Linear and nonlinear diffusion problems

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Introduction

These lectures deal with the functional analytical approach to linear and nonlinear parabolic problems.

The simplest significant example is the heat equation, either linear

$$\begin{cases} u_t(t,x) = u_{xx}(t,x) + f(t,x), \;\; 0 < t \le T, \; 0 \le x \le 1, \\ \\ u(0,x) = u_0(x), \;\; 0 \le x \le 1, \\ \\ u(t,0) = u(t,1) = 0, \;\; 0 \le t \le T, \end{cases} \tag{1}$$

or nonlinear,

$$\begin{cases} u_t(t,x) = u_{xx}(t,x) + f(u(t,x)), \;\; t > 0, \; 0 \le x \le 1, \\ \\ u(0,x) = u_0(x), \;\; 0 \le x \le 1, \\ \\ u(t,0) = u(t,1) = 0, \;\; t \ge 0. \end{cases} \tag{2}$$

In both cases, u is the unknown, and f, u_0 are given. We will write problems (1), (2) as evolution equations in suitable Banach spaces. To be definite, let us consider problem (1), and let us set

$$u(t,\cdot) = U(t), \;\; f(t,\cdot) = F(t), \;\; 0 \le t \le T,$$

so that for every $t \in [0,T]$, $U(t)$ and $F(t)$ are functions, belonging to a suitable Banach space X. The choice of X depends on the type of the results expected, or else on the regularity properties of the data. For instance, if f and u_0 are continuous functions the most natural choice is $X = C([0,1])$; if $f \in L^p((0,T)\times(0,1))$ and $u_0 \in L^p(0,1)$, $p \ge 1$, the natural choice is $X = L^p(0,1)$, and so on.

Next, we write (1) as an evolution equation in X,

$$\begin{cases} U'(t) = Au(t) + F(t), \;\; 0 < t \le T, \\ \\ U(0) = u_0, \end{cases} \tag{3}$$

where A is the realization of the second order derivative with Dirichlet boundary condition in X (that is, we consider functions that vanish at $x = 0$ and at $x = 1$). For instance, if $X = C([0,1])$ then

$$D(A) = \{\varphi \in C^2([0,1]) : \varphi(0) = \varphi(1) = 0\}, \;\; (A\varphi)(x) = \varphi''(x).$$

Problem (3) is a Cauchy problem for a linear differential equation in the space $X = C([0,1])$. However, the theory of ordinary differential equations is not easily extendable to this type of problems, because the linear operator A is defined on a proper subspace of X, and it is not continuous.

What we use is an important spectral property of A, i.e. the resolvent set of A contains a sector $S = \{\lambda \in \mathbb{C} : \lambda \neq 0,\ |\arg \lambda| < \theta\}$, with $\theta > \pi/2$ (precisely, it consists of a sequence of negative eigenvalues), and moreover

$$\|(\lambda I - A)^{-1}\|_{L(X)} \leq \frac{M}{|\lambda|}, \quad \lambda \in S. \tag{4}$$

This property will allow us to define the solution of the homogeneous problem (i.e., when $F \equiv 0$), that will be called $e^{tA}u_0$. We shall see that for each $t \geq 0$ the linear operator $u_0 \mapsto e^{tA}u_0$ is bounded. The family of operators $\{e^{tA} :\ t \geq 0\}$ is said to be an *analytic semigroup*: semigroup, because it satisfies

$$e^{(t+s)A} = e^{tA}e^{sA},\ \forall t, s \geq 0, \quad e^{0A} = I,$$

analytic, because the function $(0, +\infty) \mapsto \mathcal{L}(X)$, $t \mapsto e^{tA}$ will be shown to be analytic.

Then we shall see that the solution of (3) is given by the variation of constants formula

$$U(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}F(s)ds, \quad 0 \leq t \leq T,$$

that will let us study several properties of the solution to (3) and of u, recalling that $U(t) = u(t, \cdot)$.

We shall be able to study the asymptotic behavior of U as $t \to +\infty$, in the case that F is defined in $[0, +\infty)$. As in the case of ordinary differential equations, the asymptotic behavior depends heavily on the spectral properties of A.

Also the nonlinear problem (2) will be written as an abstract equation,

$$\begin{cases} U'(t) = AU(t) + F(U(t)), \quad t \geq 0, \\ \\ U(0) = u_0, \end{cases} \tag{5}$$

where $F : X \mapsto X$ is the composition operator, or Nemitzky operator, $F(v) = f(v(\cdot))$. After stating local existence and uniqueness results, we shall see some criteria for existence in the large. As in the case of ordinary differential equations, in general the solution is defined only in a small time interval $[0, \delta]$. The problem of existence in the large is of particular interest in equations coming from mathematical models in physics, biology, chemistry, etc., where existence in the large is expected. Some sufficient conditions for existence in the large will be given.

Then we shall study the stability of the (possible) stationary solutions, that is all the $\overline{u} \in D(A)$ such that $A\overline{u} + F(\overline{u}) = 0$. We shall see that under suitable assumptions the Principle of Linearizad Stability holds. Roughly speaking, $\overline{u}$ has the same stability properties of the solution of the linearized problem

$$V'(t) = AV(t) + F'(\overline{u})V(t)$$

If possible we shall construct the stable manifold, consisting of all the initial data such that the solution $U(t)$ exists in the large and tends to $\overline{u}$ as $t \to +\infty$, and the unstable manifold, consisting of all the initial data such that problem (5) has a backward solution going to $\overline{u}$ as $t \to -\infty$.

Chapter 1

Analytic semigroups

1.1 Introduction

Our concern in this chapter is the Cauchy problem in general Banach space X,

$$\begin{cases} u'(t) = Au(t), \ t > 0, \\ \\ u(0) = x, \end{cases} \tag{1.1}$$

where $A : D(A) \to X$ is a linear operator and $x \in X$. A solution of (1.1) is a function $u \in C([0, +\infty); X) \cap C^1((0, +\infty); X)$, verifying (1.1). Of course, the construction and the properties of the solution will depend upon the class of operators that will be considered. The most elementary case is that of a finite-dimensional X and a matrix A, which we assume to be known to the reader. The case of a bounded operator A in general Banach space X can be treated essentially in the same way, and we are going to discuss it briefly in this introduction. We shall present two formulae for the solution, a power series expansion and an integral formula based on a complex contour integral. While the first one cannot be generalized to the case of unbounded A, the contour integral admits a generalization to an integral along an unbounded curve for suitable unbounded operators, those called *sectorial*. This class of operators is discussed in section 1.2. If A is sectorial, then the solution map $x \mapsto u(t)$ of (1.1) is given by an *analytic semigroup*. Analytic semigroups are the main subject of this chapter.

Let $A : X \to X$ be a bounded linear operator. First, we give a solution of (1.1) as the sum of a power series of exponential type.

Proposition 1.1.1 *Let $A \in \mathcal{L}(X)$. Then, the series*

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}, \qquad t \in \mathbb{R}, \tag{1.2}$$

converges in $\mathcal{L}(X)$ uniformly on bounded subsets of $\mathbb{R}$. Setting $u(t) = e^{tA}x$, the Cauchy problem (1.1) admits the restriction of u to $[0, +\infty)$ as its unique solution.

Proof. *Existence.* Using Theorem A.1.2 as in the finite-dimensional case, it is easily checked that solving (1.1) is equivalent to finding a continuous function $u : [0, \infty) \mapsto X$ which solves the integral equation

$$u(t) = x + \int_0^t Au(s)ds, \ \ t \geq 0. \tag{1.3}$$

In order to show that u solves (1.3), let us fix an interval $[0, T]$ and define

$$x_0(t) = x, \ x_{n+1}(t) = x + \int_0^t Ax_n(s)ds, \ n \in \mathbb{N}. \tag{1.4}$$

We have

$$x_n(t) = \sum_{k=0}^{n} \frac{t^k A^k}{k!} x, \quad n \in \mathbb{N}.$$

Since

$$\left\| \sum_{k=0}^{n} \frac{t^k A^k}{k!} \right\| \le \sum_{k=0}^{n} \frac{t^k \|A^k\|}{k!} \le \sum_{k=0}^{n} \frac{T^k \|A\|^k}{k!},$$

the series $\sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$ converges in $\mathcal{L}(X)$, uniformly with respect to t in $[0, T]$. Moreover, the sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ converges to $x(t) = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} x$ uniformly for t in $[0, T]$. Letting $n \to \infty$ in (1.4), we conclude that u is a solution of (1.3).

Uniqueness. If x, y are two solutions of (1.3) in $[0, T]$, we have by Proposition A.1.1(d)

$$\|x(t) - y(t)\| \le \|A\| \int_0^t \|x(s) - y(s)\| ds$$

and from Gronwall's lemma (see exercise 1.1.4.2 below), the equality $x = y$ follows at once. □

As in the finite dimensional setting, we define

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}, \quad t \in \mathbb{R}. \tag{1.5}$$

It is clear that for every *bounded* operator A the above series converges in $\mathcal{L}(X)$ for each $t \in \mathbb{R}$. If A is unbounded, the domain of A^k may get smaller and smaller as k increases, and even for $x \in \cap_{k\in\mathbb{N}} D(A^k)$ it is not obvious that $\sum_{k=0}^{\infty} t^k A^k x/k!$ converges. So, we have to look for another representation of the solution to (1.1) if we want to extend it to the unbounded case. As a matter of fact, it is given in the following proposition.

Proposition 1.1.2 *Let $\gamma \subset \mathbb{C}$ be any circle with centre 0 and radius $r > \|A\|$. Then*

$$e^{tA} = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} R(\lambda, A)\, d\lambda, \qquad t \ge 0. \tag{1.6}$$

Proof. From (1.5) and the series expansion (see (B.11))

$$R(\lambda, A) = \sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}}, \qquad |\lambda| > \|A\|$$

we have

$$\begin{aligned}
\frac{1}{2\pi i} \int_\gamma e^{t\lambda} R(\lambda, A)\, d\lambda &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_\gamma \lambda^n R(\lambda, A)\, d\lambda \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_\gamma \lambda^n \sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}}\, d\lambda \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} A^k \int_\gamma \lambda^{n-k-1}\, d\lambda = e^{tA},
\end{aligned}$$

as the integrals in the last series equal $2\pi i$ if $n = k$, 0 otherwise. □

Let us see how it is possible to generalize to the infinite-dimensional setting the variation of parameters formula, that gives the solution of the non-homogeneous Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t), & 0 \le t \le T, \\ u(0) = x, \end{cases} \tag{1.7}$$

where $A \in \mathcal{L}(X)$, $x \in X$, $f \in C([0, T]; X)$ and $T > 0$.

Proposition 1.1.3 *Problem (1.7) has a unique solution in $[0,T]$, given by*

$$u(t) = e^{tA}x + \int_0^t e^{(t-s)A} f(s)ds. \tag{1.8}$$

Proof. It can be directly checked that u is a solution. Concernibg uniqueness, let u_1, u_2 be two solutions; then, $v = u_1 - u_2$ satisfies $v'(t) = Av(t)$ for $0 \le t \le T$, $v(0) = 0$. By proposition 1.1.1, $v \equiv 0$. □

Exercises 1.1.4 1. Prove that if the operators A and B commute, $AB = BA$, then $e^A e^B = e^{A+B}$, and deduce that in this case $e^{tA}e^{tB} = e^{t(A+B)}$.

2. Prove the following form of Gronwall's lemma:

   Let $u, v : [0, +\infty) \to [0, +\infty)$ be continuous, and assume that

   $$u(t) \le \alpha + \int_0^t u(s)v(s)ds$$

   for some $\alpha \ge 0$. Then, $u(t) \le \alpha \exp\{\int_0^t v(s)ds\}$.

3. Check that the function u defined in (1.8) is a solution of problem (1.7).

1.2 Sectorial operators

In this section we introduce the class of *sectorial operators* which will be proved to be suitable to extend the integral formula (1.6) in order to get a solution of (1.1).

Definition 1.2.1 *A linear operator $A : D(A) \subset X \to X$ is said to be* sectorial *if there are constants $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, $M > 0$ such that*

$$\begin{cases} (i) & \rho(A) \supset S_{\theta,\omega} = \{\lambda \in \mathbb{C} : \lambda \ne \omega, |\arg(\lambda - \omega)| < \theta\}, \\ \\ (ii) & \|R(\lambda, A)\|_{\mathcal{L}(X)} \le \dfrac{M}{|\lambda - \omega|} \quad \forall \lambda \in S_{\theta,\omega}. \end{cases} \tag{1.9}$$

For every $t > 0$, conditions (1.9) allow us to define a bounded linear operator e^{tA} on X, through an integral formula that generalizes (1.6). For $r > 0$, $\eta \in (\pi/2, \theta)$, let $\gamma_{r,\eta}$ be the curve

$$\{\lambda \in \mathbb{C} : |\arg\lambda| = \eta,\ |\lambda| \ge r\} \cup \{\lambda \in \mathbb{C} : |\arg\lambda| \le \eta,\ |\lambda| = r\},$$

oriented counterclockwise.

For each $t > 0$ set

$$e^{tA} = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}+\omega} e^{t\lambda} R(\lambda, A)\, d\lambda, \quad t > 0. \tag{1.10}$$

Lemma 1.2.2 *If $A : D(A) \subset X \to X$ satisfies (1.9), the integral in (1.10) is well defined, and it is independent of r and η.*

Proof. First of all, notice that $\lambda \mapsto e^{t\lambda}R(\lambda, A)$ is a $\mathcal{L}(X)$-valued holomorphic function in the sector $S_{\theta,\omega}$. Moreover, the estimate

$$\|e^{t\lambda}R(\lambda, A)\|_{\mathcal{L}(X)} \le \exp(t|\lambda|\cos\theta)\frac{M}{r}, \tag{1.11}$$

with $\theta > \frac{\pi}{2}$, holds for each λ in the two half-lines, and this easily implies that the improper integral is convergent. Take now different $r' > 0, \eta' \in (\pi/2, \theta)$ and consider the integral on $\gamma_{r',\eta'} + \omega$. Let Ω be the region lying between the curves $\gamma_{r,\eta} + \omega$ and $\gamma_{r',\eta'} + \omega$ and for every $n \in \mathbb{N}$ set $D_n = D \cap \{|z| \le n\}$. By Cauchy integral theorem A.1.7 we have

$$\int_{\partial D_n} e^{t\lambda} R(\lambda, A)\, d\lambda = 0.$$

By estimate (1.11) the integrals on the two circle arcs and on the halflines $\{|\lambda \ge n\} \cap \gamma_{r,\eta}$, $\{|\lambda \ge n\} \cap \gamma_{r',\eta'}$ tend to 0 as n tends to $+\infty$, so that

$$\int_{\gamma_{r,\eta}+\omega} e^{t\lambda} R(\lambda, A)\, d\lambda = \int_{\gamma_{r',\eta'}+\omega} e^{t\lambda} R(\lambda, A)\, d\lambda$$

and the proof is complete. □

Notice that using the obvious parametrization of $\gamma_{r,\eta}$ we get

$$\begin{aligned} e^{tA} \;=\; \frac{e^{\omega t}}{2\pi i}\Big(& -\int_r^{+\infty} e^{(\xi\cos\eta - i\xi\sin\eta)t} R(\omega + \xi e^{-i\eta}, A) e^{-i\eta} d\xi \\ & + \int_{-\eta}^{\eta} e^{(r\cos\alpha + ir\sin\alpha)t} R(\omega + re^{i\alpha}, A) ire^{i\alpha} d\alpha \\ & + \int_r^{+\infty} e^{(\xi\cos\eta + i\xi\sin\eta)t} R(\omega + \xi e^{i\eta}, A) e^{i\eta} d\xi \Big). \end{aligned}$$

for every $t > 0$ and for every $r > 0$, $\eta \in (\pi/2, \theta)$.

