and $u \in C_b((0,\delta]; X)$ with $0 < \delta \leq T$, then the composition $\varphi(t) := F(t, u(t))$ belongs to $C_b((0,\delta]; X)$,

   (b) if F satisfies (6.2) and (6.9), and $u \in C^\theta([a,b]; X)$ with $0 \leq a < b \leq T$, $0 < \theta < 1$, then the composition $\varphi(t) := F(t, u(t))$ belongs to $C^\theta([a,b]; X)$.

   These properties have been used in the proofs of Theorem 6.1.1 and of Proposition 6.1.2.

2. Prove that if u is a mild solution to (6.1) in an interval $[0, t_0]$ and v is a mild solution to

$$\begin{cases} v'(t) = Av(t) + F(t, v(t)), & t_0 < t \leq t_1, \\ v(t_0) = u(t_0), \end{cases}$$

   then the function z defined by $z(t) = u(t)$ for $0 \leq t \leq t_0$, and $z(t) = v(t)$ for $t_0 \leq t \leq t_1$, is a mild solution to (6.1) in the interval $[0, t_1]$.

3. Under the assumptions of Theorem 6.1.1, for $t_0 \in (0,T)$ let $u(\cdot; t_0, x) : [t_0, \tau(t_0, x)) \to X$ be the maximally defined solution to problem $u' = Au + F(t,u)$, $t > t_0$, $u(t_0) = x$.

   (a) Prove that for each $a \in (0, \tau(0,x))$ we have $\tau(a, u(a; 0, x)) = \tau(0,x)$ and for $t \in [a, \tau(0; x))$ we have $u(t; a, u(a; 0, x)) = u(t; 0, x)$.

   (b) Prove that if F does not depend on t, then $\tau(0, u(a; 0, x)) = \tau(0,x) - a$, and for $t \in [0, \tau(0,x) - a)$ we have $u(t; 0, u(a; 0, x)) = u(a + t; 0, x)$.

4. Under the assumptions of Theorem 6.1.1 and with the notation of Exercise 3, prove that for each u_0 and for each $b \in (0, \tau(0, u_0))$ there are $r > 0$, $K > 0$ such that if $\|u_0 - u_1\| \leq r$ then $\tau(0, u_1) \geq b$ and $\|u(t; 0, u_0) - u(t; 0, u_1)\| \leq K\|u_0 - u_1\|$ for each $t \in [0,b]$.

   [Hint: cover the orbit $\{u(t; 0, u_0) : \; 0 \leq t \leq b\}$ with a finite number of balls as in the statement of Theorem 6.1.1].

5. (*A variant of Theorem 6.1.1*) Let $\mathcal{O}$ be a nonempty open set in X, and let $F : [0,T] \times \mathcal{O} \to X$ be a continuous function which is locally Lipschitz continuous in x, uniformly with respect to time, i.e. for each $x_0 \in \mathcal{O}$ there are $r > 0$, $L > 0$ such that $\|F(t,x) - F(t,y)\| \leq L\|x - y\|$ for each x, $y \in B(x_0, r)$. Prove that for every $\overline{u} \in \mathcal{O}$ there exist s, $\delta > 0$, $K > 0$ such that for every $u_0 \in \overline{D(A)} \cap B(\overline{u}, s)$ the problem (6.1) has a unique mild solution $u = u(\cdot; u_0) \in C([0,\delta]; X)$. Moreover for u_0, $u_1 \in \overline{D(A)} \cap B(\overline{u}, s)$ we have

$$\|u(t;u_0) - u(t;u_1)\| \leq K\|u_0 - u_1\|, \quad 0 \leq t \leq \delta.$$

[Hint: follow the proof of Theorem 6.1.1, with $Y = B(0,\rho) \subset C([0,\delta]; X)$, but now ρ has to be small].

6. Prove that if F satisfies (6.2), then for every $u_0 \in X$, the mild solution u of problem (6.1) is bounded with values in $D_A(\beta,\infty)$ in the interval $[\varepsilon, \tau(u_0) - \varepsilon]$, for each $\beta \in (0,1)$ and $\varepsilon \in (0, \tau(u_0)/2)$.

6.2 Reaction–diffusion equations and systems

Let us consider a differential system in $[0,T] \times \mathbb{R}^n$. Let $d_1, \ldots, d_m > 0$ and let D be the diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_m)$. Consider the problem

$$\begin{cases} u_t(t,x) = D\Delta u(t,x) + f(t,x,u(t,x)), & t > 0, \quad x \in \mathbb{R}^n; \\ u(0,x) = u_0(x), & \qquad\quad\;\; x \in \mathbb{R}^n, \end{cases} \tag{6.12}$$

where $u = (u_1, \ldots, u_m)$ is unknown, and the regular function $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, the bounded and continuous $u_0 : \mathbb{R}^n \to \mathbb{R}^m$ are given.

This type of problems are often encountered as mathematical models in chemistry and in biology. The part $D\Delta u$ in the system is called the diffusion part, the numbers d_i are called the diffusion coefficients, $f(t,x,u)$ is called the reaction part. Detailed treatments of these problems may be found in the books of Rothe [14], Smoller [15], Pao [12].

Set

$$X = C_b(\mathbb{R}^n; \mathbb{R}^m).$$

The linear operator A defined by

$$\begin{cases} D(A) = \{u \in W^{2,p}_{loc}(\mathbb{R}^n; \mathbb{R}^m), \quad p \geq 1 : \; u, \; \Delta u \in X\}, \\[2ex] A : D(A) \to X, \quad Au = D\Delta u, \end{cases}$$

is sectorial in X, see Section 2.3 and Exercise 1 in §1.3.18, and

$$\overline{D(A)} = BUC(\mathbb{R}^n; \mathbb{R}^m).$$

We assume that f is continuous, and that there exists $\theta \in (0,1)$ such that for every $R > 0$ there is $K = K(R) > 0$ such that

$$|f(t,x,u) - f(s,x,v)|_{\mathbb{R}^m} \leq K((t-s)^\theta + |u - v|_{\mathbb{R}^m}), \tag{6.13}$$

for $0 \leq s < t \leq T$, $x \in \mathbb{R}^n$, u, $v \in \mathbb{R}^m$, $|u|_{\mathbb{R}^m}, |v|_{\mathbb{R}^m} \leq R$. Moreover we assume that

$$\sup_{0 \leq t \leq T,\, x \in \mathbb{R}^n} f(t,x,0) < +\infty, \tag{6.14}$$

so that for every $\varphi \in C_b(\mathbb{R}^n;\mathbb{R}^m)$ and $t \in [0,T]$ the composition $f(t,\cdot,\varphi(\cdot))$ is in $C_b(\mathbb{R}^n;\mathbb{R}^m)$. Then we may apply the general results of Section 6.1 to get a regular solution of problem (6.12).

Proposition 6.2.1 *Under the above assumptions, for each $u_0 \in C_b(\mathbb{R}^n,\mathbb{R}^m)$ there are a maximal interval $I(u_0)$ and a unique solution u to (6.12) in $I(u_0)\times\mathbb{R}^n$, such that $u \in C(I(u_0)\times\mathbb{R}^n;\mathbb{R}^m)$, u_t, D_iu, and Δu are bounded and continuous in the interval $[\varepsilon,\tau(u_0)-\varepsilon]$ for each $\varepsilon \in (0,\tau(u_0)/2)$, where $\tau(u_0) = \sup I(u_0)$.*

Proof. Setting

$$F(t,\varphi)(x) = f(t,x,\varphi(x)),\ \ 0 \le t \le T,\ x \in \mathbb{R}^n, \varphi \in X,$$

the function $F : [0,T]\times X \to X$ is continuous, and it satisfies (6.2) and (6.9). Indeed, fix any φ_1, $\varphi_2 \in B(0,R) \subset X$. Then, for all $x \in \mathbb{R}^n$, $|\varphi_1(x)|_{\mathbb{R}^m} \le R$, $|\varphi_2(x)|_{\mathbb{R}^m} \le R$, so that for $0 \le s \le t \le T$ we get from (6.13)

$$|F(t,\varphi_1)(x) - F(s,\varphi_2)(x)| \le K((t-s)^\theta + |\varphi_1(x) - \varphi_2(x)|_{\mathbb{R}^m}),$$

which implies

$$\|F(t,\varphi_1) - F(s,\varphi_2)\|_\infty \le K((t-s)^\theta + \|\varphi_1 - \varphi_2\|_\infty).$$

The local existence and uniqueness Theorem 6.1.1 implies that there exists a unique mild solution $t \mapsto u(t) \in C_b((0,\delta];X)$ of (6.1), that may be extended to a maximal time interval $I(u_0)$.

By Proposition 6.1.2, u, u', and Au are continuous in $(0,\tau(u_0))$ with values in X (in fact, they are Hölder continuous in each compact subinterval). Then the function $(t,x) \mapsto u(t,x) := u(t)(x)$ is bounded and continuous in $[0,a]\times\mathbb{R}^n$ for each $a \in I(u_0)$ (why is it continuous up to $t = 0$? Compare with Section 2.3, part (a), and Proposition 4.1.5), and it is continuously differentiable with respect to t in $I(u_0)\setminus\{0\}\times\mathbb{R}^n$.

Notice $D(A)$ is continuously embedded in $C^1_b(\mathbb{R}^n;\mathbb{R}^m)$. This may be seen as a consequence of (3.10), or it may be proved directly using estimate (3.12)(a) and then the representation formula (1.22) for the resolvent. In any case, it follows that all the first order space derivatives D_iu are continuous in $(0,\tau(u_0))\times\mathbb{R}^n$ too. The second order space derivatives $D_{ij}u(t,\cdot)$ are in $L^p_{loc}(\mathbb{R}^n;\mathbb{R}^m)$, Δu is continuous in $I(u_0)\times\mathbb{R}^n$, and u satisfies (6.12). $\square$

Concerning existence in the large, Proposition 6.1.3 implies that if u is bounded in $I(u_0)\times\mathbb{R}^n$ then $I(u_0) = [0,T]$.

A sufficient condition for u to be bounded is given by Proposition 6.1.4:

$$|f(t,x,u)|_{\mathbb{R}^m} \le C(1+|u|_{\mathbb{R}^m}),\quad t \in [0,T],\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m. \tag{6.15}$$

Indeed, in this case the nonlinear function

$$F : [0,T]\times X \to X,\quad F(t,\varphi)(x) = f(t,x,\varphi(x))$$

satisfies (6.11), for

$$\|F(t,\varphi)\|_\infty = \sup_{x\in\mathbb{R}^n} |f(t,x,\varphi(x))|_{\mathbb{R}^m} \le C(1+\|\varphi\|_\infty).$$

Estimate (6.15) is satisfied if (6.13) holds with a constant K independent of R.

Similar results hold for reaction – diffusion systems in $[0,T]\times\overline{\Omega}$, where Ω is a bounded open set in $\mathbb{R}^n$ with C^2 boundary.

The simplest case is a single equation,

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + f(t,x,u(t,x)), & t>0, \quad x\in\overline{\Omega}, \\ u(0,x) = u_0(x), & x\in\overline{\Omega}, \end{cases} \tag{6.16}$$

with Dirichlet boundary condition,

$$u(t,x) = 0, \;\; t>0, \; x\in\partial\Omega, \tag{6.17}$$

or Neumann boundary condition,

$$\frac{\partial u(t,x)}{\partial n} = 0, \;\; t>0, \; x\in\partial\Omega. \tag{6.18}$$

Here $f:[0,T]\times\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is a regular function satisfying (6.13); $u_0:\overline{\Omega}\to\mathbb{R}$ is continuous and satisfies the compatibility condition $u_0(x)=0$ for $x\in\partial\Omega$ in the case of the Dirichlet boundary condition. Such a condition is necessary to have u continuous up to $t=0$.

Again, we set our problem in the space $X = C(\overline{\Omega})$. Since the realization of the Laplacian in $C(\overline{\Omega})$ with homogeneous Dirichlet conditions is a sectorial operator (see Section 2.4), then problem (6.16) has a unique classical solution in a maximal time interval. Arguing as before, we see that if there is $C>0$ such that

$$|f(t,x,u)| \le C(1+|u|) \quad t\in[0,T], \; x\in\overline{\Omega}, \; u\in\mathbb{R}$$

then for each initial datum u_0 the solution exists globally. But this assumption is rather restrictive, and it is not satisfied in many mathematical models. In the next subsection we shall see a more general assumption that yields existence in the large.

In this section, up to now we have chosen to work with real-valued functions just because in most mathematical models the unknown u is real valued. But we could replace $C_b(\mathbb{R}^n,\mathbb{R}^m)$ and $C(\overline{\Omega};\mathbb{R})$ by $C_b(\mathbb{R}^n;\mathbb{C}^m)$ and $C(\overline{\Omega};\mathbb{C})$ as well without any modification in the proofs, getting the same results in the case of complex-valued data. On the contrary, the results of the next subsection only hold for real-valued functions.

6.2.1 The maximum principle

Using the well known properties of the first and second order derivatives of real-valued functions at relative maximum or minimum points it is possible to find estimates on the solutions to several first or second order partial differential equations. Such techniques are called *maximum principles*.

To begin with, we give a sufficient condition for the solution of (6.16)–(6.17) or of (6.16)–(6.18) to be bounded (and hence, to exist in the large).

Proposition 6.2.2 *Let Ω be a bounded open set in $\mathbb{R}^N$ with C^2 boundary, and let $f:[0,T]\times\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying*

$$|f(t,x,u)-f(s,x,v)| \le K((t-s)^\theta + |u-v|),$$

for any $0\le s<t\le T$, any $x\in\overline{\Omega}$, any $u,v\in\mathbb{R}$ such that $|u|$, $|v|\le R$ and for some positive constant $K=K(R)$. Assume moreover that

$$uf(t,x,u) \le C(1+u^2), \;\; 0\le t\le T, \; x\in\overline{\Omega}, \; u\in\mathbb{R}, \tag{6.19}$$

for some $C \geq 0$*. Then for each initial datum* u_0 *the solution to* (6.16)–(6.17) *or to* (6.16)–(6.18) *satisfies*

$$\sup_{t \in I(u_0),\, x \in \overline{\Omega}} |u(t,x)| < +\infty.$$

If $C = 0$ *in* (6.19)*, then*

$$\sup_{t \in I(u_0),\, x \in \overline{\Omega}} |u(t,x)| = \|u_0\|_\infty.$$

Proof. Fix $\lambda > C$, $a < \tau(u_0)$ and set

$$v(t,x) = u(t,x)e^{-\lambda t}, \;\; 0 \leq t \leq a, \;\; x \in \overline{\Omega}.$$

The function v satisfies

$$v_t(t,x) = \Delta v(t,x) + f(t,x,e^{\lambda t}v(t,x))e^{-\lambda t} - \lambda v(t,x), \;\; 0 < t \leq a, \;\; x \in \overline{\Omega}, \tag{6.20}$$

and it satisfies the same boundary condition as u, and $v(0,\cdot) = u_0$. Since v is continuous, there exists (t_0,x_0) such that $v(t_0,x_0) = \pm\|v\|_{C([0,a]\times\overline{\Omega})}$. (t_0,x_0) is either a point of positive maximum or of negative minimum for v. Assume for instance that (t_0,x_0) is a maximum point. If $t_0 = 0$ we have obviously $\|v\|_\infty \leq \|u_0\|_\infty$. If $t_0 > 0$ and $x_0 \in \Omega$ we rewrite (6.20) at (t_0,x_0) and we multiply both sides by $v(t_0,x_0) = \|v\|_\infty$. Since $v_t(t_0,x_0) \geq 0$ and $\Delta v(t_0,x_0) \leq 0$, we get

$$\lambda\|v\|_\infty^2 \leq C(1 + |e^{\lambda t_0}v(t_0,x_0)|^2)e^{-2\lambda t_0} = C(1 + e^{2\lambda t_0}\|v\|_\infty^2)e^{-2\lambda t_0},$$

so that

$$\|v\|_\infty^2 \leq \frac{C}{\lambda - C}.$$

Let us consider the case $t_0 > 0$, $x_0 \in \partial\Omega$. If u satisfies the Dirichlet boundary condition, then $v(t_0,x_0) = 0$. If u satisfies the Neumann boundary condition, we have $D_i v(t_0,x_0) = 0$ for each i, $\Delta v(t_0,x_0) \leq 0$ (see Exercise 2, §6.2.6), and we go on as in the case $x_0 \in \Omega$.

If (t_0,x_0) is a minimum point the proof is similar. Therefore we have

$$\|v\|_\infty \leq \max\{\|u_0\|_\infty, \sqrt{C/(\lambda - C)}\} \tag{6.21}$$

so that

$$\|u\|_\infty \leq e^{\lambda T}\max\{\|u_0\|_\infty, \sqrt{C/(\lambda - C)}\}$$

and the first statement follows.