Let us also set

$$e^{0A}x = x, \quad \forall x \in X. \tag{1.12}$$

In the following theorem the main properties of e^{tA} for $t > 0$ are summarized.

Theorem 1.2.3 *Let A be a sectorial operator and let e^{tA} be given by (1.10). Then, the following statements hold.*

(i) $e^{tA}x \in D(A^k)$ *for all* $t > 0$, $x \in X$, $k \in \mathbb{N}$. *If* $x \in D(A^k)$, *then*

$$A^k e^{tA} x = e^{tA} A^k x, \quad \forall t \ge 0.$$

(ii) $e^{tA}e^{sA} = e^{(t+s)A}, \quad \forall\, t,\ s \ge 0.$

(iii) There are constants M_0, M_1, M_2, $\ldots$, *such that*

$$\begin{cases} (a) \quad \|e^{tA}\|_{\mathcal{L}(X)} \le M_0 e^{\omega t}, \quad t > 0, \\ \\ (b) \quad \|t^k (A - \omega I)^k e^{tA}\|_{\mathcal{L}(X)} \le M_k e^{\omega t}, \quad t > 0, \end{cases} \tag{1.13}$$

where ω is the constant in (1.9). In particular, from (1.13)(b) it follows that for every $\varepsilon > 0$ and $k \in \mathbb{N}$ there is $C_{k,\varepsilon} > 0$ such that

$$\|t^k A^k e^{tA}\|_{\mathcal{L}(X)} \le C_{k,\varepsilon} e^{(\omega+\varepsilon)t}, \quad t > 0. \tag{1.14}$$

(iv) The function $t \mapsto e^{tA}$ *belongs to* $C^\infty((0, +\infty); \mathcal{L}(X))$, *and the equality*

$$\frac{d^k}{dt^k} e^{tA} = A^k e^{tA}, \quad t > 0, \tag{1.15}$$

holds for every $k \in \mathbb{N}$. Moreover, it has an analytic continuation e^{zA} in the sector $S_{0,\theta-\pi/2}$, and, for $z = \rho e^{i\alpha} \in S_{0,\theta-\pi/2}$, $\theta' \in (\pi/2, \theta - \alpha)$, the equality

$$e^{zA} = \frac{1}{2\pi i}\int_{\gamma_{r,\theta'}+\omega} e^{\lambda z} R(\lambda, A) d\lambda$$

holds.

Proof. Possibly replacing A by $A - \omega I$, we may suppose $\omega = 0$.

Proof of (i). First, let $k = 1$. Using lemma B.1.2 with $f(t) = e^{\lambda t}R(\lambda, A)$ and the resolvent identity $AR(\lambda, A) = \lambda R(\lambda, A) - I$, which holds for every $\lambda \in \rho(A)$, we deduce that $e^{tA}x$ belongs to $D(A)$ for every $x \in X$. Moreover, if $x \in D(A)$, the equality $Ae^{tA}x = e^{tA}Ax$ follows from (1.10), since $AR(\lambda, A)x = R(\lambda, A)Ax$. Note that for each $x \in X$ we have

$$Ae^{tA} = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} \lambda e^{t\lambda} R(\lambda, A) d\lambda,$$

because $\int_{\gamma_{r,\eta}} e^{t\lambda} d\lambda = 0$.

Iterating this argument, we obtain that $e^{tA}x$ belongs to $D(A^k)$ for every $k \in \mathbb{N}$; moreover

$$A^k e^{tA} = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} \lambda^k e^{t\lambda} R(\lambda, A) d\lambda,$$

and (i) can be easily proved by recurrence.

Proof of (ii). From

$$e^{tA}e^{sA} = \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{r,\eta}} e^{\lambda t}R(\lambda, A)d\lambda \int_{\gamma_{2r,\eta'}} e^{\mu t}R(\mu, A)d\mu,$$

with $\eta' \in (\frac{\pi}{2}, \eta)$, using the resolvent identity it follows that

$$\begin{aligned} e^{tA}e^{sA} &= \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{r,\eta}}\int_{\gamma_{2r,\eta'}} e^{\lambda t+\mu s}\frac{R(\lambda, A) - R(\mu, A)}{\mu - \lambda} d\lambda d\mu \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{r,\eta}} e^{\lambda t}R(\lambda, A)d\lambda \int_{\gamma_{2r,\eta'}} e^{\mu s}\frac{d\mu}{\mu - \lambda} \\ &\quad - \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{2r,\eta'}} e^{\mu s}R(\mu, A)d\mu \int_{\gamma_{r,\eta}} e^{\lambda t}\frac{d\lambda}{\mu - \lambda} = e^{(t+s)A}, \end{aligned}$$

where we have used the equalities

$$\int_{\gamma_{2r,\eta'}} e^{\mu s}\frac{d\mu}{\mu - \lambda} = 2\pi i e^{s\lambda},\ \lambda \in \gamma_{r,\eta}, \qquad \int_{\gamma_{r,\eta}} e^{\lambda t}\frac{d\lambda}{\mu - \lambda} = 0,\ \mu \in \gamma_{2r,\eta'}$$

that can be easily checked using the same domains D_n as in the proof of Lemma 1.2.2.

Proof of (iii). As a preliminary remark, let us point out that if we estimate $\|e^{tA}\|$ integrating $\|e^{\lambda t}R(\lambda, A)\|$ we get a singularity near $t = 0$, because the integrand behaves like $M/|\lambda|$ for $|\lambda|$ small. We have to be more careful. Setting $\lambda t = \xi$, we rewrite (1.10) as

$$\begin{aligned} e^{tA} &= \frac{1}{2\pi i}\int_{\gamma_{rt,\eta}} e^{\xi} R\left(\frac{\xi}{t}, A\right)\frac{d\xi}{t} = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} e^{\xi} R\left(\frac{\xi}{t}, A\right)\frac{d\xi}{t} \\ &= \frac{1}{2\pi i}\bigg(\int_r^{+\infty} e^{\xi e^{i\eta}} R\left(\frac{\xi e^{i\eta}}{t}, A\right)\frac{e^{i\eta}}{t} d\xi - \int_r^{+\infty} e^{\xi e^{-i\eta}} R\left(\frac{\xi e^{-i\eta}}{t}, A\right)\frac{e^{-i\eta}}{t} d\xi \\ &\quad + \int_{-\eta}^{\eta} e^{re^{i\theta}} R\left(\frac{re^{i\theta}}{t}, A\right) ire^{i\theta}\frac{d\theta}{t}\bigg). \end{aligned}$$

It follows that

$$\|e^{tA}\| \le \frac{1}{\pi}\left\{\int_r^{+\infty} M e^{\xi\cos\eta}\frac{d\xi}{\xi} + \int_{-\eta}^{\eta} M e^{r\cos\alpha} d\alpha\right\}.$$

In an analogous way one can prove that there exists $N > 0$ such that $\|Ae^{tA}\| \le N/t$, for all $t > 0$.

From the equality $Ae^{tA}x = e^{tA}Ax$, which is true for each $x \in D(A)$, it follows that $A^k e^{tA} = (Ae^{\frac{t}{k}A})^k$ for all $k \in \mathbb{N}$, so that

$$\|A^k e^{tA}\|_{\mathcal{L}(X)} \le (Nkt^{-1})^k := M_k t^{-k}.$$

Proof of (iv). From the definition it is clear that $t \mapsto e^{tA}$ belongs to $C^\infty(0, +\infty, \mathcal{L}(X))$; moreover, using the result of exercise A.5 we get

$$\begin{aligned}\frac{d}{dt}e^{tA} &= \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} \lambda e^{\lambda t} R(\lambda, A) d\lambda \\ &= \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} e^{\lambda t} d\lambda + \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} Ae^{\lambda t} R(\lambda, A) d\lambda \\ &= Ae^{tA}, \qquad t > 0\end{aligned}$$

because the first integral vanishes by the analyticity of the function $\lambda \mapsto e^{\lambda t}$. The equality

$$\frac{d^k}{dt^k}e^{tA} = A^k e^{tA}, \qquad t > 0$$

can be proved by the same argument, or by recurrence. Let now $0 < \alpha < \theta - \pi/2$ be given, and set $\eta = \theta - \alpha$. The function

$$z \mapsto e^{zA} = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}} e^{z\lambda} R(\lambda, A) d\lambda$$

is well defined and holomorphic in the sector

$$S_\varepsilon = \{z \in \mathbb{C} : z \ne 0,\ |\arg z| < \theta - \pi/2 - \alpha\}.$$

Since the union of the sectors S_α, for $0 < \alpha < \theta - \pi/2$, is $S_{0,\theta-\frac{\pi}{2}}$, (iv) is proved. □

Statement (ii) in theorem 1.2.3 tells us that the family of operators e^{tA} satisfies the *semigroup law*, an algebraic property which is coherent with the exponential notation. Let us give the definitions of analytic and strongly continuous semigroups.

Definition 1.2.4 *Let A be a sectorial operator. The function from $[0, +\infty)$ to $\mathcal{L}(X)$, $t \mapsto e^{tA}$ (see (1.10)–(1.12)) is called the* analytic semigroup generated by A *(in X).*

Definition 1.2.5 *Let $(T(t))_{t\ge 0}$ be a family of bounded operators on X. If $T(0) = I$, $T(t+s) = T(t)T(s)$ for all $t, s \ge 0$ and the map $t \mapsto T(t)x$ is continuous from $[0, +\infty) \to X$ then $(T(t))_{t\ge 0}$ is called a* strongly continuous semigroup.

Given $x \in X$, the function $t \mapsto e^{tA}x$ is analytic for $t > 0$. Let us consider the behavior of $e^{tA}x$ for t close to 0.

Proposition 1.2.6 *The following statements hold.*

(i) If $x \in \overline{D(A)}$, then $\lim_{t\to 0^+} e^{tA}x = x$. Conversely, if $y = \lim_{t\to 0^+} e^{tA}x$ exists, then $x \in \overline{D(A)}$ and $y = x$.

(ii) For every $x \in X$ and $t \geq 0$, the integral $\int_0^t e^{sA}xds$ belongs to $D(A)$, and

$$A\int_0^t e^{sA}xds = e^{tA}x - x. \tag{1.16}$$

If, in addition, the function $s \mapsto Ae^{sA}x$ is integrable in $(0,\varepsilon)$ for some $\varepsilon > 0$, then

$$e^{tA}x - x = \int_0^t Ae^{sA}xds, \quad t \geq 0.$$

(iii) If $x \in D(A)$ and $Ax \in \overline{D(A)}$, then $\lim_{t\to 0^+}(e^{tA}x - x)/t = Ax$. Conversely, if $z := \lim_{t\to 0}(e^{tA}x - x)/t$ exists, then $x \in D(A)$ and $Ax = z \in \overline{D(A)}$.

(iv) If $x \in D(A)$ and $Ax \in \overline{D(A)}$, then $\lim_{t\to 0^+} Ae^{tA}x = Ax$.

Proof. *Proof of (i).* Notice that we cannot let $t \to 0$ in the definition (1.10) of $e^{tA}x$, because the estimate $\|R(\lambda, A)\| \leq M/|\lambda-\omega|$ is not enough to guarantee that the improper integral is well defined for $t = 0$.

But if $x \in D(A)$ things are easier: fix ξ, r such that $\omega < \xi \in \rho(A)$ and $0 < r < \xi - \omega$. For all $x \in D(A)$, set $y = \xi x - Ax$, so that $x = R(\xi, A)y$. We have

$$\begin{aligned} e^{tA}x &= e^{tA}R(\xi,A)y = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}+\omega} e^{t\lambda}R(\lambda,A)R(\xi,A)y\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\gamma_{r,\eta}+\omega} e^{t\lambda}\frac{R(\lambda,A)}{\xi-\lambda}y\,d\lambda - \frac{1}{2\pi i}\int_{\gamma_{r,\eta}+\omega} e^{t\lambda}\frac{R(\xi,A)}{\xi-\lambda}y\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\gamma_{r,\eta}+\omega} e^{t\lambda}\frac{R(\lambda,A)}{\xi-\lambda}y\,d\lambda, \end{aligned}$$

because the other integral vanishes (why?). Here we may let $t \to 0$ because $\|R(\lambda, A)y/(\xi-\lambda)\| \leq C|\lambda|^{-2}$ for $\lambda \in \gamma_{r,\eta} + \omega$. We get

$$\lim_{t\to 0^+} e^{tA}x = \frac{1}{2\pi i}\int_{\gamma_{r,\eta}+\omega} \frac{R(\lambda,A)}{\xi-\lambda}y\,d\lambda R(\xi,A)y = x.$$

Since $D(A)$ is dense in $\overline{D(A)}$ and $\|e^{tA}\|$ is bounded by a constant independent of t for $0 < t < 1$, then $\lim_{t\to 0^+} e^{tA}x = x$ for all $x \in \overline{D(A)}$, see lemma B.1.1.

Conversely, if $y = \lim_{t\to 0^+} e^{tA}x$, then $y \in \overline{D(A)}$ because $e^{tA}x \in D(A)$ for $t > 0$, and we have $R(\xi, A)y = \lim_{t\to 0^+} R(\xi, A)e^{tA}x = \lim_{t\to 0^+} e^{tA}R(\xi, A)x = R(\xi, A)x$ as $R(\xi, A)x \in D(A)$. Therefore, $y = x$.

Proof of (ii) To prove the first statement, take $\xi \in \rho(A)$ and $x \in X$. For every $\varepsilon \in (0, t)$ we have

$$\begin{aligned} \int_\varepsilon^t e^{sA}xds &= \int_\varepsilon^t (\xi - A)R(\xi,A)e^{sA}xds \\ &= \xi\int_\varepsilon^t R(\xi,A)e^{sA}xds - \int_\varepsilon^t \frac{d}{ds}(R(\xi,A)e^{sA}x)ds \\ &= \xi R(\xi,A)\int_\varepsilon^t e^{sA}xds - e^{tA}R(\xi,A)x + e^{\varepsilon A}R(\xi,A)x. \end{aligned}$$

Since $R(\xi, A)x$ belongs to $D(A)$, letting $\varepsilon \to 0$ we get

$$\int_0^t e^{sA}xds = \xi R(\xi,A)\int_0^t e^{sA}xds - R(\xi,A)(e^{tA}x - x). \tag{1.17}$$

Then, $\int_0^t e^{sA}x ds \in D(A)$, and

$$(\xi I - A)\int_0^t e^{sA}x ds = \xi \int_0^t e^{sA}x ds - (e^{tA}x - x),$$

whence the first statement in (ii) follows. If in addition $s \mapsto \|Ae^{sA}x\|$ belongs to $L^1(0,T)$, we may commute A with the integral and the second statement in (ii) is proved.

Proof of (iii). If $x \in D(A)$ and $Ax \in \overline{D(A)}$, we have

$$\frac{e^{tA}x - x}{t} = \frac{1}{t}A\int_0^t e^{sA}x\, ds = \frac{1}{t}\int_0^t e^{sA}Ax\, ds.$$

Since the function $s \mapsto e^{sA}Ax$ is continuous on $[0,t]$ by (i), then $\lim_{t\to 0^+}(e^{tA}x - x)/t = Ax$.

Conversely, if the limit $z := \lim_{t\to 0^+}(e^{tA}x - x)/t$ exists, then $\lim_{t\to 0^+} e^{tA}x = x$, so that both x and z belong to $\overline{D(A)}$. Moreover, for every $\xi \in \rho(A)$ we have

$$R(\xi, A)z = \lim_{t\to 0} R(\xi, A)\frac{e^{tA}x - x}{t},$$

and from (ii) it follows

$$R(\xi, A)z = \lim_{t\to 0^+}\frac{1}{t}R(\xi, A)A\int_0^t e^{sA}x\, ds = \lim_{t\to 0}(\xi R(\xi, A) - I)\frac{1}{t}\int_0^t e^{sA}x\, ds.$$

Since $x \in \overline{D(A)}$, the function $s \mapsto e^{sA}x$ is continuous at $s = 0$, and then

$$R(\xi, A)z = \xi R(\xi, A)x - x.$$

In particular, $x \in D(A)$ and $z = \xi x - (\xi - A)x = Ax$.

Proof of (iv). Statement (iv) is an easy consequence of (i). □

Coming back to the Cauchy problem (1.1), let us notice that theorem 1.2.3 and proposition 1.2.6 imply that the function

$$u(t) = e^{tA}x, \quad t \geq 0$$

is analytic with values in $D(A)$ for $t > 0$, and it is a solution of the differential equation in (1.1) for $t > 0$. If, moreover, $x \in \overline{D(A)}$, then u is continuous also at $t = 0$ (with values in X) and then it is a solution of the Cauchy problem (1.1). If $x \in D(A)$ and $Ax \in \overline{D(A)}$, then u is continuously differentiable up to $t = 0$, and it solves the differential equation also at $t = 0$, i.e., $u'(0) = Ax$. Uniqueness of the solution to (1.1) will be proved in proposition 4.2.3, in a more general context.

If x does not belong to $\overline{D(A)}$, proposition 1.2.6 implies that u is not continuous at 0, hence (even though, by definition, $e^{0A}x = x$) the initial datum is not assumed in the usual sense. However, some weak continuity property holds; for instance we have

$$\lim_{t\to 0^+} R(\lambda, A)e^{tA}x = R(\lambda, A)x \tag{1.18}$$

for every $\lambda \in \rho(A)$. Indeed, $R(\lambda, A)e^{tA}x = e^{tA}R(\lambda, A)x$ for every $t > 0$, and $R(\lambda, A)x \in D(A)$.

A standard way to obtain a strongly continuous semigroup from a sectorial operator A is to consider the part of A in $\overline{D(A)}$.

Definition 1.2.7 *Let $L : D(L) \subset X \mapsto X$ be a linear operator, and let Y be a subspace of X. The* part of L in Y *is the operator L_0 defined by*

$$D(L_0) = \{x \in D(L) \cap Y : \; Lx \in Y\}, \quad L_0x = Lx.$$

It is easy to see that the part A_0 of A in $\overline{D(A)}$ is still sectorial. Since $D(A_0)$ is dense in $\overline{D(A)}$ (because for each $x \in D(A_0)$ we have $x = \lim_{t\to 0} e^{tA}x$), then the semigroup generated by A_0 is strongly continuous in $\overline{D(A)}$. The semigroup generated by A_0 coincides of course with the restriction of e^{tA} to $\overline{D(A)}$.

Let us remark that all the properties of e^{tA} have been deduced from those of the resolvent operator, through the representation formula (1.10). Conversely, the following proposition says that the resolvent is the *Laplace transform* of the semigroup; as a consequence, several properties of $R(\lambda, A)$ can be deduced from properties of e^{tA}.

Proposition 1.2.8 *Let $A : D(A) \subset X \to X$ be a sectorial operator. For every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > \omega$ we have*

$$R(\lambda, A) = \int_0^{+\infty} e^{-\lambda t} e^{tA} dt. \tag{1.19}$$

Proof. Fix $0 < r < \mathrm{Re}\,\lambda - \omega$ and $\eta \in (\pi/2, \theta)$. Then

$$\begin{aligned}\int_0^{+\infty} e^{-\lambda t} e^{tA} dt &= \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} R(z, A)\int_0^{+\infty} e^{-\lambda t + zt} dt\, dz \\ &= \frac{1}{2\pi i}\int_{\omega+\gamma_{r,\eta}} R(z, A)(\lambda - z)^{-1} dz = R(\lambda, A).\end{aligned}$$

□

Corollary 1.2.9 *For all $t \geq 0$ the operator e^{tA} is one to one.*

Proof. $e^{0A} = I$ is obviously one to one. If there are $t_0 > 0$, $x \in X$ such that $e^{t_0A}x = 0$, then for $t \geq t_0$, $e^{tA}x = e^{(t-t_0)A}e^{t_0A}x = 0$. Since the function $t \mapsto e^{tA}x$ is analytic, $e^{tA}x \equiv 0$ in $(0, +\infty)$. From Proposition 1.2.8 we get $R(\lambda, A)x = 0$ for $\lambda > \omega$, so that $x = 0$. □

Remark 1.2.10 (1.19) is the formula used to define the Laplace transform of the scalar function $t \mapsto e^{tA}$, if $A \in \mathbb{C}$. On the other hand, the classical inversion formula given by a complex integral on a suitable vertical line must be modified, and in fact to get the semigroup from the resolvent operator a complex integral on a different curve has been used, see (1.10), in such a way that the improper integral converges because of assumption (1.9).

Theorem 1.2.11 *Let $\{T(t) : t > 0\}$ be a family of bounded operators such that the function $t \mapsto T(t)$ is differentiable, and assume that*

(i) *$T(t)T(s) = T(t + s)$, for all $t,\ s > 0$;*

(ii) *there are $\omega \in \mathbb{R}$, M_0, $M_1 > 0$ such that $\|T(t)\|_{\mathcal{L}(X)} \leq M_0 e^{\omega t}$, $\|tT'(t)\|_{\mathcal{L}(X)} \leq M_1 e^{\omega t}$ for $t > 0$;*

(iii) *one of the following conditions holds:*

(a) *there is $t > 0$ such that $T(t)$ is one to one*

(b) *for every $x \in X$ we have $\lim_{t\to 0} T(t)x = x$.*

Then the function $t \mapsto T(t)$ from $(0, +\infty)$ to $\mathcal{L}(X)$ is analytic, and there is a unique sectorial operator $A : D(A) \subset X \to X$ such that $T(t) = e^{tA}$, $t > 0$.

Proof. The function

$$F(\lambda) = \int_0^{+\infty} e^{-\lambda t} T(t) dt$$

is well defined and holomorphic in the halfplane $\Pi = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > \omega\}$. To prove the statement, it suffices to show that

(a) $F(\lambda)$ can be analytically continued in a sector $S_{\beta,\omega}$ with angle $\beta > \pi/2$, and the norm $\|(\lambda-\omega)F(\lambda)\|_{\mathcal{L}(X)}$ is bounded in $S_{\beta,\omega}$;

(b) there is a linear operator $A : D(A) \subset X \to X$ such that $F(\lambda) = R(\lambda, A)$ for $\lambda \in S_{\beta,\omega}$.

To prove (a), let us show by recurrence that $t \mapsto T(t)$ is infinitely many times differentiable, and

$$T^{(n)}(t) = (T'(t/n))^n, \;\; t > 0, \; n \in \mathbb{N}. \tag{1.20}$$

Equality (1.20) is true for $n = 1$ by assumption. Moreover, if (1.20) is true for $n = n_0$, from $T(t+s) = T(t)T(s)$ we deduce $T^{(n_0)}(t+s) = T^{(n_0)}(t)T(s) = T^{(n_0)}(s)T(t)$ for all $t, \; s > 0$, and also

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{h}\left(T^{(n_0)}(t+h) - T^{(n_0)}(t)\right)\\
&= \lim_{h\to 0}\frac{1}{h}T^{(n_0)}\left(\frac{tn_0}{n_0+1}\right)\left(T\left(\frac{t}{n_0+1}+h\right) - T\left(\frac{t}{n_0+1}\right)\right)\\
&= \left(T'\left(\frac{t}{n_0+1}\right)\right)^{n_0} T'\left(\frac{t}{n_0+1}\right) = \left(T'\left(\frac{t}{n_0+1}\right)\right)^{n_0+1},
\end{aligned}$$

so that $T^{(n_0+1)}(t)$ exists and (1.20) holds for $n = n_0+1$. Therefore, (1.20) is true for every n, and it implies that

$$\|T^{(n)}(t)\|_{\mathcal{L}(X)} \le (nM_1/t)^n e^{\omega t} \le (M_1 e)^n t^{-n} n! e^{\omega t}, \;\; t > 0, \; n \in \mathbb{N}.$$

Hence, the series

$$\sum_{n=0}^{\infty}\frac{(z-t)^n}{n!}\frac{d^n}{dt^n}T(t)$$

converges for every $z \in \mathbb{C}$ such that $|z-t| < t(M_1e)^{-1}$. As a consequence, $t \mapsto T(t)$ can be analytically continued in the sector $S_{\beta_0,0}$, with $\beta_0 = \arctan(M_1e)^{-1}$, and, denoting by $T(z)$ its extension, we have

$$\|T(z)\|_{\mathcal{L}(X)} \le (1-(eM_1)^{-1}\tan\theta)^{-1} e^{\omega Re\, z}, \;\; z \in S_{\beta_0,0}, \; \theta = \arg z.$$

Shifting the half-line $\{\mathrm{Re}\,\lambda \ge 0\}$ onto the halfline $\{\arg z = \beta\}$, with $|\beta| < \beta_0$, we conclude that (a) holds for every $\beta \in (\pi/2, \beta_0)$.

Let us prove (b). It is easily seen that F verifies the resolvent identity in the half-plane Π: indeed, for $\lambda \ne \mu, \; \lambda, \mu \in \Pi$, we have

$$\begin{aligned}
F(\lambda)F(\mu) &= \int_0^{+\infty} e^{-\lambda t}T(t)dt \int_0^{+\infty} e^{-\mu s}T(s)ds\\
&= \int_0^{+\infty} e^{-\mu\sigma}T(\sigma)d\sigma \int_0^{\sigma} e^{-(\lambda+\mu)t}dt\\
&= \int_0^{+\infty} e^{-\mu\sigma}T(\sigma)\frac{e^{-(\lambda-\mu)\sigma}-1}{\lambda-\mu}d\sigma\\
&= \frac{1}{\lambda-\mu}(F(\lambda)-F(\mu)).
\end{aligned}$$

Let us prove that $F(\lambda)$ is one to one for $\lambda \in \Pi$. Suppose that there are $x \ne 0, \; \lambda_0 \in \Pi$ such that $F(\lambda_0)x = 0$. From the resolvent identity it follows that $F(\lambda)x = 0$ for all $\lambda \in \Pi$. Hence, for all $x' \in X'$

$$\langle F(\lambda)x, x'\rangle = \int_0^{+\infty} e^{-\lambda t}\langle T(t)x, x'\rangle dt = 0, \;\; \forall \lambda \in \Pi.$$

Since $\langle F(\lambda)x, x'\rangle$ is the Laplace transform of the scalar function $t \mapsto \langle T(t)x, x'\rangle$, we get $\langle T(t)x, x'\rangle \equiv 0$ in $(0, +\infty)$, and then $T(t)x \equiv 0$ in $(0, +\infty)$, by the arbitrariness of x'. This is impossible if either (iii)(a) or (iii)(b) hold, and therefore $F(\lambda)$ is one to one for all $\lambda \in \Pi$. Thus, by proposition B.1.4 there is a linear operator $A : D(A) \subset X \to X$ such that $\rho(A) \supset \Pi$ and $R(\lambda, A) = F(\lambda)$ for $\lambda \in \Pi$. Since F is holomorphic in the sector $S_{\beta_0,\omega}$, then $\rho(A) \supset S_{\beta_0,\omega}$, $R(\lambda, A) = F(\lambda)$ for $\lambda \in S_{\beta_0,\omega}$ and statement (b) is proved. □

Remark 1.2.12 *Notice that in theorem 1.2.11 hypotheses (i) and (ii) are sufficient to prove that $T(t)$ is a semigroup and that $t \mapsto T(t)$ is analytic with values in $\mathcal{L}(X)$, whereas hypothesis (iii) is used to prove the existence of a sectorial operator which is its generator.*

Let us give a sufficient condition, seemingly weaker than (1.9), in order that a linear operator be sectorial. It will be useful to prove that realizations of some elliptic partial differential operators are sectorial in the usual function spaces.

Proposition 1.2.13 *Let $A : D(A) \subset X \to X$ be a linear operator such that $\rho(A)$ contains a halfplane $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \geq \omega\}$, and*

$$\|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} \leq M, \quad \mathrm{Re}\,\lambda \geq \omega, \tag{1.21}$$

with $\omega \in \mathbb{R}$, $M > 0$. Then A is sectorial.

Proof. By the general properties of resolvent operators, for every $r > 0$ the open ball with centre $\omega + ir$ and radius $|\omega + ir|/M$ is contained in $\rho(A)$. The union of such balls contains the sector $S = \{\lambda \neq \omega : |\arg(\lambda - \omega)| < \pi - \arctan M\}$, and for λ such that $\mathrm{Re}\,\lambda < \omega$ and $|\arg(\lambda - \omega)| \leq \pi - \arctan 2M\}$, say $\lambda = \omega + ir - \theta r/M$ with $0 < \theta \leq 1/2$, the resolvent series expansion

$$R(\lambda, A) = \sum_{n=0}^{\infty} (-1)^n (\lambda - \omega)^n (R(\omega, A))^{n+1}$$

gives

$$\|R(\lambda, A)\| \leq \sum_{n=0}^{\infty} |\lambda - (\omega + ir)|^n \frac{M^{n+1}}{(\omega^2 + r^2)^{(n+1)/2}} \leq \frac{2M}{r}.$$

On the other hand, for $\lambda = \omega + ir - \theta r/M$ we have

$$r \geq (1/(4M^2) + 1)^{-1/2} |\lambda - \omega|,$$

so that $\|R(\lambda, A)\| \leq 2M(1/(4M^2) + 1)^{-1/2} |\lambda - \omega|^{-1}$, and the claim follows. □

Next, we give a useful perturbation theorem.