If $C = 0$ we obtain $\|u\|_\infty \leq e^{\lambda T}\|u_0\|_\infty$ for every $\lambda > 0$ and letting $\lambda \to 0$ the second statement follows. □

A similar result holds if Ω is replaced by the whole space $\mathbb{R}^N$, but the proof has to be adapted to the noncompact domain case. Indeed, if a function v is bounded and continuous in $[0,a] \times \mathbb{R}^N$, it may have no maximum or minimum points, in general. We state this result, without a proof, in the following proposition.

Proposition 6.2.3 *Let* $f : [0,T] \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying the assumptions of Proposition 6.2.2 with* Ω *replaced by* $\mathbb{R}^N$*. Consider problem* (6.12) *with*

$m = 1$, $d_1 = 1$. Then for each bounded and continuous initial datum u_0 the solution to (6.12) satisfies

$$\sup_{t\in I(u_0),\, x\in\mathbb{R}^N} |u(t,x)| < +\infty,$$

and therefore it exists in the large. If $C = 0$ in (6.19), then

$$\sup_{t\in I(u_0),\, x\in\mathbb{R}^N} |u(t,x)| = \|u_0\|_\infty.$$

Let us remark that (6.15) is a growth condition at infinity, while (6.19) is an algebraic condition and it is not a growth condition. For instance, it is satisfied by $f(t,x,u) = \lambda u - u^{2k+1}$ for each $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. The sign $-$ is important: for instance, in the problem

$$\begin{cases} u_t = \Delta u + |u|^{1+\varepsilon}, & t > 0, \quad x \in \overline{\Omega}, \\ \dfrac{\partial u}{\partial n}(t,x) = 0, & t > 0, \quad x \in \partial\Omega, \\ u(0,x) = \overline{u}, & \qquad\quad\; x \in \overline{\Omega}, \end{cases} \tag{6.22}$$

with $\varepsilon > 0$ and constant initial datum $\overline{u}$, the solution is independent of x and it coincides with the solution to the ordinary differential equation

$$\begin{cases} \xi'(t) = |\xi(t)|^{1+\varepsilon}, & t > 0, \\ \xi(0) = \overline{u}, \end{cases}$$

which blows up in a finite time if $\overline{u} > 0$.

In the proof of Propositions 6.2.2 and 6.2.3 we used a property of the functions $\varphi \in D(A)$, where A is either the realization of the Laplacian in $C_b(\mathbb{R}^N)$ or the realization of the Laplacian with Dirichlet or Neumann boundary condition in $C(\overline{\Omega})$: if $x \in \Omega$ (and also if $x \in \partial\Omega$ in the case of Neumann boundary conditions) is a relative maximum point for φ, then $\Delta\varphi(x) \le 0$. While this is obvious if $\varphi \in C^2(\Omega)$, it has to be proved if φ is not twice differentiable pointwise. We provide a proof only in the case of interior points.

Lemma 6.2.4 *Let $x_0 \in \mathbb{R}^N$, $r > 0$, and let $\varphi : B(x_0,r) \to \mathbb{R}$ be a continuous function. Assume that $\varphi \in W^{2,p}(B(x_0,r))$ for each $p \in [1,+\infty)$, that $\Delta\varphi$ is continuous, and that x_0 is a maximum (respectively, minimum) point for φ. Then $\Delta\varphi(x_0) \le 0$ (respectively, $\Delta\varphi(x_0) \ge 0$).*

Proof. Assume that x_0 is a maximum point. Possibly replacing φ by $\varphi + c$, we may assume $\varphi(x) \ge 0$ for $|x - x_0| \le r$. Let $\theta : \mathbb{R}^N \to \mathbb{R}$ be a smooth function with support contained in $B(x_0,r)$, such that $0 \le \theta(x) \le 1$ for each x, $\theta(x_0) > \theta(x)$ for $x \ne x_0$, and $\Delta\theta(x_0) = 0$. Define

$$\widetilde{\varphi}(x) = \begin{cases} \varphi(x)\theta(x), & x \in B(x_0,r), \\ 0, & x \in \mathbb{R}^N \setminus B(x_0,r). \end{cases}$$

Then $\widetilde{\varphi}(x_0)$ is the maximum of $\widetilde{\varphi}$, and it is attained only at $x = x_0$. Moreover, $\widetilde{\varphi}$ and $\Delta\widetilde{\varphi}$ are continuous in the whole $\mathbb{R}^N$ and vanish outside $B(x_0,r)$, so that there is a sequence

$(\widetilde{\varphi}_n)_{n\in\mathbb{N}} \subset C^2_b(\mathbb{R}^N)$ such that $\widetilde{\varphi}_n \to \widetilde{\varphi}$, $\Delta\widetilde{\varphi}_n \to \Delta\widetilde{\varphi}$ uniformly and each $\widetilde{\varphi}_n$ has support contained in the ball $B(x_0, 2r)$. For instance, we can take $\widetilde{\varphi}_n = \eta T(1/n)\widetilde{\varphi}$ where $T(t)$ is the heat semigroup defined in (2.8) and η is a smooth function with support contained in $B(x_0, 2r)$ and equal to 1 in $B(x_0, r)$. Since x_0 is the unique maximum point of $\widetilde{\varphi}$, there is a sequence $(x_n) \subset B(x_0, 2r)$ converging to x_0 as $n \to \infty$ such that x_n is a maximum point of $\widetilde{\varphi}_n$, for each n. Since $\widetilde{\varphi}_n$ is twice continuously differentiable, we have $\Delta\widetilde{\varphi}_n(x_n) \le 0$. Letting $n \to +\infty$ we get $\Delta\widetilde{\varphi}(x_0) \le 0$, and consequently $\Delta\varphi(x_0) \le 0$.

If x_0 is a minimum point the proof is similar. □

The maximum principle may be also used in some systems. For instance, let us consider

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + f(u(t,x)), & t > 0, \quad x \in \overline{\Omega}, \\ u(t,x) = 0, & t > 0, \quad x \in \partial\Omega, \\ u(0,x) = u_0(x), & \qquad\quad\; x \in \overline{\Omega}, \end{cases}$$

where the unknown u is a R^m-valued function, Ω is a bounded open set in $\mathbb{R}^N$ with C^2 boundary, $f : \mathbb{R}^m \to \mathbb{R}^m$ is a locally Lipschitz continuous function such that

$$\langle y, f(y)\rangle \le C(1 + |y|^2), \;\; y \in \mathbb{R}^m \tag{6.23}$$

and u_0 is a continuous function vanishing on $\partial\Omega$.

As in the case of a single equation, it is convenient to fix $a \in (0, \tau(u_0))$ and to introduce the function $v : [0, a] \times \overline{\Omega} \to \mathbb{R}^m$, $v(t, x) = u(t, x)e^{-\lambda t}$ with $\lambda > C$, that satisfies

$$\begin{cases} v_t(t,x) = \Delta v(t,x) + f(e^{\lambda t}v(t,x))e^{-\lambda t} - \lambda v(t,x), & t > 0, \quad x \in \overline{\Omega}, \\ v(t,x) = 0, & t > 0, \quad x \in \partial\Omega, \\ v(0,x) = u_0(x), & \qquad\quad\; x \in \overline{\Omega}. \end{cases}$$

Instead of $|v|$ it is better to work with $\varphi(t, x) = |v(t, x)|^2 = \sum_{i=1}^m v_i(t, x)^2$, which is more regular. Let us remark that

$$\varphi_t = 2\langle v_t, v\rangle, \qquad D_j\varphi = 2\langle D_j v, v\rangle, \qquad \Delta\varphi = 2\sum_{i=1}^m |Dv_i|^2 + 2\langle v, \Delta v\rangle.$$

If $(t_0, x_0) \in (0, a]\times\Omega$ is a positive maximum point for φ (i.e. for $|v|$) we have $\varphi_t(t_0, x_0) \ge 0$, $\Delta\varphi(t_0, x_0) \le 0$ and hence $\langle v(t_0, x_0), \Delta v(t_0, x_0)\rangle \le 0$. Writing the differential system at (t_0, x_0) and taking the inner product with $v(t_0, x_0)$ we get

$$\begin{aligned} 0 &\le \langle v_t(t_0,x_0), v(t_0,x_0)\rangle \\ &= \langle \Delta v(t_0,x_0), v(t_0,x_0)\rangle + \langle f(e^{\lambda t_0}v(t_0,x_0)), v(t_0,x_0)e^{-\lambda t_0}\rangle - \lambda|v(t_0,x_0)|^2 \\ &\le C(1 + |v(t_0,x_0)|^2) - \lambda|v(t_0,x_0)|^2 \end{aligned}$$

so that $\|v\|^2_\infty \le C/(\lambda - C)$. Therefore, $\|v\|_\infty \le \max\{\|u_0\|_\infty, \sqrt{C/(\lambda - C)}\}$, and consequently $\|u\|_\infty \le e^{\lambda T}\max\{\|u_0\|_\infty, \sqrt{C/(\lambda - C)}\}$, the same result as in the scalar case. Therefore, u exists in the large.

The maximum principle is used also to prove qualitative properties of the solutions, for instance to prove that the solutions are nonnegative for nonnegative initial data, or nonpositive for nonpositive initial data.

Consider for example the heat equation with Dirichlet boundary condition in a regular bounded open set $\Omega \subset \mathbb{R}^N$,

$$\begin{cases} u_t(t,x) = \Delta u(t,x), & t > 0, \quad x \in \overline{\Omega}, \\ u(t,x) = 0, & t \geq 0, \quad x \in \partial\Omega, \\ u(0,x) = u_0(x), & \qquad\quad x \in \overline{\Omega}, \end{cases}$$

with $u_0 = 0$ on $\partial\Omega$ and $u_0(x) \geq 0$ for each $x \in \overline{\Omega}$. To show that $u(t,x) \geq 0$ for each (t,x) we consider the function $v(t,x) := e^{-t}u(t,x)$ which satisfies the same boundary condition as u, $v(0,x) = u_0(x)$ and $v_t(t,x) = \Delta v(t,x) - v(t,x)$. If v has a negative minimum at (t_0,x_0), then $t_0 > 0$, $x_0 \in \Omega$ and hence $v_t(t_0,x_0) \leq 0$, $\Delta v(t_0,x_0) \geq 0$, contradicting the equation at (t_0,x_0).

More general situations, even in nonlinear problems, can be treated with the following comparison result.

Proposition 6.2.5 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, let $f \in C^1(\mathbb{R})$ and let $u, v \in C([0,a] \times \overline{\Omega}) \cap C^1((0,a] \times \Omega)$ be such that, for every $t \in (0,a]$, $u(t,\cdot), v(t,\cdot) \in W^{2,p}(\Omega)$ for every $p < +\infty$ and $\Delta u(t,\cdot), \Delta v(t,\cdot) \in C(\Omega)$.*

Assume that $u_t \geq \Delta u + f(u)$, $v_t \leq \Delta v + f(v)$ in $(0,a] \times \Omega$, that $u(0,x) \geq v(0,x)$ for $x \in \overline{\Omega}$ and that $u(t,x) \geq v(t,x)$ for $(t,x) \in (0,a] \times \partial\Omega$. Then $u(t,x) \geq v(t,x)$ in $[0,a] \times \overline{\Omega}$.

Proof. The function $w = u - v$ has the same regularity properties as u, v, and it satisfies

$$w_t(t,x) \geq \Delta w(t,x) + f(u(t,x)) - f(v(t,x)) = \Delta w(t,x) + h(t,x)w(t,x)$$

in $(0,a] \times \Omega$, where $h(t,x) = \int_0^1 f'\left(v(t,x) + \xi(u(t,x) - v(t,x))\right) d\xi$ is a bounded function. Let $\lambda > \|h\|_\infty$ and set $z(t,x) := e^{-\lambda t}w(t,x)$. Then $z_t \geq \Delta z + (h - \lambda)z$ in $(0,a] \times \Omega$, $z(0,x) \geq 0$ for any $x \in \overline{\Omega}$, $z(t,x) \geq 0$ for any $t > 0, x \in \partial\Omega$ so that, if z has a negative minimum at (t_0,x_0), then $t_0 > 0$, $x_0 \in \Omega$ and therefore $z_t(t_0,x_0) \leq 0$, $\Delta z(t_0,x_0) \geq 0$ in contradiction with the differential inequality satisfied by z at (t_0,x_0). Therefore $z \geq 0$ everywhere, i.e., $u \geq v$. □

As an application we consider the problem

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + \lambda u(t,x) - \rho u^2(t,x), & t > 0, \quad x \in \overline{\Omega}, \\ u(t,x) = 0, & t \geq 0, \quad x \in \partial\Omega, \\ u(0,x) = u_0(x), & \qquad\quad x \in \overline{\Omega}. \end{cases} \tag{6.24}$$

Here λ, $\rho > 0$. By comparing the solution u with the function $v \equiv 0$, it follows that $u(t,x) \leq 0$ if $u_0(x) \leq 0$ and $u(t,x) \geq 0$ if $u_0(x) \geq 0$. Therefore, by Proposition 6.2.2, $\tau(u_0) = +\infty$ if $u_0 \geq 0$. See Exercise 4, §6.2.6.

Finally, let us see a system from combustion theory. Here u and v are a concentration and a temperature, respectively, both normalized and rescaled. The numbers $\mathcal{L}e$, ε, q are positive parameters, $\mathcal{L}e$ is called the Lewis number. Ω is a bounded open set in $\mathbb{R}^N$ with

C^2 boundary. The system is

$$\begin{cases} u_t(t,x) = \mathcal{L}e\,\Delta u(t,x) - \varepsilon u(t,x) f(v(t,x)), & t>0, \quad x\in\overline{\Omega}, \\ v_t(t,x) = \Delta v(t,x) + q u(t,x) f(v(t,x)), & t>0, \quad x\in\overline{\Omega}, \\ \dfrac{\partial u}{\partial n}(t,x) = 0, \quad v(t,x) = 1, & t>0, \quad x\in\partial\Omega, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), & x\in\overline{\Omega}, \end{cases} \tag{6.25}$$

f is the Arrhenius function

$$f(v) = e^{-h/v},$$

with $h>0$. The initial data u_0 and v_0 are continuous nonnegative functions, with $v_0 \equiv 1$ on $\partial\Omega$. Replacing the unknowns (u,v) by $(u, v-1)$, problem (6.25) reduces to a problem with homogeneous boundary conditions, which we locally solve using the above techniques.

The physically meaningful solutions are such that $u, v \geq 0$. Using the maximum principle we can prove that for nonnegative initial data we get nonnegative solutions.

Let us consider u: if, by contradiction, there is $a>0$ such that the restriction of u to $[0,a]\times\overline{\Omega}$ has a negative minimum, say at (t_0,x_0) we have $t_0>0$, $x_0\in\Omega$ and

$$0 \geq u_t(t_0,x_0) = \mathcal{L}e\,\Delta u(t_0,x_0) - \varepsilon u(t_0,x_0) f(v(t_0,x_0)) > 0,$$

a contradiction. Therefore u cannot have negative values.

To study the sign of v it is again convenient to introduce the function $z(t,x) := e^{-\lambda t} v(t,x)$ with $\lambda>0$. If there is $a>0$ such that the restriction of z to $[0,a]\times\overline{\Omega}$ has a negative minimum, say at (t_0,x_0) we have $t_0>0$, $x_0\in\Omega$ and

$$0 \geq z_t(t_0,x_0) = \Delta z(t_0,x_0) - \lambda z(t_0,x_0) + q u(t_0,x_0) f(z(t_0,x_0)e^{\lambda t_0}) e^{-\lambda t_0} > 0,$$

again a contradiction. Therefore, v too cannot have negative values.

Exercises 6.2.6

1. Prove the following additional regularity properties of the solution to (6.12):

   (i) if $u_0 \in BUC(\mathbb{R}^n,\mathbb{R}^m)$, then $u(t,x)\to u_0(x)$ as $t\to 0$, uniformly for x in $\mathbb{R}^n$;

   (ii) if for every $R>0$ there is $K=K(R)>0$ such that

   $$|f(t,x,u) - f(s,y,v)|_{\mathbb{R}^m} \leq K((t-s)^\theta + |x-y|^\theta_{\mathbb{R}^n} + |u-v|_{\mathbb{R}^m}),$$

   for $0\leq s<t\leq T$, x, $y\in\mathbb{R}^n$, u, $v\in\mathbb{R}^m$, $|u|_{\mathbb{R}^m}$, $|v|_{\mathbb{R}^m}\leq R$, then all the second order derivatives $D_{ij}u$ are continuous in $I(u_0)\times\mathbb{R}^n$.