Theorem 1.2.14 *Let $A : D(A) \mapsto X$ be sectorial operator, and let $B : D(B) \mapsto X$ be a linear operator such that $D(A) \subset D(B)$ and*

$$\|Bx\| \leq a\|Ax\| + b\|x\| \qquad x \in D(A). \tag{1.22}$$

There is $\delta > 0$ such that if $a \in [0, \delta]$ then $A + B : D(A) \mapsto X$ is sectorial.

Proof. As a first step, we assume that the constant ω in (1.9) is zero, i.e.

$$\rho(A) \supset S_{0,\theta} = \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \theta\}, \quad \|R(\lambda, A)\| \leq \frac{M}{|\lambda|}, \;\; \lambda \in S,$$

for some $\theta \in (\pi/2, \pi)$, $M > 0$. From (1.22) we deduce that $BR(\lambda, A)$ is bounded, and for each $\lambda \in S$ we have

$$\begin{aligned} \|BR(\lambda, A)x\| &\leq a\|AR(\lambda, A)x\| + b\|R(\lambda, A)x\| \\ &\leq a(M+1)\|x\| + \frac{bM}{|\lambda|}\|x\|. \end{aligned} \tag{1.23}$$

For $\lambda \in S$, the equation

$$\lambda u - (A + B)u = x$$

is equivalent, setting $\lambda u - Au = z$, to

$$z = BR(\lambda, A)z + x.$$

If $a \leq \frac{1}{2}(M+1)^{-1}$ and $|\lambda| > 2bM$ we have $\|BR(\lambda, A)\| < 1$, hence the operator $I - BR(\lambda, A)$ is invertible, $z = (I - BR(\lambda, A))^{-1}x$, and the equality

$$(\lambda I - (A + B))^{-1} = R(\lambda, A)(I - BR(\lambda, A))^{-1}$$

holds. Thus, for $|\lambda| > 2bM$ and $\arg\lambda| \leq \theta$, using (1.23) we get $\|R(\lambda, A + B)\| \leq M'/|\lambda|$, which shows that $A + B$ is sectorial.

In the general case $\omega \neq 0$, set $A_0 = A - \omega I$. Assumption (1.22) implies

$$\|Bx\| \leq a\|A_0x\| + (a|\omega| + b)\|x\| \qquad x \in D(A).$$

Then, for a small enough the operator $A_0 + B = A + B - \omega I$ is sectorial, and so is $A + B$. □

Corollary 1.2.15 *If A is sectorial and $B : D(B) \supset D(A) \mapsto X$ is a linear operator such that for some $\theta \in (0, 1)$, $C > 0$ we have*

$$\|Bx\| \leq C\|x\|_{D(A)}^{\theta}\|x\|_X^{1-\theta}, \quad \forall x \in D(A),$$

then $A + B : D(A + B) := D(A) \mapsto X$ is sectorial.

The next theorem is sometimes useful, because it lets us work in smaller subspaces of $D(A)$. A subspace D as in the following statement is called a *core* for the operator A.

Theorem 1.2.16 *Let A be a sectorial operator with dense domain. If a subspace $D \subset D(A)$ is dense in X and e^{tA}-invariant for each $t > 0$, then D is dense in $D(A)$ with respect to the graph norm.*

Proof. Fix $x \in D(A)$ and a sequence $(x_n) \subset D$ which converges to x in X. Since $D(A)$ is dense, then

$$Ax = \lim_{t\to 0} \frac{e^{tA}x - x}{t} = \lim_{t\to 0} \frac{A}{t}\int_0^t e^{sA}x\, ds,$$

and the same formula holds with x_n instead of x. Therefore it is convenient to set

$$y_{n,t} = \frac{1}{t}\int_0^t e^{sA}x_n\, ds = \frac{1}{t}\int_0^t e^{sA}(x_n - x)\, ds + \frac{1}{t}\int_0^t e^{sA}x)\, ds - x.$$

For each n, the map $s \mapsto e^{sA}x_n$ is continuous with values in $D(A)$; it follows that $\int_0^t T(s)x_n ds$, being the limit of the Riemann sums, belongs to the closure of D in $D(A)$, and then each $y_{n,t}$ does. Moreover $\|y_{n,t} - x\|$ goes to 0 as $t \to 0$, $n \to \infty$, and

$$Ay_{n,t} - Ax = \frac{e^{tA}(x_n - x) - (x_n - x)}{t} + \frac{1}{t}\int_0^t e^{sA}Ax\, ds - Ax.$$

Given $\varepsilon > 0$, fix τ small enough, in such a way that $\|\frac{1}{\tau}\int_0^\tau e^{sA}Ax\,ds - Ax\| \leq \varepsilon$, and then choose n large, in such a way that $(M+1)\|x_n - x\|/\tau \leq \varepsilon$. For such choices of τ and n we have $\|Ay_{n,\tau} - Ax\| \leq 2\varepsilon$, and the statement follows. □

Theorem 1.2.16 implies that the operator A is the closure of the restriction of A to D, i.e. $D(A)$ is the set of all $x \in X$ such that there is a sequence $(x_n) \subset D$ such that $x_n \to x$ and Ax_n converges as $n \to \infty$; in this case we have $Ax = \lim_{n\to\infty} Ax_n$.

Remark 1.2.17 Up to now we have considered complex Banach spaces, and the operators e^{tA} have been defined through integrals over paths in $\mathbb{C}$. But in many applications we have to work in real Banach spaces.

If X is a real Banach space, and $A : D(A) \subset X \mapsto X$ is a closed linear operator, it is however convenient to consider complex spectrum and resolvent. So we introduce the complexifications of X and of A, defined by

$$\tilde{X} = \{x + iy :\ x,\ y \in X\};\ \ \|x+iy\|_{\tilde{X}} = \sup_{-\pi\leq\theta\leq\pi} \|x\cos\theta + y\sin\theta\|$$

(note that the "euclidean norm" $\sqrt{\|x\|^2 + \|y\|^2}$ is not a norm, in general), and

$$D(\tilde{A}) = \{x + iy :\ x,\ y \in D(A)\},\ \ \tilde{A}(x+iy) = Ax + iAy.$$

If the complexification $\tilde{A}$ of A is sectorial, so that the semigroup $e^{t\tilde{A}}$ is analytic in $\tilde{X}$, then the restriction of $e^{t\tilde{A}}$ to X maps X into itself for each $t \geq 0$. To prove this statement it is convenient to replace the path $\gamma_{r,\eta}$ by the path $\gamma = \{\lambda \in \mathbb{C} : \lambda = \omega' + \rho e^{\pm i\theta},\ \rho \geq 0\}$, with $\omega' > \omega$. For each $x \in X$ we get

$$e^{t\tilde{A}}x = \frac{1}{2\pi i}\int_0^{+\infty} e^{\omega' t}\left(e^{i\theta+\rho t e^{i\theta}}R(\rho e^{i\theta}, A) - e^{-i\theta+\rho t e^{-i\theta}}R(\rho e^{-i\theta}, A)\right)x\,d\rho,\ \ t > 0.$$

The real part of the function under the integral vanishes (why?), and then $e^{t\tilde{A}}x$ belongs to X. So, we have a semigroup of linear operators in X which enjoys all the properties that we have seen up to now.

Exercises 1.2.18 1. Let $A : D(A) \subset X \mapsto X$ be sectorial, let $\alpha \in \mathbb{C}$, and set $B : D(B) := D(A) \mapsto X$, $Bx = Ax - \alpha x$. For which values of α the operator B is sectorial? In this case, show that $e^{tB} = e^{-\alpha t}e^{tA}$. Use this result to complete the proof of theorem 1.2.3 in the case $\omega \neq 0$.

2. Let $A : D(A) \subset X \mapsto X$ be sectorial, and let $B : D(B) \supset D(A) \mapsto X$ be a linear operator such that $\lim_{\lambda\in S_{\theta,\omega},\,|\lambda|\to\infty} \|BR(\lambda, A)\| = 0$. Show that $A+B : D(A+B) := D(A) \mapsto X$ is sectorial.

3. Let X_k, $k = 1, \ldots, n$ be Banach spaces, and let $A_k : D(A_k) \mapsto X_k$ be sectorial operators. Set
$$X = \prod_{k=1}^n X_k,\ \ D(A) = \prod_{k=1}^n D(A_k),$$
and $A(x_1, \ldots, x_n) = (A_1x_1, \ldots, A_nx_n)$, and show that A is a sectorial operator in X.

Chapter 2

Generation of analytic semigroups by differential operators

In this chapter we show several examples of sectorial operators A, and we study the associated evolution equations $u' = Au$.

The leading example is the heat equation in one or more variables, i.e., the equation $u_t = \Delta u$, where Δ is the Laplacian in $\mathbb{R}^N$, $\Delta u = u''$ if $N = 1$ and $\Delta u = \sum_{i=1}^N D_{ii}u$ if $N > 1$. We shall see some realizations of the Laplacian in different Banach spaces, with different domains, that turn out to be sectorial operators.

2.1 The operator $\mathcal{A}u = u''$

2.1.1 The second order derivative in the real line

Let us define the realizations of the second order derivative in $L^p(\mathbb{R})$ $(1 \leq p < \infty)$, and in $C_b(\mathbb{R})$, endowed with the maximal domains

$$\begin{aligned} D(A_p) &= W^{2,p}(\mathbb{R}) \subset L^p(\mathbb{R}), \;\; A_p u = u'', \qquad 1 \leq p < \infty \\ D(A_\infty) &= C^2b(\mathbb{R}), \;\; A_\infty u = u''. \end{aligned}$$

We recall that for $p < \infty$ the Sobolev space $W^{2,p}(\mathbb{R})$ is the subspace of $L^p(\mathbb{R})$ consisting of the (classes of equivalence of) functions $f : \mathbb{R} \mapsto \mathbb{C}$ that admit first and second order weak derivatives belonging to $L^p(\mathbb{R})$; the norm is

$$\|u\|_{W^{2,p}(\mathbb{R})} = \|u\|_{L^p} + \|u'\|_{L^p} + \|u''\|_{L^p}.$$

In the definition of A_p the second order derivative is meant in the weak sense.

$C_b(\mathbb{R})$ is the space of the bounded continuous functions from $\mathbb{R}$ to $\mathbb{C}$; $C_b^2(\mathbb{R})$ is the subspace of $C_b(\mathbb{R})$ consisting of the twice continuously differentiable functions, with bounded first and second order derivatives; the norm is

$$\|u\|_{C_b^2(\mathbb{R})} = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty.$$

Let us determine the spectrum of A_p and let us estimate its resolvent.

Proposition 2.1.1 *For all $1 \leq p \leq \infty$ the spectrum of A_p is the halfline $(-\infty, 0]$. If $\lambda = |\lambda|e^{i\theta}$ with $|\theta| < \pi$ then*

$$\|R(\lambda, A)\|_{\mathcal{L}(L^p)} \leq \frac{1}{|\lambda|\cos(\theta/2)}.$$

Proof. a) First we show that $(-\infty,0]\subset\sigma(A_p)$. Fix $\lambda\le 0$ and consider the function $u(x)=\exp(i\sqrt{-\lambda}x)$ which satisfies $u''=\lambda u$. For $p=\infty$, u is an eigenfunction of A_∞ with eigenvalue λ. For $p<\infty$, u does not belong to $L^p(\mathbb{R})$. Consider a cut-off function $\psi:\mathbb{R}\mapsto\mathbb{R}$, supported in $[-2,2]$ and identically equal to 1 in $[-1,1]$ and set $\psi_n(x)=\psi(x/n)$.

If $u_n=\psi_n u$, then $u_n\in D(A_p)$ and $\|u_n\|_p\approx n^{1/p}$ as $n\to\infty$. Moreover, $\|Au_n-\lambda u_n\|_p\le Cn^{1/p-1}$, from which it follows that, setting $v_n=\frac{u_n}{\|u_n\|_p}$, $\|(\lambda-A)v_n\|_p\to 0$ as $n\to\infty$, and then $\lambda\in\sigma(A)$.

b) Let now $\lambda\notin(-\infty,0]$, $\lambda=|\lambda|e^{i\theta}$, $|\theta|<\pi$. If $p=\infty$, the equation $\lambda u-u''=0$ has no nonzero bounded solution, hence $\lambda I-A_\infty$ is one to one. If $p<\infty$, it is easy to see that all the nonzero solutions $u\in W^{2,p}_{loc}(\mathbb{R})$ to the equation $\lambda u-u''=0$ belong to $C^\infty(\mathbb{R})$ and they are classical solutions, but they do not belong to $L^p(\mathbb{R})$, and the operator $\lambda I-A_p$ is injective.

Let us show that $\lambda I-A_p$ is onto. We write $\sqrt{\lambda}=\mu$, so that $\operatorname{Re}\mu>0$. If $f\in C_b(\mathbb{R})$ the variation of constants methods gives the (unique) bounded solution to $\lambda u-u''=f$, written as

$$u(x)=\frac{1}{2\mu}\left(\int_{-\infty}^{x}e^{-\mu(x-y)}f(y)dy+\int_{x}^{+\infty}e^{\mu(x-y)}f(y)dy\right)=(f*h_\mu)(x), \tag{2.1}$$

where $h_\mu(x)=\frac{1}{2\mu}e^{-\mu|x|}$. Since $\|h_\mu\|_{L^1(\mathbb{R})}=\frac{1}{|\mu|\operatorname{Re}\mu}$, we get

$$\|u\|_\infty\le\|h_\mu\|_{L^1(\mathbb{R})}\|f\|_\infty\le\frac{1}{|\lambda|\cos(\theta/2)}\|f\|_\infty.$$

If $|\arg\lambda|\le\theta_0<\pi$ we get $\|u\|_\infty\le(|\lambda|\cos(\theta_0/2))^{-1}\|f\|_\infty$, and therefore A_∞ is sectorial, with $\omega=0$ and any $\theta\in(\pi/2,\pi)$.

If $p<\infty$ and $f\in L^p(\mathbb{R})$, the natural candidate to be $R(\lambda,A_p)f$ is still the function u defined by (2.1). We have to check that $u\in D(A_p)$ and and that $(\lambda I-A_p)u=f$. By the Young's inequality (see e.g. [3, Th. IV.15]), $u\in L^p(\mathbb{R})$ and again

$$\|u\|_p\le\|f\|_p\|h_\mu\|_1\le\frac{1}{|\lambda|\cos(\theta/2)}\|f\|_p.$$

That $u\in D(A_p)$ may be seen in several ways; all of them need some knowledge of elementary properties of Sobolev spaces. The following proof relies on the fact that smooth functions are dense in $W^{1,p}(\mathbb{R})$(1).

Approach $f\in L^p(\mathbb{R})$ by a sequence $(f_n)\subset C_0^\infty(\mathbb{R})$. The corresponding solutions u_n to $\lambda u_n-u_n''=f_n$ are smooth and they are given by formula (2.1) with f_n instead of f, therefore they converge to u by the Young's inequality. Moreover,

$$u_n'(x)=-\frac{1}{2}\int_{-\infty}^{x}e^{-\mu(x-y)}f_n(y)dy+\frac{1}{2}\int_{x}^{+\infty}e^{\mu(x-y)}f_n(y)dy$$

converge to the function

$$g(x)=-\frac{1}{2}\int_{-\infty}^{x}e^{-\mu(x-y)}f(y)dy+\frac{1}{2}\int_{x}^{+\infty}e^{\mu(x-y)}f(y)dy$$

again by the Young's inequality, hence $g=u'\in L^p(\mathbb{R})$, and $u_n''=\lambda u_n-f_n$ converge to $\lambda u-f$, hence $\lambda u-f=u''\in L^p(\mathbb{R})$. Therefore $u\in W^{2,p}(\mathbb{R})$ and the statement follows. □

[1] Precisely, a function $v\in L^p(\mathbb{R})$ belongs to $W^{1,p}(\mathbb{R})$ iff there is a sequence $(v_n)\subset C^\infty(\mathbb{R})$ with v_n, $v_n'\in L^p(\mathbb{R})$, such that $v_n\to v$ and $v_n'\to g$ in $L^p(\mathbb{R})$ as $n\to\infty$. In this case, g is the weak derivative of v.

Note that $D(A_\infty)$ is not dense in $C_b(\mathbb{R}^N)$, and its closure is $BUC(\mathbb{R})$. Therefore, the associated semigroup e^{tA_∞} is not strongly continuous. But the part of A_∞ in $BUC(\mathbb{R})$, i.e. the operator

$$BUC^2(\mathbb{R}) \mapsto BUC(\mathbb{R}), \;\; u \mapsto u''$$

has dense domain in $BUC(\mathbb{R})$ and it is sectorial, so that the restriction of e^{tA_∞} to $BUC(\mathbb{R})$ is strongly continuous. If $p < \infty$, $D(A_p)$ is dense in $L^p(\mathbb{R})$, and e^{tA_p} is strongly continuous in $L^p(\mathbb{R})$.

This is one of the few situations in which we have a nice representation formula for e^{tA_p}, for $1 \le p \le \infty$, and precisely

$$(e^{tA_p}f)(x) = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} f(y)dy, \;\; t > 0, \; x \in \mathbb{R}. \tag{2.2}$$

This formula will be discussed in subsection 2.2. In principle, since we have an explicit representation formula for the resolvent, replacing in (1.10) we should get (2.5). But the contour integral obtained in this way is not very easy. To obtain the above representation formula it is easier to argue as follows: we recall that the function $u(t,x) := (e^{tA_p}f)(x)$ is a candidate to be a solution to the Cauchy problem for the heat equation

$$\begin{cases} u_t(t,x) = u_{xx}(t,x), \;\; t > 0, x \in \mathbb{R}, \\ \\ u(0,x) = f(x), \;\; x \in \mathbb{R}. \end{cases} \tag{2.3}$$

Let us apply (just formally) the Fourier transform, denoting by $\hat{u}(t,\xi)$ the Fourier transform of u with respect to the space variable x. We get

$$\begin{cases} \hat{u}_t = -|\xi|^2 \hat{u} & \text{in } (0,+\infty) \times \mathbb{R}, \\ \hat{u}(0,\xi) = \hat{f}(\xi) & x \in \mathbb{R}, \end{cases}$$

whose solution is $\hat{u}(t,\xi) = \hat{f}(\xi)e^{-|\xi|^2 t}$. Taking the inverse Fourier transform, we obtain (2.5). Once we have a candidate for $e^{tA_p}f$ we may check directly that the formula is correct. See section 2.2 for the general N-dimensional case.

2.1.2 The operator $\mathcal{A}u = u''$ in a bounded interval, with Dirichlet boundary conditions

Without loss of generality, we fix $I = (0,1)$, and we consider the realizations of the second order derivative in $L^p(I)$, $1 \le p < \infty$,

$$D(A_p) = \{u \in W^{2,p}(I) : \; u(0) = u(1) = 0\} \subset L^p(I), \;\; A_p u = u'',$$

as well as its realization in $C([0,1])$,

$$D(A_\infty) = \{u \in C^2([0,1]) : \; u(0) = u(1) = 0\}, \;\; A_\infty u = u''.$$

We could follow the same approach of subsection 2.1.1, by computing explicitly the resolvent operator $R(\lambda, A_\infty)$ for $\lambda \notin (-\infty, 0]$ and then showing that the same formula gives $R(\lambda, A_p)$. The formula comes out to be more complicated than before, but it leads to the same final estimate, see exercise 2.5.3.1. Here we prefer to follow a slightly different approach that leads to a less precise estimate for the norm of the resolvent, but computations are simpler.

Proposition 2.1.2 *The operators $A_p : D(A_p) \mapsto L^p(0,1)$, $1 \le p < \infty$ and $A_\infty : D(A_\infty) \mapsto C([0,1])$ are sectorial, with $\omega = 0$ and any $\theta \in (\pi/2, \pi)$.*

Proof. For $\lambda \notin (-\infty, 0]$ set $\mu = \sqrt{\lambda}$, so that $\mathrm{Re}\,\mu > 0$. For every $f \in X$, $X = L^p(0,1)$ or $X = C([0,1])$, extend f to a function $\widetilde{f} \in L^p(\mathbb{R})$ or $\widetilde{f} \in C_b(\mathbb{R})$, in such a way that $\|\widetilde{f}\| = \|f\|$. For instance we may define $\widetilde{f}(x) = 0$ for $x \notin (0,1)$ if $X = L^p(0,1)$, $\widetilde{f}(x) = f(1)$ for $x > 1$, $\widetilde{f}(x) = f(0)$ for $x < 0$ if $X = C([0,1])$. Let $\widetilde{u}$ be defined by (2.1) with $\widetilde{f}$ instead of f. We already know from example 2.1.1 that $\widetilde{u}_{|[0,1]}$ is a solution of the equation $\lambda u - u'' = f$ satisfying $\|u\|_p \le \frac{\|f\|_p}{|\lambda|\cos(\theta/2)}$. However, it does not necessarily satisfy boundary condition, and we set

$$\gamma_0 = \frac{1}{2\mu}\int_{\mathbb{R}} e^{-\mu|s|}\widetilde{f}(s)\,ds$$

and

$$\gamma_1 = \frac{1}{2\mu}\int_{\mathbb{R}} e^{-\mu|1-s|}\widetilde{f}(s)\,ds.$$

Then all the solutions to $\lambda u - u'' = f$ belonging to $W^{2,p}(0,1)$ or to $C^2([0,1])$ are given by $u(x) = \widetilde{u}(x) + c_1u_1(x) + c_2u_2(x)$, where $u_1(x) = e^{-\mu x}$ and $u_2(x) = e^{\mu x}$ are two independent solutions of the homogeneous equation $\lambda u - u'' = 0$. We can determine uniquely c_1 and c_2 imposing $u(0) = u(1) = 0$ because the determinant

$$D(\mu) = e^{\mu} - e^{-\mu}$$

is nonzero since $\mathrm{Re}\,\mu > 0$. A straightforward computation yields

$$c_1 = \frac{1}{D(\mu)}\Big[\gamma_1 - e^{\mu}\gamma_0\Big],$$
$$c_2 = \frac{1}{D(\mu)}\Big[-\gamma_1 + e^{-\mu}\gamma_0\Big].$$

Explicit computations give for $1 \le p < \infty$

$$\|u_1\|_p \le \frac{1}{(p\mathrm{Re}\,\mu)^{1/p}} \qquad \|u_2\|_p \le \frac{e^{\mathrm{Re}\,\mu}}{(p\mathrm{Re}\,\mu)^{1/p}};$$

while $\|u_1\|_\infty = e^{\mathrm{Re}\,\mu}$, $\|u_2\|_\infty = 1$ and for $1 < p < \infty$ by the Hölder inequality we also obtain

$$|\gamma_0| \le \frac{1}{2|\mu|(p'\mathrm{Re}\,\mu)^{1/p'}}\|f\|_p \qquad |\gamma_1| \le \frac{1}{2|\mu|(p'\mathrm{Re}\,\mu)^{1/p'}}\|f\|_p$$

and also $|\gamma_0|$, $|\gamma_1| \le \frac{1}{2|\mu|}\|f\|_1$, if $f \in L^1$, $|\gamma_0|$, $|\gamma_1| \le \frac{1}{|\mu|\mathrm{Re}\,\mu}\|f\|_\infty$ if $f \in C([0,1])$.

Moreover $|D(\mu)| \approx e^{\mathrm{Re}\,\mu}$ for $|\mu| \to \infty$. If $\lambda = |\lambda|e^{i\theta}$ with $|\theta| \le |\theta_0| < \pi$ then $\mathrm{Re}\,\mu \ge |\mu|\cos(\theta_0/2)$ and we easily get

$$\|c_1u_1\|_p \le \frac{C}{|\lambda|}\|f\|_p \qquad \text{and} \qquad \|c_2u_2\|_p \le \frac{C}{|\lambda|}\|f\|_p$$

for a suitable $C > 0$ and λ as above, $|\lambda|$ big enough, and finally

$$\|v\|_p \le \frac{C}{|\lambda|}\|f\|_p$$

for $|\lambda|$ large, say $|\lambda| \ge R$, and $|\arg\lambda| \le \theta_0$.

For $|\lambda|$ small we may argue as follows: one checks easily that 0 is in the resolvent set of A_p; since the resolvent set is open there is a circle centered at 0 contained in the resolvent set (in fact it can be shown that the spectrum of A_p consists only of the eigenvalues $-n^2/\pi^2$, $n \in \mathbb{N}$); since $\lambda \mapsto R(\lambda, A_p)$ is holomorphic in the resolvent set it is continuous, hence it is bounded on the compact set $\{|\lambda| \le R,\ |\arg\lambda| \le \theta_0\} \cup \{0\}$. □

2.2 The Laplacian in R^N

Let us consider the heat equation

$$\begin{cases} u_t(t,x) = \Delta u(t,x), & t>0, x\in\mathbb{R}^N, \\ u(0,x) = f(x), & x\in\mathbb{R}^N, \end{cases} \tag{2.4}$$

where f is a given function in X, $X = L^p(\mathbb{R}^N)$, $1 \le p < \infty$, or $X = BUC(\mathbb{R}^N)$.

A representation formula for the solution may be deduced formally by Fourier transform, as in dimension $N = 1$, getting $u(t,x) = (T(t)f)(x)$, where the *heat semigroup* $(T(t)_{t\ge 0})$ is defined by the Gauss-Weierstrass formula

$$T(t)f(x) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)dy, \;\; t>0, \; x\in\mathbb{R}^N. \tag{2.5}$$

(as usual, we define $T(0)f(x) = f(x)$). The verification that $(T(t)_{t\ge 0})$ is a semigroup is left as an exercise, see 2.5.3.3 below.

Now, we check that formula (2.5) gives in fact a solution to (2.4).

Let us first notice that $T(t)f = G_t * f$, where

$$G_t(x) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}, \qquad \int_{\mathbb{R}^N} G_t(x)dx = 1 \;\; \forall\, t>0,$$

and $*$ denotes the convolution. The function $(t,x)\mapsto G_t(x)$ is smooth for $t>0$, and its derivative with respect to t equals its Laplacian with respect to the space variables x. By the Young inequality,

$$\|T(t)f\|_{L^p} \le \|f\|_{L^p}, \;\; t>0, \; 1\le p\le\infty. \tag{2.6}$$

Since G_t and all its derivatives belong to $C^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$, it readily follows that the function $u(t,x) := (T(t)f)(x)$ belongs to $C^\infty((0,+\infty)\times\mathbb{R}^N)$, because we can differentiate under the integral sign. Since $\partial G_t/\partial t = \Delta G_t$, then u solves the heat equation in $(0,+\infty)\times\mathbb{R}^N$.

Let us show that $T(t)f \to f$ in X as $t\to 0$. If $X = L^p(\mathbb{R}^N)$ we have

$$\begin{aligned} \|T(t)f - f\|_p^p &= \int_{\mathbb{R}^N} \Big| \int_{\mathbb{R}^N} G_t(y) f(x-y)dy - f(x)\Big|^p dx \\ &= \int_{\mathbb{R}^N} \Big| \int_{\mathbb{R}^N} G_t(y)[f(x-y) - f(x)]dy\Big|^p dx \\ &= \int_{\mathbb{R}^N} \Big| \int_{\mathbb{R}^N} G_1(z)[f(x-\sqrt{t}z) - f(x)]dz\Big|^p dx \\ &\le \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} G_1(z)|f(x-\sqrt{t}z) - f(x)|^p dz\, dx \\ &= \int_{\mathbb{R}^N} G_1(z) \int_{\mathbb{R}^N} |f(x-\sqrt{t}z) - f(x)|^p dxdz. \end{aligned}$$

Here we used twice the property that the integral of G_t is 1; the first one to put $f(x)$ under the integral and the second one to get $\Big|\int_{\mathbb{R}^N} G_1(z)[f(x-\sqrt{t}z) - f(x)]dz\Big|^p \le \int_{\mathbb{R}^N} G_1(z)|f(x-\sqrt{t}z) - f(x)|^p dz$. Now, the function $\varphi(t,z) := \int_{\mathbb{R}^N} |f(x-\sqrt{t}z) - f(x)|^p dx$ goes to zero for each z as $t\to 0$, by a well known property of the L^p functions, and it does not exceed $2^p\|f\|_p^p$. By dominated convergence, $\|T(t)f - f\|_p^p$ goes to 0 as $t\to 0$.

If $X = BUC(\mathbb{R}^N)$ and $f \in X$, we have

$$\begin{aligned}
\sup_{x\in\mathbb{R}^N} |(T(t)f - f)(x)| &\leq \sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} G_t(y)|f(x-y) - f(x)|dy \\
&= \sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} G_1(z)|f(x-\sqrt{t}z) - f(x)|dz \\
&\leq \int_{\mathbb{R}^N} G_1(z) \sup_{x\in\mathbb{R}^N} |f(x-\sqrt{t}z) - f(x)|dz.
\end{aligned}$$

Again, the function $\varphi(t,z) := \sup_{x\in\mathbb{R}^N} |f(x-\sqrt{t}z) - f(x)|$ goes to zero as $t \to 0$ for each z by the uniform continuity of f, and it does not exceed $2\|f\|_\infty$. By dominated convergence, $T(t)f - f$ goes to 0 as $t \to 0$ in the sup norm.

The proof that $T(t)$ satisfies all the assumptions of theorem 1.2.11 is left as an exercise, see exercises 2.5.3.4 and 2.5.3.5. Then, there is a sectorial operator A such that $T(t) = e^{tA}$.