   [Hint: u' and $F(t,u)$ belong to $B([\varepsilon,\tau(u_0)-\varepsilon]; D_A(\theta/2,\infty))$, hence $u\in B([\varepsilon,\tau(u_0)-\varepsilon]; C^{2+\theta}_b(\mathbb{R}^N))$. To show Hölder continuity of $D_{ij}u$ with respect to t, proceed as in Corollary 4.1.11].

2. Let Ω be an open set in $\mathbb{R}^N$ with C^1 boundary, and let $x_0\in\partial\Omega$ be a relative maximum point for a C^1 function $v:\overline{\Omega}\to\mathbb{R}$. Prove that if the normal derivative of v vanishes at x_0 then all the partial derivatives of v vanish at x_0.

   If $\partial\Omega$ and v are C^2, prove that we also have $\Delta v(x_0)\leq 0$.

3. Construct explicitly a function θ as in the proof of Lemma 6.2.4.

4. Prove that for each continuous nonnegative initial function u_0 such that $u_0 = 0$ on $\partial\Omega$, the solution to (6.24) exists in the large.

5. Show that the solution u to

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + u^2(t,x) - 1, & t \geq 0, \quad x \in \overline{\Omega}, \\ u(t,x) = 0, & t \geq 0, \quad x \in \partial\Omega \\ u(0,x) = u_0(x), & \quad x \in \overline{\Omega} \end{cases}$$

with $u_0 = 0$ on $\partial\Omega$ and $\|u_0\|_\infty \leq 1$ exists in the large.

6. Let u be the solution to

$$\begin{cases} u_t(t,x) = u_{xx}(t,x) + u^2(t,x), & t \geq 0, \quad x \in [0,1], \\ u(t,0) = u(t,1) = 0, & t \geq 0, \\ u(0,x) = u_0(x), & \quad x \in [0,1] \end{cases}$$

with $u_0(0) = u_0(1) = 0$.

(i) Prove that if $0 \leq u_0(x) \leq \pi^2 \sin(\pi x)$ for each $x \in [0,1]$, then u exists in the large. [Hint: compare u with $v(t,x) := \pi^2 \sin(\pi x)$].

(ii) Set $h(t) := \int_0^1 u(t,x)\sin(\pi x)dx$ and prove that $h'(t) \geq (\pi/2)h^2 - \pi^2 h(t)$ for each $t \in I(u_0)$. Deduce that if $h(0) > 2\pi$ then u blows up (i.e., $\|u(t,\cdot)\|_\infty$ becomes unbounded) in finite time.

6.3 Nonlinearities defined in intermediate spaces

Let $A : D(A) \subset X \to X$ be a sectorial operator, and let X_α be any space of class J_α between X and $D(A)$, with $\alpha \in (0,1)$. Consider the Cauchy problem

$$\begin{cases} u'(t) = Au(t) + F(t,u(t)), & t > 0, \\ u(0) = u_0, \end{cases} \tag{6.26}$$

where $u_0 \in X_\alpha$ and $F : [0,T] \times X_\alpha \to X$ is a continuous function, for some $T > 0$. The definition of strict, classical, or mild solution to (6.26) is similar to the definition in Section 6.1.

The Lipschitz condition (6.2) is replaced by a similar assumption: for each $R > 0$ there exists $L = L(R) > 0$ such that

$$\|F(t,x) - F(t,y)\| \leq L\|x-y\|_{X_\alpha}, \quad t \in [0,T], \quad x,y \in B(0,R) \subset X_\alpha. \tag{6.27}$$

Because of the embeddings $D(A) \subset X_\alpha \subset X$, then $t \mapsto e^{tA}$ is analytic in $(0,+\infty)$ with values in $\mathcal{L}(X_\alpha)$. But the norm $\|e^{tA}\|_{\mathcal{L}(X_\alpha)}$ could blow up as $t \to 0$, see Exercise 5 in §2.1.3. We want to avoid this situation, so we assume throughout

$$\limsup_{t\to 0} \|e^{tA}\|_{\mathcal{L}(X_\alpha)} < +\infty. \tag{6.28}$$

It follows that $\|e^{tA}\|_{\mathcal{L}(X_\alpha)}$ is bounded on every compact interval contained in $[0,+\infty)$. Moreover, we set

$$M := \sup_{0\le t\le T} \|e^{tA}\|_{\mathcal{L}(X_\alpha)}. \tag{6.29}$$

6.3.1 Local existence, uniqueness, regularity

As in the case of nonlinearities defined in the whole X, it is convenient to look for a local mild solution at first, and then to see that under reasonable assumptions the solution is classical or strict.

The proof of the local existence and uniqueness theorem for mild solutions is quite similar to the proof of Theorem 6.1.1, but we need an extension of Proposition 4.1.5. We set

$$M_{k,\alpha} := \sup\{t^{k+\alpha}\|A^k e^{tA}\|_{\mathcal{L}(X,X_\alpha)} : 0 < t \le T\}, \ \ k = 0,1,2.$$

By Proposition 3.2.2(ii), $M_{k,\alpha} < +\infty$.

In the proof of the next results, we use the following generalization of the Gronwall lemma, whose proof may be found for instance in [9, p. 188].

Lemma 6.3.1 *Let $0 \le a < b < \infty$, and let $u : [a,b] \to \mathbb{R}$ be a nonnegative function, bounded in any interval $[a, b-\varepsilon]$, integrable and such that*

$$u(t) \le k + h\int_a^t (t-s)^{-\alpha} u(s)ds, \ \ a \le t \le b,$$

with $0 \le \alpha < 1$, h, $k > 0$. Then there exists $C_1 > 0$, independent of a, b, k such that

$$u(t) \le C_1 k, \quad a \le t < b.$$

Using the generalized Gronwall Lemma and Exercise 1 in §4.1.13, the proof of the local existence and uniqueness theorem for mild solutions goes on as the proof of Theorem 6.1.1, with minor modifications.

Theorem 6.3.2 *The following statements hold.*

(a) If $u, v \in C_b((0,a];X_\alpha)$ are mild solutions of (6.26) for some $a \in (0,T]$, then $u \equiv v$.

(b) For each $\overline{u} \in X_\alpha$ there are r, $\delta > 0$, $K > 0$ such that if $\|u_0 - \overline{u}\|_{X_\alpha} \le r$ then problem (6.26) has a mild solution $u = u(\cdot; u_0) \in C_b((0,\delta];X_\alpha)$. The function u belongs to $C([0,\delta];X_\alpha)$ if and only if $u_0 \in \overline{D(A)}^{X_\alpha}$:= closure of $D(A)$ in X_α.

Moreover, for u_0, $u_1 \in B(\overline{u}, r)$ we have

$$\|u(t;u_0) - u(t;u_1)\|_{X_\alpha} \le K\|u_0 - u_1\|_{X_\alpha}, \ \ 0 \le t \le \delta. \tag{6.30}$$

Proof. *Proof of (a).* The proof can be obtained arguing as in the proof of Theorem 6.1.1(a), using the generalized Gronwall lemma 6.3.1.

Proof of (b). Let M be defined by (6.29). Fix $R > 0$ such that $R \ge 8M\|\overline{u}\|_{X_\alpha}$, so that if $\|u_0 - \overline{u}\|_{X_\alpha} \le r := R/(8M)$ then

$$\sup_{0\le t\le T} \|e^{tA}u_0\|_{X_\alpha} \le R/4.$$

Moreover, let L be such that

$$\|F(t,v) - F(t,w)\| \le L\|v-w\|_{X_\alpha} \quad 0 \le t \le T,\ v,w \in B(0,R) \subset X_\alpha.$$

We look for a local mild solution of (6.26) in the metric space $\mathcal{Y}$ defined by

$$\mathcal{Y} = \{u \in C_b((0,\delta];X_\alpha) :\ \|u(t)\|_{X_\alpha} \le R,\ \forall t \in (0,\delta]\},$$

where $\delta \in (0,T]$ will be chosen later. The space $\mathcal{Y}$ is the closed ball with centre at 0 and radius R in $C_b((0,\delta];X_\alpha)$, and for each $v \in \mathcal{Y}$ the function $t \mapsto F(t,v(t))$ belongs to $C_b((0,\delta];X)$. We define a nonlinear operator Γ in $\mathcal{Y}$,

$$\Gamma(v)(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}F(s,v(s))ds,\quad 0 \le t \le \delta.$$

A function $v \in \mathcal{Y}$ is a mild solution to (6.26) in $[0,\delta]$ if and only if it is a fixed point of Γ.

We shall show that Γ is a contraction, and it maps $\mathcal{Y}$ into itself, provided δ is small enough.

Let v_1, $v_2 \in \mathcal{Y}$. By Exercise 1 in §4.1.13, $\Gamma(v_1)$ and $\Gamma(v_2)$ belong to $C_b((0,\delta];X_\alpha)$ and

$$\begin{aligned}
\|\Gamma(v_1)-\Gamma(v_2)\|_{C([0,\delta];X_\alpha)} &\le \frac{M_{0,\alpha}}{1-\alpha}\delta^{1-\alpha}\|F(\cdot,v_1(\cdot)) - F(\cdot,v_2(\cdot))\|_{C_b((0,\delta];X)}\\
&\le \frac{M_{0,\alpha}}{1-\alpha}\delta^{1-\alpha}L\|v_1-v_2\|_{C_b((0,\delta];X_\alpha)}.
\end{aligned}$$

Therefore, if

$$\delta \le \delta_0 := \left(\frac{2M_{0,\alpha}L}{1-\alpha}\right)^{-1/(1-\alpha)},$$

then Γ is a contraction in $\mathcal{Y}$ with constant 1/2. Moreover for each $v \in \mathcal{Y}$ and $t \in [0,\delta]$, with $\delta \le \delta_0$, we have

$$\begin{aligned}
\|\Gamma(v)\|_{C_b((0,\delta];X_\alpha)} &\le \|\Gamma(v)-\Gamma(0)\|_{C_b((0,\delta];X_\alpha)} + \|\Gamma(0)\|_{C_b((0,\delta];X_\alpha)}\\
&\le R/2 + \|e^{\cdot A}u_0\|_{C_b((0,\delta];X_\alpha)} + C\delta^{1-\alpha}\|F(\cdot,0)\|_{C_b((0,\delta];X)}\\
&\le R/2 + R/4 + C\delta^{1-\alpha}\|F(\cdot,0)\|_{C([0,\delta];X)}.
\end{aligned}$$

Therefore, if $\delta \le \delta_0$ is such that

$$C\delta^{1-\alpha}\|F(\cdot,0)\|_{C([0,\delta];X)} \le R/4,$$

then Γ maps $\mathcal{Y}$ into itself, and it has a unique fixed point in $\mathcal{Y}$.

Concerning the continuity of u up to $t=0$, we remark that the function $t \mapsto v(t) := u(t) - e^{tA}u_0$ is in $C([0,\delta];X_\alpha)$, while $t \mapsto e^{tA}u_0$ belongs to $C([0,\delta];X_\alpha)$ if and only if $u_0 \in \overline{D(A)}^{X_\alpha}$. See Exercise 1, §6.3.7. Therefore, $u \in C([0,\delta];X_\alpha)$ if and only if $u_0 \in \overline{D(A)}^{X_\alpha}$.

The statements about continuous dependence on the initial data may be proved precisely as in Theorem 6.1.1. □

The local mild solution to problem (6.26) is extended to a maximal time interval $I(u_0)$ as in §6.1.1. We still define $\tau(u_0) := \sup I(u_0)$.

Without important modifications in the proofs it is also possible to deal with regularity and behavior of the solution near $\tau(u_0)$, obtaining results similar to the ones of Propositions 6.1.2 and 6.1.3.

Proposition 6.3.3 *If there exists $\theta \in (0,1)$ such that for every $R > 0$ we have*

$$\|F(t,x) - F(s,x)\| \le C(R)(t-s)^\theta, \;\; 0 \le s \le t \le T, \; \|x\|_{X_\alpha} \le R, \tag{6.31}$$

then the solution u of (6.26) belongs to $C^\theta([\varepsilon, \tau(u_0) - \varepsilon]; D(A)) \cap C^{1+\theta}([\varepsilon, \tau(u_0) - \varepsilon]; X)$, and u' belongs to $B([\varepsilon, \tau - \varepsilon]; D_A(\theta, \infty))$ for each $\varepsilon \in (0, \tau(u_0)/2)$. Moreover, if also $u_0 \in \overline{D(A)}$ then $u(\cdot; u_0)$ is a classical solution to (6.26). If $u_0 \in D(A)$ and $Au_0 + F(0, u_0) \in \overline{D(A)}$ then u is a strict solution to (6.26).

Proposition 6.3.4 *Let $u_0 \in X_\alpha$ be such that $I(u_0) \neq [0,T]$. Then $t \mapsto \|u(t)\|_{X_\alpha}$ is unbounded in $I(u_0)$.*

The simplest situation in which it is possible to show that $\|u(t)\|_{X_\alpha}$ is bounded in $I(u_0)$ for each initial datum u_0 is again the case when F grows not more than linearly with respect to x as $\|x\|_{X_\alpha} \to +\infty$.

Proposition 6.3.5 *Assume that there exists $C > 0$ such that*

$$\|F(t,x)\| \le C(1 + \|x\|_{X_\alpha}), \quad t \in [0,T], \;\; x \in X_\alpha. \tag{6.32}$$

Let $u : I(u_0) \to X_\alpha$ be the mild solution to (6.26). Then u is bounded in $I(u_0)$ with values in X_α, and hence $I(u_0) = [0,T]$.

Proof. Recall that

$$\|e^{tA}x\|_{X_\alpha} \le \frac{M_{0,\alpha}}{t^\alpha} \|x\|, \;\; x \in X, \; 0 < t \le T.$$

For each $t \in I(u_0)$ we have

$$\begin{aligned} \|u(t)\|_{X_\alpha} &\le M\|u_0\|_{X_\alpha} + M_{0,\alpha} \int_0^t (t-s)^{-\alpha} C(1 + \|u(s)\|_{X_\alpha}) ds \\ &\le M\|u_0\|_{X_\alpha} + CM_{0,\alpha} \left(\frac{T^{1-\alpha}}{1-\alpha} + \int_0^t \frac{\|u(s)\|_{X_\alpha}}{(t-s)^\alpha} ds \right). \end{aligned}$$

The generalized Gronwall lemma implies the inequality

$$\|u(t)\|_{X_\alpha} \le C_1 \left(M\|u_0\|_{X_\alpha} + \frac{CM_{0,\alpha}T^{1-\alpha}}{1-\alpha} \right), \;\; t \in I(u_0),$$

and the statement follows. □

The growth condition (6.32) is apparently rather restrictive. If we have some *a priori* estimate for the solution to (6.26) in the X-norm (this happens in several applications to PDE's), it is possible to find *a priori* estimates in the $D_A(\theta, \infty)$-norm if F satisfies suitable growth conditions, less restrictive than (6.32). Since $D_A(\theta, \infty)$ is continuously embedded in X_α for $\theta > \alpha$ by Proposition 3.2.2, we get an *a priori* estimate for the solution in the X_α-norm, that yields existence in the large.

Proposition 6.3.6 *Assume that there exists an increasing function $\mu : [0, +\infty) \to [0, +\infty)$ such that*

$$\|F(t,x)\| \le \mu(\|x\|)(1 + \|x\|_{X_\alpha}^\gamma), \;\; 0 \le t \le T, \; x \in X_\alpha, \tag{6.33}$$

with $1 < \gamma < 1/\alpha$. Let $u : I(u_0) \to X_\alpha$ be the mild solution to (6.26). If u is bounded in $I(u_0)$ with values in X, then it is bounded in $I(u_0)$ with values in X_α.

Proof. Let us fix $0 < a < I(u_0)$ and set $I_a = \{t \in I(u_0) : \ t \geq a\}$. Since $u \in C_b((0,a]; X_\alpha)$ it suffices to show that it is bounded in I_a with values in X_α. We show that it is bounded in I_a with values in $D_A(\theta, \infty)$, when $\theta = \alpha\gamma$. This will conclude the proof by Proposition 3.2.2(i).