Let us now show that the generator A of $T(t)$ is a suitable realization of the Laplacian. To begin with, we consider the case $p < \infty$. In this case the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ is invariant under the semigroup and it is dense in $L^p(\mathbb{R}^N)$ because it contains $C_0^\infty(\mathbb{R}^N)$. Then, by theorem 1.2.16, it is dense in the domain of the generator. For $f \in \mathcal{S}(\mathbb{R}^N)$, it can be easily checked that $u(t,x) = T(t)f(x)$ belongs to $C^2([0,\infty) \times \mathbb{R}^N)$ (in fact, it belongs to $C^\infty([0,\infty) \times \mathbb{R}^N)$). Recalling that u satisfies the heat equation for $t > 0$, we get

$$\frac{u(t,x) - u(0,x)}{t} = \frac{1}{t}\int_0^t u_t(s,x)ds = \frac{1}{t}\int_0^t \Delta u(s,x)ds \to \Delta f(x) \text{ as } t \to 0 \tag{2.7}$$

pointwise and also in $L^p(\mathbb{R}^N)$, because

$$\frac{1}{t}\int_0^t \|\Delta u(s,\cdot) - \Delta f\|_p ds \leq \sup_{0<s<t} \|T(s)\Delta f - \Delta f\|_p.$$

For $p = \infty$, we argue in the same way, using $BUC^2(\mathbb{R}^N)$ instead of $\mathcal{S}(\mathbb{R}^N)$, and observing that it is dense in $BUC(\mathbb{R}^N)$, that it is invariant under the semigroup, and that in this case the convergence in (2.7) is uniform in $\mathbb{R}^N$.

From theorem 1.2.16 it follows that the generator A of $T(t)$ is the closure of the Laplacian with domain $D = \mathcal{S}(\mathbb{R}^N)$, if $X = L^p(\mathbb{R}^N)$, with domain $D = BUC^2(\mathbb{R}^N)$, if $X = BUC(\mathbb{R}^N)$. So, $D(A)$ is the set of the functions u in X such that there is a sequence $u_n \in D$ that converge to u in X and such that Δu_n converge in X as $n \to \infty$; in other words $D(A)$ is the completion of D with respect to the graph norm $u \mapsto \|u\|_X + \|\Delta u\|_X$. If $N = 1$ we conclude rather easily that $D(A) = W^{2,p}(\mathbb{R})$ if $X = L^p(\mathbb{R})$, and $D(A) = BUC^2(\mathbb{R}^N)$, if $X = BUC(\mathbb{R})$. The problem of giving an explicit characterization of $D(A)$ in terms of known functional spaces is more difficult if $N > 1$. The answer is nice, i.e. $D(A) = W^{2,p}(\mathbb{R}^N)$ if $X = L^p(\mathbb{R}^N)$ and $1 < p < \infty$, but the proof is not easy in general. There is an easy proof, that we give below, for $p = 2$.

The domain of A in L^2 is the closure of $\mathcal{S}(\mathbb{R}^N)$ with respect to the graph norm $u \mapsto \|u\|_{L^2(\mathbb{R}^N)} + \|\Delta u\|_{L^2(\mathbb{R}^N)}$, which is weaker than the H^2-norm. Hence, to conclude it suffices to show that the two norms are in fact equivalent. The main point to be proved is that $\|D_{ij}u\|_{L^2(\mathbb{R}^N)} \leq \|\Delta u\|_{L^2(\mathbb{R}^N)}$ for each $u \in \mathcal{S}$ and $i,\ j = 1,\ldots,N$. Integrating by parts twice we get

$$\| \, |D^2u| \, \|^2_{L^2(\mathbb{R}^N)} = \sum_{i,j=1}^N \int_{\mathbb{R}^N} D_{ij}u\overline{D_{ij}u}\,dx = -\sum_{i,j=1}^N \int_{\mathbb{R}^N} D_{ijj}u\overline{D_iu}\,dx \tag{2.8}$$

$$= \sum_{i,j=1}^N \int_{\mathbb{R}^N} D_{ii}u\overline{D_{jj}u}\,dx = \|\Delta u\|^2_{L^2(\mathbb{R}^N)}. \tag{2.9}$$

The L^2 norm of the first order derivatives of u may be estimated in several ways; since we already have the semigroup $T(t)$ at our disposal we may argue as follows. For $t > 0$ and for each $f \in L^2(\mathbb{R}^N)$ we have

$$D_iT(t)f(x) = \frac{1}{(4\pi t)^{N/2}}\int_{\mathbb{R}^n}\frac{1}{2}(x_i - y_i)e^{-\frac{|x-y|^2}{4t}}f(y)dy = (D_iG_t * f)(x), \;\; t > 0, \; x \in \mathbb{R}^N,$$

so that

$$\|D_iT(t)f\|_2 \le \|D_iG_t\|_1\|f\|_2 \le \frac{C}{t^{1/2}}\|f\|_2, \;\; t > 0.$$

From the obvious equality $D_iu = D_i(u - T(t)u) + D_iT(t)u$ we get, for each $u \in \mathcal{S}(\mathbb{R}^N)$,

$$D_iu = D_i\int_0^t T(s)\Delta u\, ds + D_iT(t)u = \int_0^t D_iT(s)\Delta u\, ds + D_iT(t)u,$$

and using the above estimate we obtain

$$\|D_iu\|_2 \le C_1t^{1/2}\|\Delta u\|_2 + C_2t^{-1/2}\|u\|_2, \;\; t > 0. \tag{2.10}$$

Taking $t = 1$ we see that the L^2 norm of each D_iu is estimated by the graph norm of the Laplacian at u, which is what we needed.

In addition, taking the minimum for $t > 0$, we get another estimate of independent interest,

$$\|D_iu\|_2 \le C_3\|\Delta u\|_2^{1/2}\|u\|_2^{1/2}. \tag{2.11}$$

Estimates (2.10) and (2.11) are then extended by density to the whole domain of the Laplacian, that is to $H^2(\mathbb{R}^N)$.

2.3 Some abstract examples

The realization of the Laplacian in $L^2(\mathbb{R}^N)$ is a particular case of the following general situation. Recall that, if H is a Hilbert space, an operator $A : D(A) \subset H \to H$ with dense domain is said to be *self-adjoint* if $D(A) = D(A^\star)$ and $A = A^\star$, and that A is *dissipative* if

$$\|(\lambda - A)x\| \ge \lambda\|x\|^2, \tag{2.12}$$

for all $x \in D(A)$ and $\lambda > 0$, or equivalently (see exercise 2.5.3.6) if Re $\langle Ax, x\rangle \le 0$ for every $x \in D(A)$.

The following proposition holds.

Proposition 2.3.1 *Let H be a Hilbert space, and let $A : D(A) \subset H \mapsto H$ be a self-adjoint dissipative operator. Then A is sectorial, with arbitrary $\theta < \pi$ and $\omega = 0$.*

Proof. Let us first show that $\sigma(A) \subset \mathbb{R}$. For, let $\lambda = a + ib \in \mathbb{C}$. Since $\langle Ax, x\rangle \in \mathbb{R}$, for every $x \in D(A)$ we have

$$\|(\lambda I - A)x\|^2 = (a^2 + b^2)\|x\|^2 - 2a\langle x, Ax\rangle + \|Ax\|^2 \ge b^2\|x\|^2, \tag{2.13}$$

so that if $b \ne 0$ then $\lambda I - A$ is one to one. Let us check that the range is both closed and dense in H, so that A is onto. Take $x_n \in D(A)$ such that $\lambda x_n - Ax_n$ converges as $n \to \infty$. From the inequality

$$\|(\lambda I - A)(x_n - x_m)\|^2 \ge b^2\|x_n - x_m\|^2, \; n, m \in \mathbb{N},$$

it follows that (x_n) is a Cauchy sequence, and by difference (Ax_n) is a Cauchy sequence too. Hence there are $x, y \in H$ such that $x_n \to x$, $Ax_n \to y$. Since A is self-adjoint, it is closed, and then $x \in D(A)$, $Ax = y$, and $\lambda x_n - Ax_n$ converges to $\lambda x - Ax \in \text{Range}\,(\lambda I - A)$. Therefore, the range of $\lambda I - A$ is closed.

If y is orthogonal to the range of $(\lambda I - A)$, then for every $x \in D(A)$ we have $\langle y, \lambda x - Ax\rangle = 0$, hence $y \in D(A^\star) = D(A)$ and $\overline{\lambda} y - A^\star y = \overline{\lambda} y - Ay = 0$. Since $\overline{\lambda} I - A$ is one to one, then $y = 0$, and the range of $(\lambda I - A)$ is dense.

Let us check that $\sigma(A) \subset (-\infty, 0]$. Indeed, if $\lambda > 0$ and $x \in D(A)$, we have

$$\|(\lambda I - A)x\|^2 = \lambda^2\|x\|^2 - 2\lambda\langle x, Ax\rangle + \|Ax\|^2 \geq \lambda^2\|x\|^2, \tag{2.14}$$

and arguing as above we get $\lambda \in \rho(A)$.

Let us now verify condition (1.9)(ii) for $\lambda = \rho e^{i\theta}$, with $\rho > 0$, $-\pi < \theta < \pi$. Take $x \in H$ and $u = R(\lambda, A)x$. From the equality $\lambda u - Au = x$, multiplying by $e^{-i\theta/2}$ and taking the inner product with u, we deduce

$$\rho e^{i\theta/2}\|u\|^2 - e^{-i\theta/2}\langle Au, u\rangle = e^{-i\theta/2}\langle x, u\rangle,$$

from which, taking the real part,

$$\rho\cos(\theta/2)\|u\|^2 - \cos(\theta/2)\langle Au, u\rangle = \text{Re}(e^{-i\theta/2}\langle x, u\rangle) \leq \|x\|\,\|u\|$$

and therefore, taking into account that $\cos(\theta/2) > 0$ and $\langle Ax, x\rangle \leq 0$, we get

$$\|u\| \leq \frac{\|x\|}{|\lambda|\cos(\theta/2)},$$

with $\theta = \arg\lambda$. □

Let us see two further examples.

Proposition 2.3.2 *Let A be a linear operator such that the resolvent set $\rho(A)$ contains $\mathbb{C}\setminus i\mathbb{R}$, and there exists $M > 0$ such that $\|R(\lambda, A)\| \leq M/|Re\,\lambda|$ for $Re\,\lambda \neq 0$. Then A^2 is sectorial, with $\omega = 0$ and any $\theta < \pi$.*

Proof. For every $\lambda \in \mathbb{C}\setminus(-\infty, 0]$ and for every $y \in X$, the resolvent equation $\lambda x - A^2 x = y$ is equivalent to

$$(\sqrt{\lambda}I - A)(\sqrt{\lambda}I + A)x = y.$$

Since $\text{Re}\sqrt{\lambda} > 0$, then $\sqrt{\lambda} \in \rho(A) \cap (\rho(-A))$, so that

$$x = -R(-\sqrt{\lambda}, A)R(\sqrt{\lambda}, -A)y$$

and, since $|\text{Re}\,\lambda| = \sqrt{|\lambda|}\cos\eta/2$ if $\arg\lambda = \eta$, we get

$$\|x\| \leq \frac{M^2}{|\lambda|(\cos\theta/2)^2}\|y\|,$$

for $\lambda \in S_{\theta,0}$, and the statement follows. □

Proposition 2.3.2 gives us an alternative way to show that the realization of the second order derivative in $L^p(\mathbb{R})$, or in $C_b(\mathbb{R})$, is sectorial. But there are also other interesting applications.

Proposition 2.3.3 *Let A be a sectorial operator. Then $-A^2$ is sectorial.*

Proof. As a first step we prove the statement assuming that the constant ω in (1.9) vanishes. In this case, for every $\lambda \in S_{\theta,0}$ and for every $y \in X$, the resolvent equation $\lambda x + A^2 x = y$ is equivalent to $(i\sqrt{\lambda}I - A)(-i\sqrt{\lambda}I - A)x = y$. We can solve it and estimate the norm of the solution because both $i\sqrt{\lambda}$ and $-i\sqrt{\lambda}$ belong to $S_{\theta,0}$. We get

$x = R(-i\sqrt{\lambda}, A)R(i\sqrt{\lambda}, A)y$ and $\|x\| \leq M^2\|y\|/|\lambda|$. Therefore, $-A^2$ is sectorial, with the same sector of A.

If $\omega \neq 0$, we consider as usual the operator $B = A - \omega I : D(B) = D(A) \mapsto X$. B and B^2 are sectorial, with sector $S_{\theta,0}$. Since $R(\lambda, B^2) = R(-i\sqrt{\lambda}, B)R(i\sqrt{\lambda}, B)$ for $\lambda \in S_{\theta,0}$, then $\|BR(\lambda, B^2)\| \leq M(M+1)/\sqrt{|\lambda|}$; hence $B^2 + 2\omega B$ is sectorial, and $B^2 + 2\omega B + \omega^2 I = A^2$ is sectorial. See exercises 1.2.18. □

Using proposition 2.3.3 and the examples that we have seen up to now, we obtain other examples of sectorial operators. For instance, the realizations of $u \mapsto -u^{(iv)}$ in $L^p(\mathbb{R})$, in $BUC(\mathbb{R})$, in $C_b(\mathbb{R})$, with respective domains $W^{4,p}(\mathbb{R})$, $BUC^4(\mathbb{R})$, $C_b^4(\mathbb{R})$ are sectorial, and so on.

2.4 The Dirichlet Laplacian in a bounded open set

We now consider the Laplacian in an open bounded set $\Omega \subset \mathbb{R}^N$ with C^2 boundary $\partial\Omega$ and Dirichlet boundary condition, in $L^p(\Omega)$, $1 < p < \infty$. Even for $p = 2$ the theory is much more difficult that in the case $\Omega = \mathbb{R}^N$. In fact, the Fourier transform is useless, and estimates such as (2.8) are not available integrating by parts because boundary integrals appear.

In order to prove that the operator A_p defined by

$$D(A_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \qquad A_p u = \Delta u \ , \quad u \in D(A_p)$$

is sectorial, one shows that the resolvent set $\rho(A_p)$ contains a sector

$$S_\theta = \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg(\lambda)| < \theta\}$$

for some $\theta \in (\pi/2, \pi)$, and that the resolvent estimate

$$\|R(\lambda, A_p)\|_{\mathcal{L}(L^p(\Omega))} \leq \frac{M}{|\lambda|}$$

holds for some $M > 0$ and for all $\lambda \in S_{\theta,\omega}$. The hard part is the proof of existence of a solution $u \in D(A_p)$ to $\lambda u - \Delta u = f$, i.e. the following theorem that we give without any proof.

Theorem 2.4.1 *Let $\Omega \subset \mathbb{R}^N$ be open and bounded with C^2 boundary, and let $f \in L^p(\Omega)$, $\lambda \notin (-\infty, 0]$. Then, there is $u \in D(A_p)$ such that $\lambda u - \Delta u = f$, and the estimate*

$$\|u\|_{W^{2,p}} \leq C_1\|f\|_{L^p} + C_2\|u\|_{L^p} \tag{2.15}$$

holds, with C_1, C_2 depending only upon Ω and λ. For $Re\,\lambda \geq 0$ inequality (2.15) holds with $C_2 = 0$.

The resolvent estimate is much easier. Its proof is quite simple for $p \geq 2$, and in fact we shall consider only this case. For $1 < p < 2$ the method still works, but some technical problems occur.