Set

$$K : \sup_{t\in I(u_0)} \|u(t)\|.$$

Observe that $u(a) \in D_A(\theta,\infty)$ and that it satisfies the variation of constants formula

$$u(t) = e^{(t-a)A}u(a) + \int_a^t e^{(t-s)A}F(s,u(s))ds, \quad t \in I(a).$$

Using the interpolatory estimate

$$\|x\|_{X_\alpha} \leq c\|x\|^{1-\alpha/\theta}\|x\|^{\alpha/\theta}_{D_A(\theta,\infty)},$$

with $c = c(\alpha,\theta)$, that holds for every $x \in D_A(\theta,\infty)$, see Exercise 4(b) in §3.2.3, we get

$$\|u(s)\|^\gamma_{X_\alpha} \leq c\|u(s)\|^{\gamma(1-\alpha/\theta)}\|u(s)\|^{\alpha\gamma/\theta}_{D_A(\theta,\infty)} \leq cK^{\gamma(1-\alpha/\theta)}\|u(s)\|_{D_A(\theta,\infty)}, \ \ s \in I_a,$$

so that

$$\|F(s,u(s))\| \leq \mu(K)(1 + cK^{\gamma(1-\alpha/\theta)}\|u(s)\|_{D_A(\theta,\infty)}), \ \ s \in I_a.$$

Let $M_\theta > 0$ be such that for all $t \in (0,T]$ we have $\|t^\theta e^{tA}x\|_{D_A(\theta,\infty)} \leq M_\theta\|x\|$ for $x \in X$, and $\|e^{tA}x\|_{D_A(\theta,\infty)} \leq M_\theta\|x\|_{D_A(\theta,\infty)}$ for $x \in D_A(\theta,\infty)$. Then for $t \in I_a$ we have

$$\begin{aligned}\|u(t)\|_{D_A(\theta,\infty)} \ &\leq \ M_\theta\|u(a)\|_{D_A(\theta,\infty)} \\ &\quad + M_\theta\mu(K)\int_a^t (t-s)^{-\theta}(1 + cK^{\gamma(1-\alpha/\theta)}\|u(s)\|_{D_A(\theta,\infty)})ds, \quad (6.34)\end{aligned}$$

and the generalized Gronwall lemma implies that u is bounded in I_a with values in $D_A(\theta,\infty)$. □

The exponent $\gamma = 1/\alpha$ is called *critical growth exponent*. If $\gamma = 1/\alpha$ the above method does not work: one should replace $D_A(\alpha\gamma,\infty)$ by $D(A)$ or by $D_A(1,\infty)$, and the integral in (6.34) would be $+\infty$. We already know that in general we cannot estimate the $D(A)$-norm (and, similarly, the $D_A(1,\infty)$ norm) of $v(t) = (e^{tA} * \varphi)(t)$ in terms of $\sup\|\varphi(t)\|$.

Exercises 6.3.7

1. Show that the function $t \mapsto e^{tA}u_0$ belongs to $C([0,\delta]; X_\alpha)$ if and only if $u_0 \in \overline{D(A)}^{X_\alpha}$. This fact has been used in Proposition 6.3.2.

2. Prove Propositions 6.3.3 and 6.3.4.

3. Let $F : [0,T] \times X_\alpha \to X$ satisfy (6.27). Prove that, for any $u_0 \in X_\alpha$, the mild solution of (6.26) is bounded in the interval $[\varepsilon, \tau(u_0) - \varepsilon]$ with values in $D_A(\beta,\infty)$ for any $\beta \in (0,1)$ and any $\varepsilon \in (0, \tau(u_0)/2)$.

6.3.2 Second order PDE's

Let Ω be a bounded open set in $\mathbb{R}^N$ with regular boundary. Let us consider the problem

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + f(t,x,u(t,x),Du(t,x)), & t>0, \quad x\in\overline{\Omega}, \\ u(t,x)=0, & t>0, \quad x\in\partial\Omega, \\ u(0,x)=u_0(x), & x\in\overline{\Omega}, \end{cases} \tag{6.35}$$

We denote by Du the gradient of u with respect to the space variables, $Du = (\partial u/\partial x_1, \ldots, \partial u/\partial x_N)$. We assume that the function

$$(t,x,u,p)\mapsto f(t,x,u,p),\;\; t\in[0,T],\; x\in\overline{\Omega},\; u\in\mathbb{R},\; p\in\mathbb{R}^N,$$

is continuous, Hölder continuous with respect to t, locally Lipschitz continuous with respect to (u,p). More precisely, we assume that there exists $\theta\in(0,1)$ such that for every $R>0$ there is $K=K(R)>0$ such that

$$|f(t,x,u,p)-f(s,x,v,q)| \le K((t-s)^\theta + |u-v| + |p-q|_{\mathbb{R}^N}), \tag{6.36}$$

for $0\le s<t\le T$, (u,p), $(v,q)\in B(0,R)\subset\mathbb{R}^{N+1}$.

We choose as X the space of the continuous functions in $\overline{\Omega}$. Then the realization A of the Laplacian with Dirichlet boundary condition is sectorial in X, and Theorem 3.1.10(ii) implies that for $\alpha\in(1/2,1)$ we have

$$D_A(\alpha,\infty) = C^{2\alpha}_0(\overline{\Omega}) = \{u\in C^{2\alpha}(\overline{\Omega}) :\; u(x)=0\;\; x\in\partial\Omega\}.$$

Therefore, choosing $X_\alpha = D_A(\alpha,\infty)$ with $\alpha>1/2$, the nonlinear function

$$F(t,\varphi)(x) = f(t,x,\varphi(x),D\varphi(x))$$

is well defined in $[0,T]\times X_\alpha$, with values in X. We recall that the part of A in $D_A(\alpha,\infty)$ is sectorial in $D_A(\alpha,\infty)$ and hence (6.28) holds.

We could also take $\alpha=1/2$ and $X_{1/2} = \{\varphi\in C^1(\overline{\Omega}) : \varphi=0 \text{ on } \partial\Omega\}$. Indeed, it is possible to show that assumption (6.28) holds in such a space.

If the initial datum u_0 is in $C^{2\alpha}_0(\overline{\Omega})$ with $\alpha\in(1/2,1)$, we may rewrite problem (6.35) in the abstract formulation (6.26). The local existence and uniqueness theorem 6.3.2 yields a local existence and uniqueness result for problem (6.35).

Proposition 6.3.8 *Under the above assumptions, for ach $u_0\in C^{2\alpha}_0$ there exists a maximal time interval $I(u_0)$ such that problem* (6.35) *has a unique solution $u : I(u_0)\times\overline{\Omega}\to\mathbb{R}$, such that u and the space derivatives D_iu, $i=1,\ldots,N$, are continuous in $I(u_0)\times\overline{\Omega}$, and u_t, Δu are continuous in $(\varepsilon,\tau(u_0)-\varepsilon)\times\overline{\Omega}$ for any $\varepsilon\in(0,\tau(u_0)/2)$. Here $\tau(u_0)=\sup I(u_0)$, as usual.*

Proof. With the above choice, the assumptions of Theorem 6.3.2 are satisfied, so that problem (6.35) has a unique local solution $u=u(t;u_0)\in C_b((0,a];C^{2\alpha}_0(\overline{\Omega}))$ for each $a<\tau(u_0)$, that belongs to $C([\varepsilon,\tau(u_0)-\varepsilon];D(A))\cap C^1([\varepsilon,\tau(u_0)-\varepsilon];X)$ for each $\varepsilon\in(0,\tau(u_0))$, by Proposition 6.3.3. Consequently, the function

$$u(t,x;u_0) = u(t;u_0)(x),\;\; 0\le t\le\delta,\; x\in\overline{\Omega},$$

is a solution to (6.35) with the claimed regularity properties. The continuity of the first order space derivatives $D_i u$ up to $t = 0$ follows from Exercise 4(c) in §3.2.3 and from the continuous embedding $D_A(\beta, \infty) \subset C^1(\overline{\Omega})$ for $\beta > 1/2$. □

By Proposition 6.3.5, a sufficient condition for existence in the large is

$$|f(t, x, u, p)| \le C(1 + |u| + |p|_{\mathbb{R}^N}), \quad t \in [0, T],\ x \in \overline{\Omega},\ u \in \mathbb{R},\ p \in \mathbb{R}^N. \tag{6.37}$$

Indeed, in this case the nonlinear function

$$F : [0, T] \times X_\alpha \to X, \quad F(t, u)(x) = f(t, x, u(x), Du(x))$$

satisfies condition (6.32).

In general, one can find an *a priori* estimate for the sup norm of the solution provided that

$$uf(t, x, u, 0) \le C(1 + u^2),\ \ 0 \le t \le T,\ x \in \overline{\Omega},\ u \in \mathbb{R}. \tag{6.38}$$

Indeed, in this case we may use again the procedure of Proposition 6.2.2. Once we know that u is bounded in $I(u_0)$ with values in X, we may use Proposition 6.3.6. Assume that there is an increasing function $\mu : [0, +\infty) \to [0, +\infty)$ such that for some $\varepsilon > 0$ we have

$$|f(t, x, u, p)| \le \mu(|u|)(1 + |p|^{2-\varepsilon}),\ \ 0 \le t \le T,\ x \in \overline{\Omega},\ u \in \mathbb{R},\ p \in \mathbb{R}^N. \tag{6.39}$$

Then the nonlinearity

$$F(t, u)(x) = f(t, x, u(x), Du(x)),\ \ 0 \le t \le T,\ u \in C^{2\alpha}_0(\overline{\Omega}),\ x \in \overline{\Omega},$$

satisfies (6.33) with $\gamma = 2 - \varepsilon$, because

$$\|F(t, u)\|_\infty \le \mu(\|u\|_\infty)(1 + \|u\|^{2-\varepsilon}_{C^1}) \le \mu(\|u\|_\infty)(1 + \|u\|^{2-\varepsilon}_{C^{2\alpha}}),\ \ 0 \le t \le T,\ u \in C^{2\alpha}_0(\overline{\Omega}).$$

Then, Proposition 6.3.6 yields existence in the large provided that $(2 - \varepsilon)\alpha < 1$.

A class of equations that fits the general theory are the equations in divergence form,

$$\begin{cases} u_t = \displaystyle\sum_{i=1}^N D_i(\varphi_i(u) + D_i u) = \Delta u + \sum_{i=1}^N \varphi_i'(u) D_i u, & t > 0, \quad x \in \overline{\Omega}, \\ u(t, x) = 0, & t > 0, \quad x \in \partial\Omega, \\ u(0, x) = u_0(x), & \qquad\quad x \in \overline{\Omega}, \end{cases} \tag{6.40}$$

for which we have existence in the large for all initial data if the functions $\varphi_i : \mathbb{R} \to \mathbb{R}$ are differentiable with locally Lipschitz continuous derivatives. Indeed, the function

$$f(t, x, u, p) = \sum_{i=1}^N \varphi_i'(u) p_i$$

satisfies conditions (6.38) and (6.39).

6.3.3 The Cahn-Hilliard equation

Let us consider a one dimensional Cahn-Hilliard equation,

$$\begin{cases} u_t = \Big(-u_{xx} + f(u) \Big)_{xx}, & t > 0, \quad x \in [0,1], \\ u_x(t,0) = u_x(t,1) = u_{xxx}(t,0) = u_{xxx}(t,1) = 0, & t > 0, \\ u(0,x) = u_0(x), & x \in [0,1], \end{cases} \tag{6.41}$$

under the following assumptions on f and u_0:

$$f \in C^3(\mathbb{R}), \;\; f \text{ has a nonnegative primitive } \Phi,$$

$$u_0 \in C^2([0,1]), \;\; u_0'(0) = u_0'(1) = 0.$$

Assumption $f \in C^3(\mathbb{R})$ and the assumptions on u_0 are sufficient to obtain a local solution. The positivity of a primitive of f will be used to get *a priori* estimates on the solution that guarantee existence in the large.

Set $X = C([0,1])$ and

$$D(B) = \{\varphi \in C^2([0,1]) : \varphi'(0) = \varphi'(1) = 0\}, \;\; B\varphi = \varphi'',$$

$$D(A) = \{\varphi \in C^4([0,1]) : \varphi'(0) = \varphi'(1) = \varphi'''(0) = \varphi'''(1) = 0\}, \;\; A\varphi = -\varphi''''.$$

The operator A has a very special form; specifically $A = -B^2$, where B is sectorial by Exercise 4, §2.1.3, and (1.9) holds with any $\theta \in (\pi/2, \pi)$. Then A is sectorial in X by Exercise 1, §2.2.4, and $D(B)$ is of class $J_{1/2}$ between X and $D(A)$ by Exercise 1, §3.2.3. Therefore we may choose

$$\alpha = 1/2, \;\; X_{1/2} = D(B).$$

Note that both $D(B)$ and $D(A)$ are dense in X. Since B commutes with $R(\lambda, A)$ on $D(B)$ for each $\lambda \in \rho(A)$, then it commutes with e^{tA} on $D(B)$, and for each $\varphi \in D(B)$ and $t \in [0,T]$ we have

$$\|e^{tA}\varphi\|_{D(B)} = \|e^{tA}\varphi\|_\infty + \left\| \frac{d^2}{dx^2} e^{tA}\varphi \right\|_\infty = \|e^{tA}\varphi\|_\infty + \|e^{tA}\varphi''\|_\infty \le M_0 \|\varphi\|_{D(B)},$$

for some $M_0 > 0$, so that condition (6.28) is satisfied.

The function

$$F : X_{1/2} \to X,$$

$$F(\varphi) = \frac{d^2}{dx^2} f(\varphi) = f'(\varphi)\varphi'' + f''(\varphi)(\varphi')^2$$

is Lipschitz continuous on each bounded subset of $X_{1/2}$, because f'' is locally Lipschitz continuous.

Theorem 6.3.2 implies that for each $u_0 \in D(B)$ there is a maximal $\tau = \tau(u_0) > 0$ such that problem (6.41) has a unique solution $u : [0,\tau) \times [0,1] \to \mathbb{R}$, such that u, u_x, u_{xx} are continuous in $[0,\tau) \times [0,1]$, and u_t, u_{xxx}, u_{xxxx} are continuous in $(0,\tau) \times [0,1]$. Notice that, since $D(B)$ is dense in X, then $D(A) = D(B^2)$ is dense in $D(B)$. In other words, the closure of $D(A)$ in $X_{1/2}$ is the whole $X_{1/2}$.

Since we have a fourth order differential equation, the maximum principles are not of help to prove that u is bounded. We shall prove that the norm $\|u_x(t,\cdot)\|_{L^2}$ is bounded in $I(u_0)$; this will imply that u is bounded in $I(u_0)$ through a Poincaré-Sobolev inequality.

Since $u_t = (-u_{xx} + f(u))_{xx}$ for each $t > 0$, for $\varepsilon \in (0, \tau(u_0))$ we have

$$\int_0^1 (u(t,x) - u(\varepsilon,x))dx = \int_\varepsilon^t dt \int_0^1 u_t(s,x)dx = 0, \quad \varepsilon \le t < \tau(u_0). \tag{6.42}$$

Letting ε tend to 0 we get

$$\int_0^1 u(t,x)dx = \int_0^1 u_0(x)dx, \quad 0 < t < \tau(u_0),$$

so that the mean value of $u(t,\cdot)$ is a constant, independent of t[1].

Fix again $\varepsilon \in (0, \tau(u_0))$, multiply both sides of the equation by $-u_{xx} + f(u)$, and integrate over $[\varepsilon, t] \times [0,1]$ for $t \in (\varepsilon, \tau(u_0))$. We get

$$-\int_\varepsilon^t \int_0^1 u_t u_{xx} ds\, dx + \int_\varepsilon^t \int_0^1 u_t f(u) ds\, dx = \int_\varepsilon^t \int_0^1 (-u_{xx} + f(u))(-u_{xx} + f(u))_{xx} ds\, dx$$

Note that we may integrate by parts with respect to x in the first integral, because u_{tx} exists and it is continuous in $[\varepsilon, t] \times [0,1]$, see Exercise 2(a), §6.3.9. Hence, we integrate by parts in the first integral, we rewrite the second integral recalling that $f = \Phi'$, and we integrate by parts in the third integral too. We get

$$\begin{aligned} &\int_\varepsilon^t \int_0^1 u_x(s,x) u_{tx}(s,x) ds\, dx + \int_\varepsilon^t \frac{d}{ds} \int_0^1 \Phi(u(s,x)) dx\, ds \\ &= -\int_\varepsilon^t \int_0^1 \Big((-u_{xx}(s,x) + f(u(s,x))_x \Big)^2 dx\, ds \end{aligned}$$

so that

$$\frac{1}{2}\int_0^1 u_x(t,x)^2 dx - \frac{1}{2}\int_0^1 u_x(\varepsilon,x)^2 dx + \int_0^1 [\Phi(u(t,x) - \Phi(u(\varepsilon,x))]dx \le 0,$$

and letting $\varepsilon \to 0$ we get

$$\|u_x(t,\cdot)\|_{L^2}^2 + 2\int_0^1 \Phi(u(t,x))dx \le \|u_0'\|_{L^2}^2 + 2\int_0^1 \Phi(u_0(x))dx, \quad 0 < t < \tau(u_0).$$

Since Φ is nonnegative, then $u_x(t,\cdot)$ is bounded in L^2 for $t \in I(u_0)$. Since $u(t,\cdot)$ has constant mean value, inequality (6.45) yields that $u(t,\cdot)$ is bounded in the sup norm.

Now we may use Proposition 6.3.6, because F satisfies (6.33) with $\gamma = 1$. Indeed, for each $\varphi \in X_{1/2}$ we have

$$\begin{aligned} \|F(\varphi)\| &\le \sup_{|\xi| \le \|\varphi\|_\infty} |f'(\xi)| \cdot \|\varphi''\|_\infty + \sup_{|\xi| \le \|\varphi\|_\infty} |f''(\xi)| \cdot \|\varphi'\|_\infty^2 \\ &\le \sup_{|\xi| \le \|\varphi\|_\infty} |f'(\xi)| \cdot \|\varphi''\|_\infty + \sup_{|\xi| \le \|\varphi\|_\infty} |f''(\xi)| \cdot C\|\varphi\|_\infty \|\varphi''\|_\infty \\ &\le \mu(\|\varphi\|)\|\varphi\|_{D(B)} \end{aligned}$$

where $\mu(s) = \max\{\sup_{|\xi| \le s} |f'(\xi)|,\ Cs \sup_{|\xi| \le s} |f''(\xi)|\}$, and C is the constant in Exercise 2(b), §6.3.9. Therefore F has subcritical growth (the critical growth exponent is 2). By Proposition 6.3.6, the solution exists in the large.

[1]We take $\varepsilon > 0$ in (6.42) because our solution is just classical and it is not strict in general, so that it is not obvious that u_t is in $L^1((0,t) \times (0,1))$.

6.3.4 The Kuramoto-Sivashinsky equation

This equation arises as a mathematical model in a two dimensional combustion phenomenon. At time t, the combustion takes place along an unknown curve with equation $x = \xi(t, y)$, and the open set $\{(x, y) \in \mathbb{R}^2 : \ x < \xi(t, y)\}$ is the fresh region, the open set $\{(x, y) \in \mathbb{R}^2 : \ x > \xi(t, y)\}$ is the burnt region at time t. As time increases, the curve moves to the left, and under suitable assumptions the function $\Phi(t, y) = \xi(t, y) + t$ satisfies the Kuramoto-Sivashinsky equation

$$\Phi_t(t, y) + 4\Phi_{yyyy}(t, y) + \Phi_{yy}(t, y) + \frac{1}{2}(\Phi_y)^2 = 0, \ \ t \geq 0, \ y \in \mathbb{R}. \tag{6.43}$$

The Cauchy problem

$$\Phi(0, y) = \Phi_0(y), \ \ y \in \mathbb{R} \tag{6.44}$$

for equation (6.43) may be treated with the methods of §6.3.3. Set

$$X = C_b(\mathbb{R}),$$

and

$$A : D(A) = C_b^4(\mathbb{R}), \quad Au = -4u'''' - u''.$$

To prove that A is sectorial, it is convenient to write it as

$$A = -4B^2 - B$$

where B is the realization of the second order derivative in X, that is sectorial by §2.1.1. By Exercise 1, §2.2.4, $-4B^2$ is sectorial, and by Exercise 1, §3.2.3, the domain $D(B)$ is of class $J_{1/2}$ between X and $D(B^2)$. Then Proposition 3.2.2(iii) yields that A is sectorial.

Since the nonlinearity $\frac{1}{2}(\Phi_y)^2$ depends on the first order space derivative, it is convenient to choose $\alpha = 1/4$ and

$$X_{1/4} := C_b^1(\mathbb{R}).$$

Such a space belongs to the class $J_{1/4}$ between X and $D(A)$, by Exercise 3, §3.2.3. The nonlinear function

$$F(u)(y) = -\frac{1}{2}(u'(y))^2, \ \ u \in X_{1/4}, \ y \in \mathbb{R},$$

is Lipschitz continuous on the bounded subsets of $X_{1/4}$, and it is not hard to prove that $\|e^{tA}\|_{\mathcal{L}(X_{1/4})} \leq \|e^{tA}\|_{\mathcal{L}(X)}$ for each $t > 0$, see Exercise 4 below.

So, we may rewrite problem (6.43)–(6.44) in the form (6.26), with F independent of t. All the assumptions of Theorem 6.3.2 are satisfied. Moreover, $X_{1/4}$ is contained in $\overline{D(A)} = BUC(\mathbb{R})$, since all the elements of $X_{1/4}$ are bounded and Lipschitz continuous functions.

It follows that for each $\Phi_0 \in C_b^1(\mathbb{R})$ problem (6.43)–(6.44) has a unique classical solution $\Phi(t, y)$, defined for t in a maximal time interval $[0, \tau(\Phi_0))$ and for y in $\mathbb{R}$, such that for every $a \in (0, \tau(\Phi_0))$, $\Phi \in C_b([0, a] \times \mathbb{R})$, and there exist Φ_t, $\Phi_{yyyy} \in C((0, a] \times \mathbb{R})$, that are bounded in each $[\varepsilon, a] \times \mathbb{R}$ with $0 < \varepsilon < a$.

Exercises 6.3.9

1. Prove that the conclusions of Proposition 6.2.2 hold for the solution of problem (6.35), provided that (6.38) holds.

2. (a) Referring to §6.3.3, prove that u_{tx} exists and it is continuous in $[\varepsilon, \tau-\varepsilon] \times [0,1]$ for each $\varepsilon \in (0, \tau/2)$.

   [Hint: use Proposition 6.3.3 to get $t \mapsto u_t(t,\cdot) \in B([\varepsilon, \tau-\varepsilon]; D_A(\theta,\infty))$ for each $\theta \in (0,1)$, and then Exercise 4(c), §3.2.3, to conclude].

   (b) Prove that there is $C > 0$ such that for each $\varphi \in C^2([0,1])$ satisfying $\varphi'(0) = \varphi'(1) = 0$ we have
$$\|\varphi'\|_\infty^2 \leq C\|\varphi\|_\infty \|\varphi''\|_\infty.$$

3. Prove the inequality
$$\|\varphi - \int_0^1 \varphi(y)dy\|_\infty \leq \left(\int_0^1 (\varphi'(x))^2 dx \right)^{1/2}, \tag{6.45}$$
   for each $\varphi \in C^1([0,1])$.

4. Referring to §6.3.4, prove that $\|e^{tA}\|_{\mathcal{L}(X_{1/4})} \leq \|e^{tA}\|_{\mathcal{L}(X)}$, for every $t > 0$.

   [Hint: show that e^{tA} commutes with the first order derivative on $X_{1/4}$. For this purpose, use Exercise 1.3.18].

Chapter 7

Behavior near stationary solutions

7.1 The principle of linearized stability

Let $A : D(A) \subset X \to X$ be a sectorial operator. We use the notation of Chapter 6, so X_α is a space of class J_α between X and $D(A)$, that satisfies (6.28), and $0 < \alpha < 1$.

Let us consider the nonlinear equation

$$u'(t) = Au(t) + F(u(t)), \quad t > 0, \tag{7.1}$$

where $F : X \to X$, or $F : X_\alpha \to X$ satisfies the assumptions of the local existence Theorems 6.1.1 or 6.3.2. Throughout this section we assume that $F(0) = 0$, so that problem (7.1) admits the stationary (:= constant in time) solution $u \equiv 0$, and we study the stability of the null solution.

From the point of view of the stability, the case where F is defined in X_α does not differ much from the case where it is defined in the whole space X, and they will be treated together, setting $X_0 := X$ and considering $\alpha \in [0, 1)$.

In any case we assume that the Lipschitz constant

$$K(\rho) := \sup\left\{\frac{\|F(x) - F(y)\|}{\|x - y\|_{X_\alpha}} : x,\ y \in B(0,\rho) \subset X_\alpha\right\} \tag{7.2}$$

satisfies

$$\lim_{\rho \to 0^+} K(\rho) = 0. \tag{7.3}$$

This implies that F is Fréchet differentiable at 0, with null derivative.

We recall that if X, Y are Banach spaces and $y \in Y$, we say that a function G defined in a neighborhood of y with values in X is *Fréchet differentiable* at y if there exists a linear bounded operator $L \in \mathcal{L}(Y, X)$ such that

$$\lim_{h \to 0} \frac{\|G(y + h) - G(y) - Lh\|_X}{\|h\|_Y} = 0.$$

In this case, L is called the derivative of G at y and we set $L = G'(y)$. If $\mathcal{O} \subset Y$ is an open set, we say that $G : \mathcal{O} \to X$ is continuously differentiable in $\mathcal{O}$ if it is differentiable at each $y \in \mathcal{O}$ and the function $G' : \mathcal{O} \to \mathcal{L}(Y, X)$ is continuous in $\mathcal{O}$.

It is clear that if F is Fréchet continuously differentiable in a neighborhood of 0, and $F'(0) = 0$, then $\lim_{\rho \to 0^+} K(\rho) = 0$.

By Theorem 6.1.1 (if $\alpha = 0$) or Theorem 6.3.2 (if $\alpha \in (0,1)$), for every initial datum $u_0 \in X_\alpha$ the Cauchy problem for equation (7.1) has a unique solution $u(\cdot; u_0)$ defined in a maximal time interval $[0, \tau(u_0))$.

Definition 7.1.1 *We say that the null solution of* (7.1) *is* stable *(in X_α) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$u_0 \in X_\alpha, \ \ \|u_0\|_{X_\alpha} \leq \delta \Longrightarrow \tau(u_0) = +\infty, \ \ \|u(t; u_0)\|_{X_\alpha} \leq \varepsilon, \ \ \forall t \geq 0.$$

The null solution of (7.1) *is said to be* asymptotically stable *if it is stable and moreover there exists $\delta_0 > 0$ such that if $\|u_0\|_{X_\alpha} \leq \delta_0$ then $\lim_{t\to+\infty} \|u(t; u_0)\|_{X_\alpha} = 0$.*

The null solution of (7.1) *is said to be* unstable *if it is not stable.*

The principle of linearized stability says that in the noncritical case $s(A) \neq 0$ the null solution to the nonlinear problem (7.1) has the same stability properties of the null solution to the linear problem $u' = Au$. Note that by assumption (7.3) the linear part of $Ax + F(x)$ near $x = 0$ is just Ax, so that the nonlinear part $F(u)$ in problem (7.1) looks like a small perturbation of the linear part $u' = Au$, at least for solutions close to 0. In the next two subsections we make this argument rigorous.

The study of the stability of other possible stationary solutions, that is of the $\overline{u} \in D(A)$ such that

$$A\overline{u} + F(\overline{u}) = 0,$$

can be reduced to the case of the null stationary solution by defining a new unknown

$$v(t) = u(t) - \overline{u},$$

and studying the problem

$$v'(t) = \widetilde{A}v(t) + \widetilde{F}(v(t)), \quad t > 0,$$

where $\widetilde{A} = A + F'(\overline{u})$ and $\widetilde{F}(v) = F(v + \overline{u}) - F(\overline{u}) - F'(\overline{u})v$, provided that F is Fréchet differentiable at $\overline{u}$. Note that in this case the Fréchet derivative of $\widetilde{F}$ vanishes at 0.

7.1.1 Linearized stability

The main assumption is

$$s(A) < 0. \tag{7.4}$$

(The spectral bound $s(A)$ is defined in (5.1)). In the proof of the linearized stability theorem we shall use the next lemma, which is a consequence of Proposition 5.1.1.

Lemma 7.1.2 *Let* (7.4) *hold, and fix $\omega \in [0, -s(A))$. If $f \in C_{-\omega}((0,+\infty); X)$ and $x \in X_\alpha$ then the function*

$$v(t) = e^{tA}x + \int_0^t e^{(t-s)A} f(s) ds, \quad t > 0,$$

belongs to $C_{-\omega}((0,+\infty); X_\alpha)$, and there is a constant $C = C(\omega)$ such that

$$\sup_{t>0} e^{\omega t} \|v(t)\|_{X_\alpha} \leq C(\|x\|_{X_\alpha} + \sup_{t>0} e^{\omega t} \|f(t)\|).$$

Proof. By Proposition 5.1.1, for each $\omega \in [0, -s(A))$ there is $M(\omega) > 0$ such that $\|e^{tA}\|_{\mathcal{L}(X)} \leq M(\omega)e^{-\omega t}$, for every $t > 0$. Therefore, for $t \geq 1$,

$$\|e^{tA}\|_{\mathcal{L}(X,X_\alpha)} \leq \|e^{A}\|_{\mathcal{L}(X,X_\alpha)}\|e^{(t-1)A}\|_{\mathcal{L}(X)} \leq Ce^{-\omega t} \tag{7.5}$$

with $C = M(\omega)e^{\omega}\|e^{A}\|_{\mathcal{L}(X,X_\alpha)}$, while for $0 < t < 1$ we have $\|e^{tA}\|_{\mathcal{L}(X,X_\alpha)} \leq Ct^{-\alpha}$ for some constant $C > 0$, by Proposition 3.2.2(ii). Since $\omega \in [0, -s(A))$ is arbitrary, this implies

$$\sup_{t>0} e^{\omega t}t^{\alpha}\|e^{tA}\|_{\mathcal{L}(X,X_\alpha)} := C_\omega < +\infty.$$

Since X_α is continuously embedded in X, (7.5) implies also that $\|e^{tA}\|_{\mathcal{L}(X_\alpha)} \leq \widehat{C}e^{-\omega t}$ for $t \geq 1$ and some positive constant $\widehat{C}$. Since $\|e^{tA}\|_{\mathcal{L}(X_\alpha)}$ is bounded for $t \in (0,1)$ by a constant independent of t by assumption (6.28), we get

$$\sup_{t>0} e^{\omega t}\|e^{tA}\|_{\mathcal{L}(X_\alpha)} := \widetilde{C}_\omega < +\infty.$$

Therefore $\|e^{tA}x\|_{X_\alpha} \leq \widetilde{C}_\omega e^{-\omega t}\|x\|_{X_\alpha}$, and for any fixed $\omega' \in (\omega, -s(A))$,

$$\|e^{\omega t}(e^{tA} * f)(t)\|_{X_\alpha} \leq C_{\omega'}e^{\omega t}\int_0^t \frac{e^{-\omega' s}}{s^\alpha}\|f(t-s)\|ds \leq \frac{C_{\omega'}\Gamma(1-\alpha)}{(\omega'-\omega)^{1-\alpha}}\sup_{r>0} e^{\omega r}\|f(r)\|,$$

for every $t > 0$, and the statement follows. □

Theorem 7.1.3 *Let A satisfy (7.4), and let $F : X_\alpha \to X$ be Lipschitz continuous in a neighborhood of 0 and satisfy (7.3). Then for every $\omega \in [0, -s(A))$ there exist positive constants $M = M(\omega)$, $r = r(\omega)$ such that if $u_0 \in X_\alpha$, $\|u_0\|_{X_\alpha} \leq r$, we have $\tau(u_0) = +\infty$ and*

$$\|u(t; u_0)\|_{X_\alpha} \leq Me^{-\omega t}\|u_0\|_{X_\alpha}, \quad t \geq 0. \tag{7.6}$$

Therefore the null solution is asymptotically stable.