Proposition 2.4.2 *Let $2 \leq p < \infty$, and let $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $\lambda \in \mathbb{C}$ with $Re\,\lambda \geq 0$, be such that $\lambda u - \Delta u = f \in L^p(\Omega)$. Then*

$$\|u\|_{L^p} \leq C_p \frac{\|f\|_{L^p}}{|\lambda|},$$

with $C_p = (1 + p^2/4)^{1/2}$.

Proof. If $u = 0$ the statement is obvious. If $u \neq 0$, we multiply the equation $\lambda u - \Delta u = f$ by $|u|^{p-2}\overline{u}$, which belongs to $W^{1,p'}(\Omega)$ (see exercise 2.5.3.7), and we integrate over Ω. We have

$$\lambda\|u\|^p + \int_\Omega \sum_{k=1}^n \frac{\partial u}{\partial x_k}\,\frac{\partial}{\partial x_k}\Big(|u|^{p-2}\overline{u}\Big)\,dx = \int_\Omega f\,|u|^{p-2}\overline{u}\,dx.$$

Notice that

$$\frac{\partial}{\partial x_k}|u|^{p-2}\overline{u} = |u|^{p-2}\frac{\partial \overline{u}}{\partial x_k} + \frac{1}{2}(p-2)\overline{u}|u|^{p-4}\Big(\overline{u}\frac{\partial u}{\partial x_k} + u\frac{\partial \overline{u}}{\partial x_k}\Big).$$

Setting

$$|u|^{\frac{p-4}{2}}\overline{u}\frac{\partial u}{\partial x_k} = a_k + ib_k$$

with a_k, $b_k \in \mathbb{R}$, we have

$$\begin{aligned}
&\int_\Omega \sum_{k=1}^n \frac{\partial u}{\partial x_k}\,\frac{\partial}{\partial x_k}\Big(|u|^{p-2}\overline{u}\Big)\,dx \\
&= \int_\Omega \sum_{k=1}^n \Big((|u|^{\frac{p-4}{2}})^2 u\overline{u}\frac{\partial u}{\partial x_k}\frac{\partial \overline{u}}{\partial x_k} + \frac{p-2}{2}(|u|^{\frac{p-4}{2}})^2\overline{u}\frac{\partial u}{\partial x_k}\Big(\overline{u}\frac{\partial u}{\partial x_k} + u\frac{\partial \overline{u}}{\partial x_k}\Big)\Big)\,dx \\
&= \int_\Omega \sum_{k=1}^n \Big(a_k^2 + b_k^2 + (p-2)a_k(a_k + ib_k)\Big)\,dx,
\end{aligned}$$

whence

$$\lambda\|u\|^p + \int_\Omega \sum_{k=1}^n ((p-1)a_k^2 + b_k^2)dx + i(p-2)\int_\Omega \sum_{k=1}^n a_k b_k\,dx = \int_\Omega f|u|^{p-2}\overline{u}\,dx.$$

Taking the real part we get

$$\mathrm{Re}\,\lambda\|u\|^p + \int_\Omega \sum_{k=1}^n ((p-1)a_k^2 + b_k^2)dx = \ \mathrm{Re}\int_\Omega f|u|^{p-2}\overline{u}\,dx \leq \|f\|_p\,\|u\|_p^{p-1},$$

and then

$$\begin{cases} (a) & \mathrm{Re}\,\lambda\|u\| \leq \|f\|. \\ (b) & \displaystyle\int_\Omega \sum_{k=1}^n ((p-1)a_k^2 + b_k^2)dx \leq \|f\|\,\|u\|^{p-1}. \end{cases}$$

Taking the imaginary part we get

$$\mathrm{Im}\,\lambda\,\|u\|^p + (p-2)\int_\Omega \sum_{k=1}^n a_k b_k\,dx = \ \mathrm{Im}\int_\Omega f|u|^{p-2}\overline{u}\,dx$$

and then

$$|\mathrm{Im}\,\lambda|\,\|u\|^p \leq \frac{p-2}{2}\int_\Omega \sum_{k=1}^n (a_k^2 + b_k^2)dx + \|f\|\,\|u\|^{p-1},$$

so that, using (b),

$$|\mathrm{Im}\,\lambda|\,\|u\|^p \leq \Big(\frac{p-2}{2} + 1\Big)\|f\|\,\|u\|^{p-1},$$

i.e.,

$$|\mathrm{Im}\,\lambda|\,\|u\| \leq \frac{p}{2}\|f\|.$$

From this inequality and from (a), squaring and summing up, we obtain

$$|\lambda|^2\|u\|^2 \le \left(1+\frac{p^2}{4}\right)\|f\|^2,$$

and the statement follows. □

2.5 More general operators

Let us consider general second order elliptic operators, both in $\mathbb{R}^N$ and in a bounded open set Ω with C^2 boundary $\partial\Omega$. Let us denote by $\nu(x)$ the outer unit vector normal to $\partial\Omega$ at x.

Let $\mathcal{A}$ be the differential operator

$$(\mathcal{A}u)(x) = \sum_{i,j=1}^N a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^N b_i(x)D_iu(x) + c(x)u(x) \tag{2.16}$$

with real, uniformly continuous coefficients and bounded a_{ij}, b_i, c on $\overline{\Omega}$. We assume that for every $x \in \overline{\Omega}$ the matrix $[a_{ij}(x)]_{i,j=1,\dots,N}$ is symmetric and strictly positive definite, i.e.,

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \ge \nu|\xi|^2, \ \ x \in \overline{\Omega},\ \xi \in \mathbb{R}^n, \tag{2.17}$$

for some $\nu > 0$. The following results hold.

Theorem 2.5.1 (S. Agmon, [1]) *Let $p \in (1,\infty)$.*

(i) Let $A_p : W^{2,p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ be defined by $(A_pu)(x) = (\mathcal{A}u)(x)$. The operator A_p is sectorial in $L^p(\mathbb{R}^N)$.

(ii) Let Ω and $\mathcal{A}$ be as above, and let A_p be defined by

$$D(A_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \ \ (A_pu)(x) = (\mathcal{A}u)(x).$$

Then, the operator A_p is sectorial in $L^p(\Omega)$, and $D(A_p)$ is dense in $L^p(\Omega)$.

(iii) Let Ω and $\mathcal{A}$ be as above, and let A_p be defined by

$$D(A_p) = \{u \in W^{2,p}(\Omega) : \mathcal{B}u_{|\partial\Omega} = 0\}, A_pu = \mathcal{A}u, \ \ u \in D_p(A),$$

where

$$\mathcal{B}u = b_0(x)u(x) + \sum_{i=1}^N b_i(x)D_iu(x),$$

the coefficients b_i, $i = 1,\dots,N$ are in $C^1(\overline{\Omega})$ and the transversality condition

$$\sum_{i=1}^n b_i(x)\nu_i(x) \ne 0, \ \ x \in \partial\Omega$$

holds. Then, the operator A_p is sectorial in $L^p(\Omega)$, and $D(A_p)$ is dense in $L^p(\Omega)$.

We have also the following result.

Theorem 2.5.2 (H. B. Stewart, [15, 16]) *Let $\mathcal{A}$ be the differential operator in (2.16).*

(i) Consider the operator $A : D(A) \to X = C_b(\mathbb{R}^N)$ defined by

$$\begin{aligned} D(A) &= \{u \in C_b(\mathbb{R}^N) \cap_{p\ge 1} W^{2,p}_{loc}(\mathbb{R}^N) : \mathcal{A}u \in C_b(\mathbb{R}^N)\}, \\ (Au)(x) &= (\mathcal{A}u)(x), \ \ u \in D(A). \end{aligned} \tag{2.18}$$

Then, A is sectorial in X, and $\overline{D(A)} = BUC(\mathbb{R}^N)$.

(ii) Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with C^2 boundary $\partial\Omega$, and consider the operator

$$\begin{aligned} D(A) &= \{u \in \cap_{p\ge 1} W^{2,p}(\Omega) : u_{|\partial\Omega} = 0, \ \mathcal{A}u \in C(\overline{\Omega})\}, \\ (Au)(x) &= (\mathcal{A}u)(x), \ \ u \in D(A). \end{aligned} \tag{2.19}$$

Then, the operator A is sectorial in X, and $\overline{D(A)} = C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : \ u = 0 \text{ at } \partial\Omega\}$.

(iii) Let Ω be as in (ii), and let $X = C(\overline{\Omega})$,

$$\begin{aligned} D(A) &= \{u \in \cap_{p\ge 1} W^{2,p}(\Omega) : \mathcal{B}u_{|\partial\Omega} = 0, \ \mathcal{A}u \in C(\overline{\Omega})\}, \\ (Au)(x) &= (\mathcal{A}u)(x), \ \ u \in D(A), \end{aligned} \tag{2.20}$$

where

$$\mathcal{B}u = b_0(x)u(x) + \sum_{i=1}^N b_i(x)D_iu(x),$$

the coefficients b_i, $i = 1, \dots, N$ are in $C^1(\overline{\Omega})$ and the transversality condition

$$\sum_{i=1}^n b_i(x)\nu_i(x) \neq 0, \ \ x \in \partial\Omega$$

holds. Then, the operator A is sectorial in X, and $D(A)$ is dense in X.

Moreover, in all the cases above there is $M > 0$ such that $\lambda \in S_{\theta,\omega}$ implies

$$\|D_iR(\lambda, A)f\|_\infty \le \frac{M}{|\lambda|^{1/2}} \|f\|_\infty, \ \ \forall f \in X, \ i = 1, \dots, n. \tag{2.21}$$

Exercises 2.5.3

1. Consider again the operator $u \mapsto u''$ in I as in subsection 2.1.2, with the domains $D(A_p)$ defined there, $1 \le p \le \infty$. Solving explicitly the differential equation $\lambda u - u'' = f$ in $D(A_p)$, show that the eigenvalues are $-n^2\pi^2$, $n \in \mathbb{N}$, and express the resolvent as an integral operator. Then, estimate the kernel of this operator to get

$$\|R(\lambda, A_p)\|_{\mathcal{L}(X)} \le \frac{1}{|\lambda|\cos(\theta/2)}, \quad \theta = \arg\lambda, \quad X = L^p(I) \text{ or } X = C(\overline{I}).$$

2. Consider the operator $Au = u''$ in $L^p(I)$, with the domain

$$D(A_p) = \{u \in W^{2,p}(I) : \ u'(0) = u'(1) = 0\} \subset L^p(I), \qquad 1 \le p < \infty,$$

or

$$D(A_\infty) = \{u \in C^2(I) \cap C(\overline{I}) : \ u'(0) = u'(1) = 0\} \subset C(\overline{I}),$$

corresponding to the Neumann boundary condition. Use the same perturbation argument as in subsection 2.1.2 to show that it is sectorial.

3. Use the properties of the Fourier transform and formula (2.5) that defines the heat semigroup $T(t)$ to check that $T(t+s)f(x) = T(t)T(s)f(x)$ for all $f \in \mathcal{S}(\mathbb{R}^N)$ and $t, s \geq 0$, $x \in \mathbb{R}^N$. By approximation, show that this is true for each $f \in L^p(\mathbb{R}^N)$ and for each $f \in C_b(\mathbb{R}^N)$.

4. Use the Fourier transform to prove the resolvent estimate for the Laplacian in $L^2(\mathbb{R}^N)$, $\|u\|_{L^2(\mathbb{R}^N)} \leq \|f\|_{L^2(\mathbb{R}^N)}/|\lambda|$, where $\lambda u - \Delta u = f$, $\pi/2 < \arg\lambda < \pi$.

5. Prove that the heat semigroup is analytic in $X = L^p(\mathbb{R}^N)$, $1 \leq p < \infty$, and in $X = C_b(\mathbb{R}^N)$, showing that $\|d/dt\, T(t)\|_{\mathcal{L}(X)} \leq c/t$. If $X = C_b(\mathbb{R}^N)$, show that $T(t)$ is one to one for each $t > 0$.

6. Show that the dissipativity condition (2.12) is equivalent to Re $\langle Ax, x\rangle \leq 0$ for all $x \in D(A)$.

7. Show that if $p \geq 2$ and $u \in W^{1,p}(\Omega)$ then the function $|u|^{p-2}u$ belongs to $W^{1,p'}(\Omega)$. Is this true for $1 < p < 2$?

8. (a) Using the representation formula (2.5), prove the following estimates for the heat semigroup $T(t)$ in $L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$:

$$\|D^\alpha T(t)f\|_{L^p(\mathbb{R}^N)} \leq \frac{c_\alpha}{t^{|\alpha|/2}}\|f\|_{L^p(\mathbb{R}^N)}$$

   for every multiindex α, $1 \leq p \leq \infty$ and suitable constants c_α.

   (b) Use the fact that $D_i G_t$ is odd with respect to x_i to prove that for each $f \in C^\theta(\mathbb{R}^N)$, $0 < \theta < 1$, and for each $i = 1, \ldots, N$

$$\|D_i T(t)f\|_\infty \leq \frac{C}{t^{1/2-\theta/2}}[f]_{C^\theta(\mathbb{R}^N)}, \quad t > 0.$$

   (c) Use the estimates in (a) for $|\alpha| = 1$ to prove that

$$\|D_i u\|_X \leq C_1 t^{1/2}\|\Delta u\|_X + C_2 t^{-1/2}\|u\|_X, \quad t > 0,$$

$$\|D_i u\|_X \leq C_3 \|\Delta u\|_X^{1/2}\|u\|_X^{1/2},$$

   for $X = L^p(\mathbb{R}^N)$, $1 \leq p < \infty$, $X = C_b(\mathbb{R}^N)$, and u in the domain of the Laplacian in X.

9. Prove the following generalization of proposition 2.3.2: *Let A be a linear operator such that the resolvent set $\rho(A)$ contains two halfplanes* Re $\lambda > \omega$ *and* Re $\lambda, -\omega$, *with* $\omega \geq 0$, *and there exists $M > 0$ such that $\|R(\lambda, A)\| \leq M/(Re\,\lambda - \omega)$ for Re $\lambda > \omega$ and $\|R(\lambda, A)\| \leq M/(\omega -$ Re $\lambda)$ for Re $\lambda < -\omega$. Then A^2 is sectorial, with any $\theta < \pi$.*

10. Show that the operator $A : D(A) = \{f \in C_b(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\}) : x \mapsto xf'(x) \in C_b(\mathbb{R}), \lim_{x\to 0} xf'(x) = 0\}$, $Af(x) = xf'(x)$ for $x \neq 0$, $Af(0) = 0$, satisfies the assumptions of proposition 2.3.2, so that A^2 is sectorial in $C_b(\mathbb{R})$. Using the results of the exercises 1.2.18 to prove that for each a, $b \in \mathbb{R}$ a suitable realization of the operator $\mathcal{A}$ defined by $(\mathcal{A}f)(x) = x^2 f''(x) + axf'(x) + bf(x)$ is sectorial.