Proof. Let Y be the closed ball centered at 0 with small radius ρ in the space $C_{-\omega}((0,+\infty); X_\alpha)$, namely

$$Y = \{u \in C_{-\omega}((0,+\infty); X_\alpha) : \sup_{t\geq 0}\|e^{\omega t}u(t)\|_{X_\alpha} \leq \rho\}.$$

We look for the mild solution to (7.1) with initial datum u_0 as a fixed point of the operator $\mathcal{G}$ defined on Y by

$$(\mathcal{G}u)(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}F(u(s))ds, \quad t \geq 0. \tag{7.7}$$

If $u \in Y$, by (7.2) we get

$$\|F(u(t))\| = \|F(u(t)) - F(0)\| \leq K(\rho)\|u(t)\|_{X_\alpha} \leq K(\rho)\rho e^{-\omega t}, \quad t \geq 0, \tag{7.8}$$

so that $F(u(\cdot)) \in C_{-\omega}((0,+\infty); X)$. Using Lemma 7.1.2 we get

$$\|\mathcal{G}u\|_{C_{-\omega}((0,+\infty);X_\alpha)} \leq C\left(\|u_0\|_{X_\alpha} + \|F(u(\cdot))\|_{C_{-\omega}((0,+\infty);X)}\right) \leq C\left(\|u_0\|_{X_\alpha} + \rho K(\rho)\right). \tag{7.9}$$

If ρ is so small that
$$K(\rho) \le \frac{1}{2C},$$
and
$$\|u_0\|_{X_\alpha} \le r := \frac{\rho}{2C},$$
then $\mathcal{G}u \in Y$. Moreover, for u_1, $u_2 \in Y$ we have, again by Lemma 7.1.2,
$$\|\mathcal{G}u_1 - \mathcal{G}u_2\|_{C_{-\omega}((0,+\infty);X_\alpha)} \le C\|F(u_1(\cdot)) - F(u_2(\cdot))\|_{C_{-\omega}((0,+\infty);X)},$$
and (7.2) yields
$$\|F(u_1(t)) - F(u_2(t))\| \le K(\rho)\|u_1(t) - u_2(t)\|_{X_\alpha}, \;\; t > 0.$$
It follows that
$$\|\mathcal{G}u_1 - \mathcal{G}u_2\|_{C_{-\omega}((0,+\infty);X_\alpha)} \le \frac{1}{2}\|u_1 - u_2\|_{C_{-\omega}((0,+\infty);X_\alpha)},$$
so that $\mathcal{G}$ is a contraction with constant $1/2$. Consequently there exists a unique fixed point of $\mathcal{G}$ in Y, which is the solution of (7.1) with initial datum u_0. Note that the Contraction Theorem gives a unique solution in Y, but we already know by Theorems 6.1.1 and 6.3.2 that the mild solution is unique.

Moreover from (7.8), (7.9) we get
$$\|u\|_{C_{-\omega}} = \|\mathcal{G}u\|_{C_{-\omega}} \le C(\|u_0\|_{X_\alpha} + K(\rho)\|u\|_{C_{-\omega}}) \le C\|u_0\|_{X_\alpha} + \frac{1}{2}\|u\|_{C_{-\omega}}$$
which implies (7.6), with $M = 2C$. $\square$

Remark 7.1.4 Note that any mild solution to problem (7.1) is smooth for $t > 0$. Precisely, Proposition 6.1.2 if $\alpha = 0$ and Proposition 6.3.3 if $\alpha > 0$ imply that for each $\theta \in (0,1)$ and for each interval $[a,b] \subset (0, \tau(u_0))$, the restriction of $u(\cdot; u_0)$ to $[a,b]$ belongs to $C^{1+\theta}([a,b]; X) \cap C^{\theta}([a,b]; D(A))$.

7.1.2 Linearized instability

Assume now that
$$\begin{cases} \sigma_+(A) := \sigma(A) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} \neq \varnothing, \\ \\ \inf\{\operatorname{Re}\lambda : \lambda \in \sigma_+(A)\} := \omega_+ > 0. \end{cases} \tag{7.10}$$

Then it is possible to prove an instability result for the null solution. We shall use the projection P defined by (5.6), i.e.
$$P = \frac{1}{2\pi i}\int_{\gamma_+} R(\lambda, A)d\lambda,$$
γ_+ being any closed regular path with range in $\{\operatorname{Re}\lambda > 0\}$, with index 1 with respect to each $\lambda \in \sigma_+$.

For the proof of the instability theorem we need the next lemma, which is a corollary of Theorem 5.4.1(ii). It is a counterpart of Lemma 7.1.2 for the unstable case.

Lemma 7.1.5 *Let (7.10) hold, and fix $\omega \in [0, \omega_+)$. If $g \in C_\omega((-\infty, 0); X)$ and $x \in P(X)$, then the function*

$$v(t) = e^{tA}x + \int_0^t e^{(t-s)A} P g(s) ds + \int_{-\infty}^t e^{(t-s)A}(I - P) g(s) ds, \quad t \le 0 \tag{7.11}$$

is a mild solution to $v'(t) = Av(t) + g(t)$, $t \le 0$, it belongs to $C_\omega((-\infty, 0]; X_\alpha)$, and there is a constant $C = C(\omega)$ such that

$$\sup_{t\le 0} e^{-\omega t} \|v(t)\|_{X_\alpha} \le C(\|x\| + \sup_{t<0} e^{-\omega t} \|g(t)\|). \tag{7.12}$$

Conversely, if v is a mild solution belonging to $C_\omega((-\infty, 0]; X_\alpha)$ then there is $x \in P(X)$ such that v has the representation (7.11).

Proof. That v is a mild solution belonging to $C_\omega((-\infty, 0]; X)$ follows as in Theorem 5.4.1(ii), because the vertical line $\mathrm{Re}\,\lambda = \omega$ does not intersect the spectrum of A.

Conversely, if v is a mild solution in $C_\omega((-\infty, 0]; X_\alpha)$ then it is in $C_\omega((-\infty, 0]; X)$ and Theorem 5.4.1(ii) implies (7.11).

Now we prove (7.12). Let $w(t) = e^{-\omega t} v(t)$. Then

$$w(t) = e^{t(A-\omega)}x + \int_0^t e^{(t-s)(A-\omega)} P g(s) e^{-\omega s} ds + \int_{-\infty}^t e^{(t-s)(A-\omega)}(I - P) g(s) e^{-\omega s} ds$$

and $A - \omega I$ is hyperbolic with $\sigma_+(A - \omega I) = \sigma_+(A) - \omega$ and $\sigma_-(A - \omega I) = (\sigma(A) \setminus \sigma_+(A)) - \omega$. Using Proposition 5.2.1 we take a small $\sigma > 0$ such that

$$\|e^{t(A-\omega I)}(I - P)\|_{\mathcal{L}(X)} \le Ce^{-2\sigma t}, \quad t \ge 0,$$

$$\|e^{t(A-\omega I)}P\|_{\mathcal{L}(X)} \le Ce^{\sigma t}, \quad t \le 0.$$

Since the part of A in $P(X)$ is bounded,

$$\|e^{t(A-\omega I)}P\|_{\mathcal{L}(X,D(A))} \le C'e^{\sigma t}, \quad t \le 0,$$

hence

$$\|e^{t(A-\omega I)}P\|_{\mathcal{L}(X,X_\alpha)} \le C''e^{\sigma t}, \quad t \le 0.$$

Moreover, if $t \ge 1$,

$$\|e^{t(A-\omega I)}(I - P)\|_{\mathcal{L}(X,X_\alpha)} \le \|e^{A-\omega}\|_{\mathcal{L}(X,X_\alpha)} \|e^{(t-1)(A-\omega I)}(I - P)\|_{\mathcal{L}(X)} \le C_1 e^{-\sigma t}$$

and for $0 < t \le 1$

$$\|e^{t(A-\omega I)}(I - P)\|_{\mathcal{L}(X,X_\alpha)} \le C_2 t^{-\alpha},$$

so that

$$\|e^{t(A-\omega I)}(I - P)\|_{\mathcal{L}(X,X_\alpha)} \le C_3 t^{-\alpha} e^{-\sigma t}, \quad t \ge 0.$$

Therefore, for $t \le 0$

$$\begin{aligned} \|w(t)\|_{X_\alpha} \quad &\le \quad Ce^{\sigma t}\|x\| + C\|P\| \sup_{s\le 0}(e^{-\omega s}\|g(s)\|) \int_t^0 e^{\sigma s} ds \\ &\quad +C_3\|I - P\| \sup_{s\le 0}(e^{-\omega s}\|g(s)\|) \int_{-\infty}^t e^{-\sigma(t-s)}(t - s)^{-\alpha} ds \end{aligned}$$

and (7.12) follows easily.

□

Theorem 7.1.6 *Let A satisfy* (7.10), *and let $F : X_\alpha \to X$ be Lipschitz continuous in a neighborhood of* 0 *and satisfy* (7.3). *Then there exists $r_+ > 0$ such that for every $x \in P(X)$ satisfying $\|x\| \leq r_+$, the problem*

$$\begin{cases} v'(t) = Av(t) + F(v(t)), & t \leq 0, \\ Pv(0) = x, \end{cases} \tag{7.13}$$

has a backward solution v such that $\lim_{t\to-\infty} v(t) = 0$.

Proof. Let Y_+ be the closed ball centered at 0 with small radius ρ_+ in $C_\omega((-\infty, 0]; X_\alpha)$. In view of Lemma 7.1.5, we look for a solution to (7.13) as a fixed point of the operator $\mathcal{G}_+$ defined on Y_+ by

$$(\mathcal{G}_+ v)(t) = e^{tA}x + \int_0^t e^{(t-s)A} PF(v(s))ds + \int_{-\infty}^t e^{(t-s)A}(I - P)F(v(s))ds, \;\; t \leq 0.$$

If $v \in Y_+$, then $F(v(\cdot)) \in C_\omega((-\infty, 0]; X)$ and Lemma 7.1.5 implies $\mathcal{G}_+ v \in C_\omega((-\infty, 0]; X)$, with

$$\|\mathcal{G}_+ v\|_{C_\omega((-\infty,0];X_\alpha)} \leq C \left(\|x\| + \|F(v(\cdot))\|_{C_\omega((-\infty,0];X)}\right).$$

The rest of the proof is quite similar to the proof of Theorem 7.1.3 and it is left as an exercise. □

Remark 7.1.7 The existence of a backward mild solution v to problem (7.13) implies that the null solution to (7.1) is unstable. For, let $x_n = v(-n)$. Of course $x_n \to 0$ as n tends to $+\infty$. For any $n \in \mathbb{N}$ consider the forward Cauchy problem $u(0) = x_n$ for the equation (7.1), and as usual denote by $u(\cdot; x_n)$ its mild solution. Then $\tau(x_n) \geq n$ and $u(t; x_n) = v(t - n)$ for any $t \in [0, n]$. Hence

$$\sup_{t\in I(x_n)} \|u(t; x_n)\|_{X_\alpha} \geq \sup_{0\leq t\leq n} \|u(t; x_n)\|_{X_\alpha} = \sup_{-n\leq t\leq 0} \|v(t)\|_{X_\alpha} \geq \|v(0)\|_{X_\alpha} > 0$$

which implies that the null solution is unstable since $\sup_{t\geq 0} \|u(t; x_n)\|_{X_\alpha}$ does not tend to 0 as n tends to $+\infty$.

7.2 A Cauchy-Dirichlet problem

In order to give some examples of PDE's to which the results of this chapter can be applied, we need some comments on the spectrum of the Laplacian with Dirichlet boundary conditions.

Let Ω be a bounded open set in $\mathbb{R}^N$ with C^2 boundary $\partial\Omega$. We choose $X = C(\overline{\Omega})$ and define

$$D(A) = \left\{\varphi \in \bigcap_{1\leq p<+\infty} W^{2,p}(\Omega) : \Delta\varphi \in C(\overline{\Omega}), \varphi_{|\partial\Omega} = 0\right\}$$

and $A\varphi = \Delta\varphi$ for $\varphi \in D(A)$.

From Exercise 3, §5.4.4, we know that the spectrum of A consists of isolated eigenvalues and that $s(A)$ is negative. In order to give an explicit estimate of $s(A)$ we recall the so called Poincaré inequality: there is a constant $C_\Omega > 0$ such that

$$\int_\Omega |\varphi|^2 dx \leq C_\Omega \int_\Omega |D\varphi|^2 dx, \;\; \varphi \in W_0^{1,2}(\Omega). \tag{7.14}$$

A proof of (7.14) as well as the inequality $C_\Omega \le 4d^2$, where d is the diameter of Ω, is outlined in Exercise 4 below.

If $\varphi \in D(A)$ and $-\lambda\varphi - \Delta\varphi = 0$, then $\varphi \in W^{1,2}_0(\Omega)$. Multiplying by $\overline{\varphi}$ and integrating over Ω we find

$$\int_\Omega |D\varphi|^2 dx = \lambda \int_\Omega |\varphi|^2 dx$$

and therefore $\lambda \ge C_\Omega^{-1}$, that is $s(A) \le -C_\Omega^{-1}$.

We now study the stability of the null solution of

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + f(u(t,x), Du(t,x)), & t>0, \quad x \in \overline{\Omega}, \\ u(t,x) = 0, & t > 0, \quad x \in \partial\Omega, \end{cases} \tag{7.15}$$

where $f = f(u,p) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuously differentiable and $f(0,0) = 0$. The local existence and uniqueness Theorem 6.3.2 may be applied to the initial value problem for equation (7.15),

$$u(0,x) = u_0(x), \quad x \in \overline{\Omega}, \tag{7.16}$$

choosing $X = C(\overline{\Omega})$, $X_\alpha = C_0^{2\alpha}(\overline{\Omega})$ with $1/2 < \alpha < 1$. The function

$$F : X_\alpha \to X, \qquad (F(\varphi))(x) = f(\varphi(x), D\varphi(x)),$$

is continuously differentiable, and

$$F(0) = 0, \qquad (F'(0)\varphi)(x) = a\varphi(x) + \langle b, D\varphi(x)\rangle, \quad \varphi \in X_\alpha.$$

Here $a = f_u(0,0)$, $b = D_p f(0,0)$.

Then, set $D(B) = D(A)$ and $B\varphi = \Delta\varphi + \langle b, D\varphi\rangle + a\varphi$. The operator B is sectorial, see Corollary 1.3.14, and

$$s(B) \le -C_\Omega^{-1} + a. \tag{7.17}$$

Indeed we observe that the resolvent of B is compact and therefore its spectrum consists of isolated eigenvalues. Moreover, if $\lambda \in \sigma(B)$, $\varphi \in D(B)$, and $\lambda\varphi - \Delta\varphi - \langle b, D\varphi\rangle - a\varphi = 0$, then multiplying by $\overline{\varphi}$ and integrating over Ω we get

$$\int_\Omega \big((\lambda - a)|\varphi|^2 + |D\varphi|^2 - \langle b, D\varphi\rangle\, \overline{\varphi} \big)\, dx = 0.$$

Taking the real part

$$\int_\Omega \left((\mathrm{Re}\,\lambda - a)|\varphi|^2 + |D\varphi|^2 - \frac{1}{2} b \cdot D|\varphi|^2 \right) dx = \int_\Omega \big((\mathrm{Re}\,\lambda - a)|\varphi|^2 + |D\varphi|^2 \big)\, dx = 0$$

and hence $\mathrm{Re}\,\lambda - a \le -C_\Omega^{-1}$. Therefore (7.17) holds.

Since $u_0 \in X_\alpha \subset \overline{D(A)}$, Theorem 6.3.2 and Proposition 6.3.3 guarantee the existence of a unique local classical solution $u : [0, \tau(u_0)) \to X_\alpha$ of the abstract problem (7.1) with $u(0) = u_0$ having the regularity properties specified in Proposition 6.3.3. Setting as usual

$$u(t,x) := u(t)(x), \quad t \in [0, \tau(u_0)), \quad x \in \overline{\Omega},$$

the function u is continuous in $[0, \tau(u_0)) \times \overline{\Omega}$, continuously differentiable with respect to time for $t > 0$, and it satisfies (7.15), (7.16).

Concerning the stability of the null solution, Theorem 7.1.3 implies that if $s(B) < 0$, in particular if $a < C_\Omega^{-1}$, then the null solution of (7.15) is exponentially stable: for every $\omega \in (0, -s(B))$ there exist r, $C > 0$ such that if $\|u_0\|_{X_\alpha} \leq r$, then

$$\tau(u_0) = +\infty, \;\; \|u(t)\|_{X_\alpha} \leq Ce^{-\omega t}\|u_0\|_{X_\alpha}.$$

On the contrary, if $s(B) > 0$ then there are elements in the spectrum of B with positive real part. Since they are isolated they satisfy condition (7.10). Theorem 7.1.6 implies that the null solution of (7.15) is unstable: there exist $\varepsilon > 0$ and initial data u_0 with $\|u_0\|_{X_\alpha}$ arbitrarily small, but $\sup_{t\geq 0}\|u(t)\|_{X_\alpha} \geq \varepsilon$.

Finally we remark that if f is independent of p, i.e. the nonlinearity in (7.15) does not depend on Du, we can take $\alpha = 0$ and work in the space X.

Exercises 7.2.1

1. Complete the proof of Theorem 7.1.6.

2. Prove that the stationary solution $(u \equiv 0, v \equiv 1)$ to system (6.25) is asymptotically stable in $C(\overline{\Omega}) \times C(\overline{\Omega})$.

3. Assume that the functions φ_i in problem (6.40) are twice continuously differentiable and that $\varphi_i'(0) = 0$ for each $i = 1, \ldots, N$. Prove that the null solution to problem (6.40) is asymptotically stable in $C^{1+\theta}(\overline{\Omega})$, for each $\theta \in (0, 1)$.

4. Let Ω be a bounded set in $\mathbb{R}^N$ and let d be its diameter. Prove the Poincaré inequality (7.14) with $C_\Omega \leq 4d^2$.

   [Hint: assume that $\Omega \subset B(0, d)$ and for $\varphi \in C_0^\infty(\Omega)$ write
   $$\varphi(x_1, \ldots, x_N) = \int_{-d}^{x_1} \frac{\partial\varphi}{\partial x_1}(s, x_2, \ldots, x_N)ds].$$

5. Let X be a Banach space and Ω be an open set in $\mathbb{R}$ (or in $\mathbb{C}$). Moreover let $\Gamma : X \times \Omega \to X$ be such that
   $$\|\Gamma(y, \lambda) - \Gamma(x, \lambda)\| \leq C(\lambda)\|y - x\|$$
   for any $\lambda \in \Omega$, any $x, y \in X$ and some continuous function $C : \Omega \to [0, 1)$. Further, suppose that the function $\lambda \mapsto \Gamma(\lambda, x)$ is continuous in Ω for any $x \in X$. Prove that for any $\lambda \in \Omega$ the equation $x = \Gamma(x, \lambda)$ admits a unique solution $x = x(\lambda)$ and that the function $\lambda \mapsto x(\lambda)$ is continuous in Ω.

6. Let u be the solution to the problem
   $$\begin{cases} u_t = u_{xx} + u^2, & t \geq 0, \quad x \in [0, 1], \\ u(t, 0) = u(t, 1) = 0, & t \geq 0, \\ u(0, x) = u_0(x), & \qquad\;\; x \in [0, 1] \end{cases}$$
   with $u_0(0) = u_0(1) = 0$. Show that if $\|u_0\|_\infty$ is sufficiently small, then u exists in the large.

   [Hint: use the exponential decay of the heat semigroup in the variation of constants formula].

Appendix A

Linear operators and vector-valued calculus

In this appendix we collect a few basic results on linear operators in Banach spaces and on calculus for Banach space valued functions defined in a real interval or in an open set in $\mathbb{C}$. These results are assumed to be either known to the reader, or at least not surprising at all, as they follow quite closely the finite-dimensional theory.

Let X be a Banach space with norm $\|\cdot\|$. We denote by $\mathcal{L}(X)$ the Banach algebra of linear bounded operators $T : X \to X$, endowed with the norm

$$\|T\|_{\mathcal{L}(X)} = \sup_{x\in X:\, \|x\|=1} \|Tx\| \sup_{x\in X\setminus\{0\}} \frac{\|Tx\|}{\|x\|}.$$

If no confusion may arise, we write $\|T\|$ for $\|T\|_{\mathcal{L}(X)}$.

Similarly, if Y is another Banach space we denote by $\mathcal{L}(X,Y)$ the Banach space of linear bounded operators $T : X \to Y$, endowed with the norm $\|T\|_{\mathcal{L}(X,Y)} = \sup_{x\in X:\, \|x\|=1} \|Tx\|_Y$.

If $D(A)$ is a vector subspace of X and $A : D(A) \to X$ is linear, we say that A is *closed* if its graph

$$\mathcal{G}_A = \{(x,y) \in X\times X : x \in D(A),\ y = Ax\}$$

is a closed set of $X \times X$. In an equivalent way, A is closed if and only if the following implication holds:

$$\{x_n\} \subset D(A),\ \ x_n \to x,\ \ Ax_n \to y \qquad \Longrightarrow \qquad x \in D(A),\ y = Ax.$$

We say that A is *closable* if there is an (obviously unique) operator $\overline{A}$, whose graph is the closure of $\mathcal{G}_A$. It is readily checked that A is closable if and only if the implication

$$\{x_n\} \subset D(A),\ \ x_n \to 0,\ \ Ax_n \to y \qquad \Longrightarrow \qquad y = 0.$$

holds. If $A : D(A) \subset X \to X$ is a closed operator, we endow $D(A)$ with its *graph norm*

$$\|x\|_{D(A)} = \|x\| + \|Ax\|.$$

$D(A)$ turns out to be a Banach space and $A : D(A) \to X$ is continuous.

Next lemma is used in Chapter 1.

Lemma A.1 *Let X, Y be two Banach spaces, let D be a subspace of X, and let $\{A_n\}_{n\geq 0}$ be a sequence of linear bounded operators from X to Y such that*

$$\|A_n\| \leq M, \quad n \in \mathbb{N}, \qquad \lim_{n\to+\infty} A_n x = A_0 x, \quad x \in D.$$

Then

$$\lim_{n\to\infty} A_n x = A_0 x \quad x \in \overline{D},$$

where $\overline{D}$ is the closure of D in X.

Proof. Let $x \in \overline{D}$ and $\varepsilon > 0$ be given. For $y \in D$ with $\|x - y\| \leq \varepsilon$ and for every $n \in \mathbb{N}$ we have

$$\|A_n x - A_0 x\| \leq \|A_n(x - y)\| + \|A_n y - A_0 y\| + \|A_0(y - x)\|.$$

If n_0 is such that $\|A_n y - A_0 y\| \leq \varepsilon$ for every $n > n_0$, we have

$$\|A_n x - A_0 x\| \leq M\varepsilon + \varepsilon + \|A_0\|\varepsilon$$

for all $n \geq n_0$. $\square$

Let $I \subset \mathbb{R}$ be an interval. We denote by $C(I;X)$ the vector space of the continuous functions $u : I \to X$, by $B(I;X)$ the space of the bounded functions, endowed with the supremum norm

$$\|u\|_\infty = \sup_{t\in I} \|u(t)\|.$$

We also set $C_b(I;X) = C(I;X) \cap B(I;X)$. The definition of the derivative is readily extended to the present situation: a function $f \in C(I;X)$ is differentiable at an interior point $t_0 \in I$ if the following limit exists,

$$\lim_{t\to t_0} \frac{f(t) - f(t_0)}{t - t_0}.$$

As usual, the limit is denoted by $f'(t_0)$ and is it called derivative of f at t_0. In an analogous way we define right and left derivatives.

For every $k \in \mathbb{N}$ (resp., $k = +\infty$), $C^k(I;X)$ denotes the space of X-valued functions with continuous derivatives in I up to the order k (resp., of any order). We write $C^k_b(I;X)$ to denote the space of all the functions $f \in C^k(I;X)$ which are bounded in I together with their derivatives up to the k-th order.

Note that if $A : D(A) \to X$ is a linear closed operator, then a function $u : I \to D(A)$ belongs to $B(I;D(A))$ (resp., to $C(I;D(A))$, $C^k(I;D(A))$) if and only if both u and Au belong to $B(I;X)$ (resp., to $C(I;X)$, $C^k(I;X)$).

Let us define the Riemann integral of an X-valued function on a real interval.

Let $f : [a,b] \to X$ be a bounded function. We say that f is *integrable* on $[a,b]$ if there is $x \in X$ with the following property: for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every partition $\mathcal{P} = \{a = t_0 < t_1 < \ldots < t_n = b\}$ of $[a,b]$ with $t_i - t_{i-1} < \delta$ for all i, and for any choice of the points $\xi_i \in [t_{i-1}, t_i]$ we have

$$\left\| x - \sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}) \right\| < \varepsilon.$$

In this case we set

$$\int_a^b f(t)dt = x.$$

From the above definition we obtain immediately the following

Proposition A.2 *Let $\alpha, \beta \in \mathbb{C}$, and let f, g be integrable on $[a,b]$ with values in X. Then*

(a) $\int_a^b (\alpha f(t) + \beta g(t))dt = \alpha \int_a^b f(t)dt + \beta \int_a^b g(t)dt$;

(b) $|| \int_a^b f(t)dt|| \leq \sup_{t \in [a,b]} ||f(t)||(b-a)$;

(c) $|| \int_a^b f(t)dt|| \leq \int_a^b ||f(t)||dt$;

(d) if $A \in \mathcal{L}(X,Y)$, where Y is another Banach space, then Af is integrable with values in Y and $A \int_a^b f(t)dt = \int_a^b Af(t)dt$;

(e) if (f_n) is a sequence of continuous functions and there is f such that

$$\lim_{n \to +\infty} \max_{t \in [a,b]} ||f_n(t) - f(t)|| = 0,$$

then $\lim_{n \to +\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt$.

It is also easy to generalize to the present situation the Fundamental Theorem of Calculus. The proof is the same as for the real-valued case.

Theorem A.3 (Fundamental Theorem of Calculus) *Let $f : [a,b] \to X$ be continuous. Then the integral function*

$$F(t) = \int_a^t f(s)\, ds$$

is differentiable, and $F'(t) = f(t)$ for every $t \in [a,b]$.

Improper integrals of unbounded functions, or on unbounded intervals are defined as in the real-valued case. Precisely, If $I = (a,b)$ is a (possibly unbounded) interval and $f : I \to X$ is integrable on each compact interval contained in I, we set

$$\int_a^b f(t)dt := \lim_{r \to a^+,\, s \to b^-} \int_r^s f(t)dt,$$

provided that the limit exists in X. Note that statements (a), (d) of Proposition A.2 still hold for improper integrals. Statement (d) may be extended to closed operators too, as follows.

Lemma A.4 *Let $A : D(A) \subset X \to X$ be a closed operator, let I be a real interval with $\inf I = a$, $\sup I = b$ ($-\infty \leq a < b \leq +\infty$) and let $f : I \to D(A)$ be such that the functions $t \mapsto f(t)$, $t \mapsto Af(t)$ are integrable on I. Then*

$$\int_a^b f(t)dt \in D(A), \quad A \int_a^b f(t)dt = \int_a^b Af(t)dt.$$

Proof. Assume first that I is compact. Set $x = \int_a^b f(t)dt$ and choose a sequence $\mathcal{P}_k = \{a = t_0^k < \ldots < t_{n_k}^k = b\}$ of partitions of $[a,b]$ such that $\max_{i=1,\ldots,n_k}(t_i^k - t_{i-1}^k) < 1/k$. Let $\xi_i^k \in [t_i^k, t_{i-1}^k]$ for $i = 0, \ldots, n_k$, and consider

$$S_k = \sum_{i=1}^{n_k} f(\xi_i)(t_i - t_{i-1}).$$

All S_k are in $D(A)$, and

$$AS_k = \sum_{i=1}^{n_k} Af(\xi_i)(t_i - t_{i-1}).$$

Since both f and Af are integrable, S_k tends to x and AS_k tends to $y := \int_a^b Af(t)dt$. Since A is closed, x belongs to $D(A)$ and $Ax = y$.

Now let I be unbounded, say $I = [a, +\infty)$; then, for every $b > a$ the equality

$$A \int_a^b f(t)dt = \int_a^b Af(t)dt$$

holds. By hypothesis,

$$\int_a^b Af(t)dt \ \to \ \int_a^{+\infty} Af(t)dt \quad \text{and} \quad \int_a^b f(t)dt \ \to \ \int_a^{+\infty} f(t)dt \quad \text{as } b \to +\infty,$$

hence

$$A \int_a^b f(t)dt \to \int_a^{+\infty} Af(t)dt$$

and the thesis follows since A is closed. □

Now we review some basic facts concerning vector-valued functions of a complex variable.

Let Ω be an open subset of $\mathbb{C}$, $f : \Omega \to X$ be a continuous function and $\gamma : [a, b] \to \Omega$ be a piecewise C^1-curve. The integral of f along γ is defined by

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Let Ω be an open subset of $\mathbb{C}$ and let $f : \Omega \to X$ be a continuous function.

As usual, we denote by X' the dual space of X consisting of all linear bounded operators from X to $\mathbb{C}$. For each $x \in X$, $x' \in X'$ we set $x'(x) = \langle x, x' \rangle$.

Definition A.5 *f is holomorphic in Ω if for each $z_0 \in \Omega$ the limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} := f'(z_0)$$

exists in X. f is weakly holomorphic in Ω if it is continuous in Ω and the complex-valued function $z \mapsto \langle f(z), x' \rangle$ is holomorphic in Ω for every $x' \in X'$.

Clearly, any holomorphic function is weakly holomorphic; actually, the converse is also true, as the following theorem shows.

Theorem A.6 *Let $f : \Omega \to X$ be a weakly holomorphic function. Then f is holomorphic.*

Proof. Let $\overline{B(z_0, r)}$ be a closed ball contained in Ω; we prove that for all $z \in B(z_0, r)$ the following Cauchy integral formula holds:

$$f(z) = \frac{1}{2\pi i} \int_{\partial B(z_0, r)} \frac{f(\xi)}{\xi - z} d\xi. \tag{A.1}$$

First of all, we observe that the right hand side of (A.1) is well defined because f is continuous. Since f is weakly holomorphic in Ω, the complex-valued function $z \mapsto \langle f(z), x'\rangle$ is holomorphic in Ω for all $x' \in X'$, and hence the ordinary Cauchy integral formula in $B(z_0, r)$ holds, i.e.,

$$\langle f(z), x'\rangle = \frac{1}{2\pi i}\int_{\partial B(z_0,r)} \frac{\langle f(\xi), x'\rangle}{\xi - z}\, d\xi = \left\langle \frac{1}{2\pi i}\int_{\partial B(z_0,r)} \frac{f(\xi)}{\xi - z}\, d\xi, x'\right\rangle.$$

Since $x' \in X'$ is arbitrary, we obtain (A.1). We can differente with respect to z under the integral sign, so that f is holomorphic and

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial B(z_0,r)} \frac{f(\xi)}{(\xi - z)^{n+1}}\, d\xi$$

for all $z \in B(z_0, r)$ and $n \in \mathbb{N}$. □

Definition A.7 *Let $f : \Omega \to X$ be a vector-valued function. We say that f admits a power series expansion around a point $z_0 \in \Omega$ if there exist a X-valued sequence (a_n) and $r > 0$ such that $B(z_0, r) \subset \Omega$ and*

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n \quad \text{in } B(z_0, r).$$

Theorem A.8 *Let $f : \Omega \to X$ be a continuous function; then f is holomorphic if and only if f has a power series expansion around every point of Ω.*

Proof. Assume that f is holomorphic in Ω. Then, if $z_0 \in \Omega$ and $B(z_0, r) \subset \Omega$, the Cauchy integral formula (A.1) holds for every $z \in B(z_0, r)$.

Fix $z \in B(z_0, r)$ and observe that the series

$$\sum_{n=0}^{+\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}} = \frac{1}{\xi - z}$$

converges uniformly for ξ in $\partial B(z_0, r)$, since $\big|(z-z_0)/(\xi-z_0)\big| = r^{-1}|z-z_0|$. Consequently, by (A.1) and Proposition A.2(e), we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial B(z_0,r)} f(\xi) \sum_{n=0}^{+\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}}\, d\xi \\ &= \sum_{n=0}^{+\infty} \Big[\frac{1}{2\pi i}\int_{\partial B(z_0,r)} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi\Big] (z - z_0)^n, \end{aligned}$$

the series being convergent in X.

Conversely, suppose that

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n, \quad z \in B(z_0, r),$$

where (a_n) is a sequence with values in X. Then f is continuous, and for each $x' \in X'$,

$$\langle f(z), x'\rangle = \sum_{n=0}^{+\infty} \langle a_n, x'\rangle (z - z_0)^n, \quad z \in B(z_0, r).$$

This implies that the complex-valued function $z \mapsto \langle f(z), x' \rangle$ is holomorphic in $B(z_0, r)$ for all $x' \in X'$ and hence f is holomorphic by Theorem A.6. □

Now we extend some classical theorems of complex analysis to the case of vector-valued holomorphic functions.

Theorem A.9 (Cauchy) *Let $f : \Omega \to X$ be holomorphic in Ω and let D be a regular domain contained in Ω. Then*

$$\int_{\partial D} f(z)dz = 0.$$

Proof. For each $x' \in X'$ the complex-valued function $z \mapsto \langle f(z), x' \rangle$ is holomorphic in Ω and hence

$$0 = \int_{\partial D} \langle f(z), x' \rangle dz \left\langle \int_{\partial D} f(z)dz, x' \right\rangle.$$

□

Remark A.10 **[improper complex integrals]** As in the case of vector-valued functions defined on a real interval, it is possible to define *improper complex integrals* in an obvious way. Let $f : \Omega \to X$ be holomorphic, with $\Omega \subset \mathbb{C}$ possibly unbounded. If $I = (a, b)$ is a (possibly unbounded) interval and $\gamma : I \to \mathbb{C}$ is a piecewise C^1 curve in Ω, then we set

$$\int_\gamma f(z)dz := \lim_{s \to a^+,\, t \to b^-} \int_s^t f(\gamma(\tau))\gamma'(\tau)d\tau,$$

provided that the limit exists in X.

Theorem A.11 (Laurent expansion) *Let $f : D := \{z \in \mathbb{C} : r < |z - z_0| < R\} \to X$ be holomorphic. Then, for every $z \in D$*

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_{\partial B(z_0, \varrho)} \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad n \in \mathbb{Z},$$

and $r < \varrho < R$.

Proof. Since for each $x' \in X'$ the function $z \mapsto \langle f(z), x' \rangle$ is holomorphic in D the usual Laurent expansion holds, that is

$$\langle f(z), x' \rangle = \sum_{n=-\infty}^{+\infty} a_n(x')(z - z_0)^n$$

where the coefficients $a_n(x')$ are given by

$$a_n(x') = \frac{1}{2\pi i} \int_{\partial B(z_0, \varrho)} \frac{\langle f(z), x' \rangle}{(z - z_0)^{n+1}} dz, \quad n \in \mathbb{Z}.$$

By Proposition A.2(d), it follows that

$$a_n(x') = \langle a_n, x' \rangle, \quad n \in \mathbb{Z},$$

where the a_n are those indicated in the statement. □

Exercises

A.1 Given a function $u : [a,b] \times [0,1] \to \mathbb{R}$, set $U(t)(x) = u(t,x)$. Show that $U \in C([a,b]; C([0,1]))$ if and only if u is continuous, and that $U \in C^1([a,b]; C([0,1]))$ if and only if u is continuous, differentiable with respect to t and the derivative u_t is continuous.

If $[0,1]$ is replaced by $\mathbb{R}$, show that if $U \in C([a,b]; C_b(\mathbb{R}))$ then u is continuous and bounded, but the converse is not true.

A.3 Let $f : [a,b] \to X$ be a continuous function. Show that f is integrable.

A.4 Prove Proposition A.2.

A.5 Show that if $f : (a,b] \to X$ is continuous and $\|f(t)\| \leq g(t)$ for all $t \in (a,b]$, with $g \in L^1(a,b)$, then the improper integral of f on $[a,b]$ is well defined.

A.6 Let I_1 and I_2 be, respectively, an open set in $\mathbb{R}$ (or in $\mathbb{C}$) and a real interval. Moreover, let $g : I_1 \times I_2 \to X$ be a continuous function and set

$$G(\lambda) = \int_{I_2} g(\lambda, t)dt, \qquad \lambda \in I_1.$$

(a) Show that if the inequality $\|g(\lambda,t)\| \leq \varphi(t)$ holds for every $(\lambda,t) \in I_1 \times I_2$ and some function $\varphi \in L^1(I_2)$, then G is continuous in I_1.

(b) Show that if g is differentiable with respect to λ, g_λ is continuous and $\|g_\lambda(\lambda,t)\| \leq \psi(t)$ for every $(t,\lambda) \in I_1 \times I_2$ and some function $\psi \in L^1(I_2)$, then G is differentiable in I_1 and

$$G'(\lambda) = \int_{I_2} g_\lambda(\lambda,t)dt, \ \ \lambda \in I_1.$$

Appendix B

Basic Spectral Theory

In this appendix we collect a few basic results on elementary spectral theory. To begin with, we introduce the notions of resolvent and spectrum of a linear operator.

Definition B.1 *Let $A : D(A) \subset X \to X$ be a linear operator. The* resolvent set $\rho(A)$ *and the* spectrum $\sigma(A)$ *of A are defined by*

$$\rho(A) = \{\lambda \in \mathbb{C} : \exists\, (\lambda I - A)^{-1} \in \mathcal{L}(X)\}, \quad \sigma(A) = \mathbb{C} \backslash \rho(A). \tag{B.1}$$

If $\lambda \in \rho(A)$, the resolvent operator *(or briefly* resolvent*) $R(\lambda, A)$ is defined by*

$$R(\lambda, A) = (\lambda I - A)^{-1}. \tag{B.2}$$

The complex numbers $\lambda \in \sigma(A)$ such that $\lambda I - A$ is not injective are the eigenvalues *of A, and the elements $x \in D(A)$ such that $x \neq 0$, $Ax = \lambda x$ are the* eigenvectors *(or* eigenfunctions*, when X is a function space) of A relative to the eigenvalue λ. The set $\sigma_p(A)$ whose elements are the eigenvalues of A is the* point spectrum *of A.*

It is easily seen (see Exercise B.1 below) that if $\rho(A) \neq \varnothing$ then A is closed.

Let us recall some simple properties of resolvent and spectrum. First of all, it is clear that if $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ are linear operators such that $R(\lambda_0, A) = R(\lambda_0, B)$ for some $\lambda_0 \in \mathbb{C}$, then $D(A) = D(B)$ and $A = B$. Indeed,

$$D(A) = \text{Range}\, R(\lambda_0, A) = \text{Range}\, R(\lambda_0, B) = D(B),$$

and for every $x \in D(A) = D(B)$, setting $y = \lambda_0 x - Ax$, one has $x = R(\lambda_0, A)y = R(\lambda_0, B)y$. Applying $\lambda_0 I - B$, we get $\lambda_0 x - Bx = y$, so that $\lambda_0 x - Ax = \lambda_0 x - Bx$ and therefore $Ax = Bx$.

The following formula, called the *resolvent identity*, can be easily verified:

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A), \quad \lambda,\ \mu \in \rho(A). \tag{B.3}$$

In fact, write

$$R(\lambda, A) = [\mu R(\mu, A) - AR(\mu, A)] R(\lambda, A),$$
$$R(\mu, A) = [\lambda R(\lambda, A) - AR(\lambda, A)] R(\mu, A),$$

and subtract the above equalities; taking into account that $R(\lambda, A)$ and $R(\mu, A)$ commute, we get (B.3).

The resolvent identity characterizes the resolvent operators, as specified in the following proposition.

Proposition B.2 *Let $\Omega \subset \mathbb{C}$ be an open set, and let $\{F(\lambda) : \lambda \in \Omega\} \subset \mathcal{L}(X)$ be linear operators verifying the resolvent identity*

$$F(\lambda) - F(\mu) = (\mu - \lambda)F(\lambda)F(\mu), \quad \lambda, \mu \in \Omega.$$

If for some $\lambda_0 \in \Omega$, the operator $F(\lambda_0)$ is invertible, then there is a linear operator $A : D(A) \subset X \to X$ such that $\rho(A)$ contains Ω, and $R(\lambda, A) = F(\lambda)$ for all $\lambda \in \Omega$.

Proof. Fix $\lambda_0 \in \Omega$, and set

$$D(A) = \text{Range } F(\lambda_0), \;\; Ax = \lambda_0 x - F(\lambda_0)^{-1}x, \quad x \in D(A).$$

For $\lambda \in \Omega$ and $y \in X$ the resolvent equation $\lambda x - Ax = y$ is equivalent to $(\lambda - \lambda_0)x + F(\lambda_0)^{-1}xy$. Applying $F(\lambda)$ we obtain $(\lambda-\lambda_0)F(\lambda)x + F(\lambda)F(\lambda_0)^{-1}x = F(\lambda)y$, and using the resolvent identity it is easily seen that

$$F(\lambda)F(\lambda_0)^{-1} = F(\lambda_0)^{-1}F(\lambda) = (\lambda_0 - \lambda)F(\lambda) + I.$$

Hence, if x is a solution of the resolvent equation, then $x = F(\lambda)y$. Let us check that $x = F(\lambda)y$ is actually a solution. In fact, $(\lambda-\lambda_0)F(\lambda)y + F(\lambda_0)^{-1}F(\lambda)y = y$, and therefore λ belongs to $\rho(A)$ and the equality $R(\lambda, A) = F(\lambda)$ holds. □

Next, let us show that $\rho(A)$ is an open set.

Proposition B.3 *Let λ_0 be in $\rho(A)$. Then, $|\lambda-\lambda_0| < 1/\|R(\lambda_0, A)\|$ implies that λ belongs to $\rho(A)$ and the equality*

$$R(\lambda, A) = R(\lambda_0, A)(I + (\lambda - \lambda_0)R(\lambda_0, A))^{-1} \tag{B.4}$$

holds. As a consequence, $\rho(A)$ is open and $\sigma(A)$ is closed.

Proof. In fact,

$$(\lambda - A) = (I + (\lambda - \lambda_0)R(\lambda_0, A))(\lambda_0 - A)$$

on $D(A)$. Since $\|(\lambda - \lambda_0)R(\lambda_0, A)\| < 1$, the operator $I + (\lambda - \lambda_0)R(\lambda_0, A)$ is invertible and it has a continuous inverse (see Exercise B.2). Hence,

$$R(\lambda, A) = R(\lambda_0, A)(I + (\lambda - \lambda_0)R(\lambda_0, A))^{-1}.$$

□

Further properties of the resolvent operator are listed in the following proposition.

Proposition B.4 *The function $R(\cdot, A)$ is holomorphic in $\rho(A)$ and the equalities*

$$R(\lambda, A) = \sum_{n=0}^{+\infty} (-1)^n (\lambda - \lambda_0)^n R^{n+1}(\lambda_0, A), \tag{B.5}$$

$$\left.\frac{d^n R(\lambda, A)}{d\lambda^n}\right|_{\lambda=\lambda_0} (-1)^n n! R^{n+1}(\lambda_0, A), \tag{B.6}$$

hold.

Proof. (i) If $|\lambda - \lambda_0| < \frac{1}{\|R(\lambda_0,A)\|}$, from $(B.4)$ we deduce

$$R(\lambda, A) = R(\lambda_0, A)\sum_{n=0}^{+\infty}(-1)^n(\lambda-\lambda_0)^n R(\lambda_0, A)^n \sum_{n=0}^{+\infty}(-1)^n(\lambda-\lambda_0)^n R(\lambda_0, A)^{n+1}$$

and the statement follows. □

Proposition B.3 implies also that the resolvent set is the domain of analyticity of the function $\lambda \mapsto R(\lambda, A)$.

Corollary B.5 *The domain of analyticity of the function $\lambda \mapsto R(\lambda, A)$ is $\rho(A)$, and the estimate*

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \geq \frac{1}{\mathrm{dist}(\lambda, \sigma(A))} \tag{B.7}$$

holds.

Proof. It suffices to prove (B.7), because it shows that $R(\cdot, A)$ is unbounded approaching $\sigma(A)$. From Proposition B.3 for every $\lambda \in \rho(A)$ we get that if $|z-\lambda| < 1/\|R(\lambda, A)\|_{\mathcal{L}(X)}$ then $z \in \rho(A)$, and $\mathrm{dist}\,(\lambda, \sigma(A)) \geq 1/\|R(\lambda, A)\|_{\mathcal{L}(X)}$, that implies (B.7). □

Let us recall also some spectral properties of bounded operators.

Proposition B.6 *If $T \in \mathcal{L}(X)$ the power series*

$$F(z) = \sum_{k=0}^{+\infty} z^k T^k, \quad z \in \mathbb{C}, \tag{B.8}$$

(called the Neumann series of $(I - zT)^{-1}$*) is convergent in the disk $B(0, 1/r(T))$, where*

$$r(T) = \limsup_{n\to+\infty} \sqrt[n]{\|T^n\|}.$$

Moreover, $|z| < 1/r(T)$ implies $F(z) = (I - zT)^{-1}$, and $|z| < 1/\|T\|$ implies

$$\|(I - zT)^{-1}\| \leq \frac{1}{1 - |z|\,\|T\|}. \tag{B.9}$$

Proof. To prove the convergence of (B.8) in the disk $B(0, 1/r(T))$ it suffices to use Exercise B.2, whereas (B.9) follows from the inequality

$$\|F(z)\| \leq \sum_{k=0}^{+\infty} |z|^k \|T\|^k \frac{1}{1 - |z|\,\|T\|}.$$

□

Proposition B.7 *Let $T \in \mathcal{L}(X)$. Then the following properties hold.*

(i) $\sigma(T)$ is contained in the disk $\overline{B}(0, r(T))$ and if $|\lambda| > r(T)$ then

$$R(\lambda, T) = \sum_{k=0}^{+\infty} T^k \lambda^{-k-1}. \tag{B.10}$$

For this reason, $r(T)$ is called the spectral radius of T. *Moreover, $|\lambda| > \|T\|$ implies*

$$\|R(\lambda, T)\| \leq \frac{1}{|\lambda| - \|T\|}. \tag{B.11}$$

(ii) $\sigma(T)$ *is non-empty.*

Proof. (i) follows from Proposition B.6, noticing that, for $\lambda \neq 0$, $\lambda - T = \lambda(I - (1/\lambda)T)$.
(ii) Suppose by contradiction that $\sigma(T) = \varnothing$. Then, $R(\cdot, T)$ is an entire function, and then for every $x \in X$, $x' \in X'$ the function $\langle R(\cdot, T)x, x' \rangle$ is entire (i.e., holomorphic on the whole $\mathbb{C}$), it tends to 0 at infinity, and then it is constant by the Liouville theorem. As a consequence, $R(\lambda, T) = 0$ for all $\lambda \in \mathbb{C}$, which is a contradiction. □

Exercises

B.1 Show that if $A : D(A) \subset X \to X$ has non-empty resolvent set, then A is closed.

B.2 Show that if $A \in \mathcal{L}(X)$ and $\|A\| < 1$ then $I + A$ is invertible, and

$$(I + A)^{-1} = \sum_{k=0}^{+\infty} (-1)^k A^k.$$

B.3 Show that for every $\alpha \in \mathbb{C}$ the equalities $\sigma(\alpha A) = \alpha\sigma(A)$, $\sigma(\alpha I - A) = \alpha - \sigma(A)$ hold. Prove also that if $0 \in \rho(A)$ then $\sigma(A^{-1}) \setminus \{0\} = 1/\sigma(A)$, and that $\rho(A+\alpha I) = \rho(A) + \alpha$, $R(\lambda, A + \alpha I) = R(\lambda - \alpha, A)$ for all $\lambda \in \rho(A) + \alpha$.

B.4 Let $\varphi : [a, b] \to \mathbb{C}$ be a continuous function, and consider the multiplication operator $A : C([a, b]; \mathbb{C}) \to C([a, b]; \mathbb{C})$, $(Af)(x) = f(x)\varphi(x)$. Compute the spectrum of A. In which cases are there eigenvalues in $\sigma(A)$?

B.5 Let $C_b(\mathbb{R})$ be the space of bounded and continuous functions on $\mathbb{R}$, endowed with the supremum norm, and let A be the operator defined by

$$D(A) = C^1_b(\mathbb{R}) = \{f \in C_b(\mathbb{R}) : \ \exists f' \in C_b(\mathbb{R})\} \to C_b(\mathbb{R}), \ \ Af = f'.$$

Compute $\sigma(A)$ and $R(\lambda, A)$, for $\lambda \in \rho(A)$. Which are the eigenvalues of A?

B.6 Let $P \in \mathcal{L}(X)$ be a projection, i.e., $P^2 = P$. Find $\sigma(A)$, find the eigenvalues and compute $R(\lambda, P)$ for $\lambda \in \rho(P)$.

B.8 Let $X = C([0, 1])$, and consider the operators A, B, C on X defined by

$$\begin{aligned} D(A) &= C^1([0,1]) : \ Au = u', \\ D(B) &= \{u \in C^1([0,1]) : \ u(0) = 0\}, \ Bu = u', \\ D(C) &= \{u \in C^1([0,1]);\ u(0) = u(1)\}, \ Cu = u'. \end{aligned}$$

Show that

$$\begin{aligned} &\rho(A) = \varnothing, \ \ \sigma(A) = \mathbb{C}, \\ &\rho(B) = \mathbb{C}, \ \ \sigma(B) = \varnothing, \ \ (R(\lambda, B)f)(\xi) = -\int_0^\xi e^{\lambda(\xi-\eta)} f(\eta) d\eta, \ \ 0 \le \xi \le 1, \\ &\rho(C) = \mathbb{C} \setminus \{2k\pi i : \ k \in \mathbb{Z}\}, \ \ \sigma(C)\{2k\pi i : \ k \in \mathbb{Z}\}. \end{aligned}$$

Show that $2k\pi i$ is an eigenvalue of C, with eigenfunction $\xi \mapsto ce^{2k\pi i\xi}$, and that for $\lambda \in \rho(C)$,

$$(R(\lambda, C)f)(\xi) = \frac{e^{\lambda\xi}}{e^\lambda - 1} \int_0^1 e^{\lambda(1-\eta)} f(\eta) d\eta - \int_0^\xi e^{\lambda(\xi-\eta)} f(\eta) d\eta.$$

B.9 Let $A : D(A) \subset X \to X$ be a linear operator and let $\lambda \in \mathbb{C}$. Prove that, if there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $\|u_n\| = 1$ for any $n \in \mathbb{N}$ and $\lambda u_n - Au_n$ tends to 0 as n tends to $+\infty$, then $\lambda \in \sigma(A)$.

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