Chapter 3

Intermediate spaces

3.1 The interpolation spaces $D_A(\theta,\infty)$

Let $A : D(A) \subset X \to X$ be a sectorial operator, and set

$$M_0 = \sup_{0<t\leq 1} \|e^{tA}\|, \quad M_1 = \sup_{0<t\leq 1} \|tAe^{tA}\|.$$

We have seen in proposition 1.2.6 that for all $x \in \overline{D(A)}$ the function $t \mapsto u(t) = e^{tA}x$ belongs to $C([0,T];X)$, and for all $x \in D(A)$ such that $Ax \in \overline{D(A)}$, it belongs to $C^1([0,T];X)$. We also know that for $x \in X$ the function $t \mapsto v(t) = \|Ae^{tA}x\|$ has in general a singularity of order 1 as $t \to 0$, whereas for $x \in D(A)$ it is bounded near 0. It is then natural to raise the following related questions:

1. Is there a class of initial data such that the function $u(t) = e^{tA}x$ has an intermediate regularity, e.g., it is α-Hölder continuous for some $0 < \alpha < 1$?

2. Is there a class of initial data x such that the function $t \mapsto \|Ae^{tA}x\|$ has a singularity of order α, with $0 < \alpha < 1$?

To answer such questions, we introduce some intermediate Banach spaces between X and $D(A)$.

Definition 3.1.1 *Let $A : D(A) \subset X \to X$ be a sectorial operator, and fix $0 < \alpha < 1$. Let us set*

$$\begin{cases} D_A(\alpha,\infty) = \{x \in X : \ [x]_\alpha = \sup_{0<t\leq 1} \|t^{1-\alpha}Ae^{tA}x\| < \infty\}, \\ \\ \|x\|_{D_A(\alpha,\infty)} = \|x\| + [x]_\alpha. \end{cases}$$

Note that what characterizes $D_A(\alpha,\infty)$ is the behavior of $\|t^{1-\alpha}Ae^{tA}x\|$ near $t = 0$. Indeed, for $0 < a < b < \infty$ and for each $x \in X$ estimate (1.14) with $k = 1$ implies that $\sup_{a\leq t\leq b} \|t^{1-\alpha}Ae^{tA}x\| \leq C\|x\|$, with $C = C(a,b,\alpha)$. Therefore, the interval $(0,1]$ in the definition of $D_A(\alpha,\infty)$ could be replaced by any $(0,T]$ with $T > 0$, and for each $T > 0$ the norm $x \mapsto \|x\| + \sup_{0<t\leq T} \|t^{1-\alpha}Ae^{tA}x\|$ is equivalent to the $D_A(\alpha,\infty)$ norm in $D_A(\alpha,\infty)$.

Once we have an estimate for the norm $\|Ae^{tA}\|_{\mathcal{L}(D_A(\alpha,\infty);X)}$ we get estimates for the norms $\|A^k e^{tA}\|_{\mathcal{L}(D_A(\alpha,\infty);X)}$ with any $k \in \mathbb{N}$ just using the semigroup law and (1.14). For instance for $k = 2$ and for each $x \in D_A(\alpha,\infty)$ we obtain

$$\sup_{0<t\leq T} \|t^{2-\alpha}A^2e^{tA}x\| \leq \sup_{0<t\leq T} \|tAe^{t/2\,A}\|_{\mathcal{L}(X)}\|t^{1-\alpha}Ae^{t/2\,A}x\| \leq C\|x\|_{D_A(\alpha,\infty)}.$$

It is clear that if $x \in D_A(\alpha,\infty)$ and $T > 0$, then the function $s \mapsto \|Ae^{sA}x\|$ belongs to $L^1(0,T)$, so that, by proposition 1.2.6(ii),

$$e^{tA}x - x = \int_0^t Ae^{sA}x\,ds \ \ \forall t \geq 0, \quad x = \lim_{t\to 0} e^{tA}x.$$

In particular, all the spaces $D_A(\alpha,\infty)$ are contained in the closure of $D(A)$. It follows that
$$D_A(\alpha,\infty) = D_{A_0}(\alpha,\infty),$$
where A_0 is the part of A in $\overline{D(A)}$ (see definition 1.2.7).

Proposition 3.1.2 *For $0<\alpha<1$ the equality*
$$D_A(\alpha,\infty) = \{x \in X:\ [[x]]_{D_A(\alpha,\infty)} = \sup_{0<t\le 1} t^{-\alpha}\|e^{tA}x - x\| < \infty\}$$
holds, and the norm
$$x \mapsto \|x\| + [[x]]_{D_A(\alpha,\infty)}$$
is equivalent to the norm of $D_A(\alpha,\infty)$.

Proof. Let $x\in D_A(\alpha,\infty)$ be given. For $0<t\le 1$ we have
$$t^{-\alpha}(e^{tA}x - x) = t^{-\alpha}\int_0^t s^{1-\alpha}Ae^{sA}x\frac{1}{s^{1-\alpha}}ds, \tag{3.1}$$
so that
$$[[x]]_{D_A(\alpha,\infty)} = \|t^{-\alpha}(e^{tA}x-x)\|_{L^\infty(0,1)} \le \alpha^{-1}[x]_{D_A(\alpha,\infty)}, \tag{3.2}$$
Conversely, let $[[x]]_{D_A(\alpha,\infty)}<\infty$, and write
$$Ae^{tA}x = Ae^{tA}\frac{1}{t}\int_0^t (x - e^{sA}x)ds + e^{tA}\frac{1}{t}A\int_0^t e^{sA}xds.$$
It follows
$$\|t^{1-\alpha}Ae^{tA}x\| \le t^{1-\alpha}\frac{M_1}{t^2}\int_0^t s^\alpha\frac{\|x-e^{sA}x\|}{s^\alpha}ds + M_0t^{-\alpha}\|e^{tA}x - x\|, \tag{3.3}$$
and the function $s\mapsto \|x-e^{sA}x\|/s^\alpha$ is bounded, so that $t\mapsto t^{1-\alpha}Ae^{tA}x$ is bounded, too, and
$$\|t^{1-\alpha}Ae^{tA}x\|_{L^\infty(0,1)} = [x]_{D_A(\alpha,\infty)} \le (M_1(\alpha+1)^{-1} + M_0)[[x]]_{D_A(\alpha,\infty)} \tag{3.4}$$
We can conclude that the seminorms $[\,\cdot\,]_{D_A(\alpha,\infty)}$ and $[[\,\cdot\,]]_{D_A(\alpha,\infty)}$ are equivalent. □

From the semigroup law the next corollary follows, and it gives an answer to the first question at the beginning of this section.

Corollary 3.1.3 *Given $x\in X$, the function $t\mapsto e^{tA}x$ belongs to $C^\alpha([0,1];X)$ if and only if x belongs to $D_A(\alpha,\infty)$. In this case, $t\mapsto e^{tA}x$ belongs to $C^\alpha([0,T];X)$ for every $T>0$.*

Proof. The proof follows from the equality
$$e^{tA}x - e^{sA}x = e^{sA}(e^{(t-s)A}x - x),\ \ 0\le s<t,$$
recalling that $\|e^{\xi A}\|_{\mathcal{L}(X)}$ is bounded by a constant independent of ξ if ξ runs in any bounded interval. □

It is easily seen that the spaces $D_A(\alpha,\infty)$ are Banach spaces. Moreover, it can be proved that they do not depend explicitly on the operator A, but only on its domain $D(A)$ and on the graph norm of A. More precisely, for every sectorial operator $B: D(B)\to X$ such that $D(B)=D(A)$, with equivalent graph norms, the equality $D_A(\alpha,\infty)=D_B(\alpha,\infty)$ holds, with equivalent norms.

An important feature of spaces $D_A(\alpha,\infty)$ is that the part of A in $D_A(\alpha,\infty)$, defined by
$$\begin{cases} D(A_\alpha) = D_A(\alpha+1,\infty) := \{x\in D(A):\ Ax\in D_A(\alpha,\infty)\},\\[2ex] A_\alpha: D_A(\alpha+1,\infty)\to D_A(\alpha,\infty),\ \ A_\alpha x = Ax,\end{cases}$$
is a sectorial operator.

Proposition 3.1.4 *For $0 < \alpha < 1$ the resolvent set of A_α contains $\rho(A)$, $R(\lambda, A_\alpha)$ is the restriction of $R(\lambda, A)$ to $D_A(\alpha, \infty)$, and the inequality*

$$\|R(\lambda, A_\alpha)\|_{\mathcal{L}(D_A(\alpha,\infty))} \le \|R(\lambda, A)\|_{\mathcal{L}(X)}$$

holds for every $\lambda \in \rho(A)$. In particular, A_α is a sectorial operator in $D_A(\alpha, \infty)$.

Proof. Fix $\lambda \in \rho(A)$ and $x \in D_A(\alpha, \infty)$. The resolvent equation $\lambda y - Ay = x$ has a unique solution $x \in D(A)$, and since $D(A) \subset D_A(\alpha, \infty)$ then $Ay \in D_A(\alpha, \infty)$ and therefore $y = R(\lambda, A)x \in D_A(\alpha + 1, \infty)$.

Moreover for $0 < t \le 1$ the equality

$$\|t^{1-\alpha} A e^{tA} R(\lambda, A)x\| = \|R(\lambda, A) t^{1-\alpha} A e^{tA} x\| \le \|R(\lambda, A)\|_{\mathcal{L}(X)} \|t^{1-\alpha} A e^{tA} x\|$$

holds. Therefore,

$$[R(\lambda, A)x]_{D_A(\alpha,\infty)} \le \|R(\lambda, A)\|_{\mathcal{L}(X)} [x]_{D_A(\alpha,\infty)},$$

and the claim is proved. □

From corollary 3.1.3 it follows that the function $t \mapsto U(t) := e^{tA}x$ belongs to $C^\alpha([0, 1]; D(A))$ (and then to $C^\alpha([0, T]; D(A))$ for all $T > 0$) if and only if x belongs to $D_A(\alpha+1, \infty)$. Similarly, since $\frac{d}{dt} e^{tA}x = e^{tA}Ax$ for $x \in D(A)$, U belongs to $C^{1+\alpha}([0, 1]; X)$ (and then to $C^{1+\alpha}([0, T]; X)$ for all $T > 0$) if and only if x belongs to $D_A(\alpha + 1, \infty)$.

Let us see an interpolation property of the spaces $D_A(\alpha, \infty)$.

Proposition 3.1.5 *For every $x \in D(A)$ we have*

$$[x]_{D_A(\alpha,\infty)} \le M_0^\alpha M_1^{1-\alpha} \|Ax\|^\alpha \|x\|^{1-\alpha}.$$

Proof. For all $t \in (0, 1)$ we have

$$\|t^{1-\alpha} A e^{tA} x\| \le \begin{cases} M_0 t^{1-\alpha} \|Ax\|, \\ \\ M_1 t^{-\alpha} \|x\|. \end{cases}$$

It follows

$$\|t^{1-\alpha} A e^{tA} x\| \le (M_0 t^{1-\alpha} \|Ax\|)^\alpha (M_1 t^{-\alpha} \|x\|)^{1-\alpha} = M_0^\alpha M_1^{1-\alpha} \|Ax\|^\alpha \|x\|^{1-\alpha}.$$

□

Definition 3.1.6 *Given three Banach spaces $Z \subset Y \subset X$ (with continuous embeddings), and given $\alpha \in (0, 1)$, we say that Y is of class J_α between X and Z if there is $C > 0$ such that*

$$\|y\|_Y \le C \|y\|_Z^\alpha \|y\|_X^{1-\alpha}, \ \ \forall y \in Z.$$

From proposition 3.1.5 it follows that for all $\alpha \in (0, 1)$ the space $D_A(\alpha, \infty)$ is of class J_α between X and the domain of A. Another example is already in chapter 2; estimate (2.11) implies that $H^1(\mathbb{R}^N)$ is in the class $J_{1/2}$ between $L^2(\mathbb{R}^N)$ and the domain of the Laplacian, i.e. $H^2(\mathbb{R}^N)$. Arguing similarly (see exercises 2.5.3) we obtain that $W^{1,p}(\mathbb{R}^N)$ is in the class $J_{1/2}$ between $L^p(\mathbb{R}^N)$ and $W^{2,p}(\mathbb{R}^N)$ for each $p \in [1, \infty)$, and that $C_b^1(\mathbb{R}^N)$ is in the class $J_{1/2}$ between $C_b(\mathbb{R}^N)$ and the domain of the Laplacian in $C_b(\mathbb{R}^N)$.

Let us discuss in detail a fundamental example.

Example 3.1.7 *Let us consider $X = C_b(\mathbb{R}^N)$, and let $A : D(A) \mapsto X$ be the realization of the Laplacian in X. For $0 < \alpha < 1$, $\alpha \neq 1/2$, we have*

$$D_A(\alpha, \infty) = C^{2\alpha}(\mathbb{R}^N), \tag{3.5}$$

$$D_A(\alpha + 1, \infty) = C^{2\alpha+2}(\mathbb{R}^N), \tag{3.6}$$

with equivalence of the respective norms.

Proof. We prove the statement for $\alpha < 1/2$.

Recall that the heat semigroup is given by (2.5), which we rewrite for convenience:

$$(T(t)f)(x) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} f(y)dy, \;\; t > 0, \; x \in \mathbb{R}^N.$$

Differentiating we obtain

$$(DT(t)f)(x) = -\frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} \frac{x-y}{2t} e^{-\frac{|x-y|^2}{4t}} f(y)dy,$$

and hence

$$\| \, |DT(t)f| \, \|_\infty \leq \frac{c}{\sqrt{t}} \|f\|_\infty$$

for some $c > 0$ (see exercise 2.5.3(8)).

Let us first prove the inclusion $D_A(\alpha, \infty) \supset C^{2\alpha}(\mathbb{R}^N)$.

For $f \in C^{2\alpha}(\mathbb{R}^N)$ we denote by

$$[f]_{2\alpha} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^{2\alpha}}$$

the Hölder seminorm of f, and we write

$$T(t)f(x) - f(x) = \frac{1}{(4\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4}} \left[f(x - \sqrt{t}y) - f(x) \right] dy,$$

hence

$$\|T(t)f - f\|_\infty \leq \frac{1}{(4\pi)^{N/2}} [f]_{2\alpha} t^\alpha \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4}} |y|^{2\alpha} dy$$

and therefore $[[f]]_{D_A(\alpha,\infty)} \leq c[f]_{2\alpha}$.

Conversely, let $f \in D_A(\alpha, \infty)$. Then, for every $t > 0$ we have

$$\begin{aligned} |f(x) - f(y)| &\leq |T(t)f(x) - f(x)| + |T(t)f(x) - T(t)f(y)| + |T(t)f(y) - f(y)| && (3.7) \\ &\leq 2[[f]]_{D_A(\alpha,\infty)} t^\alpha + \| \, |DT(t)f| \, \|_\infty |x-y|. && (3.8) \end{aligned}$$

The estimate $\| \, |DT(t)f| \, \|_\infty \leq ct^{-1/2} \|f\|_\infty$, that we already know, is not sufficient for our purpose. To get a better estimate we use the equality

$$T(n)f - T(t)f = \int_t^n AT(s)f \, ds, \;\; 0 < t < n,$$

that implies, for each $i = 1, \ldots, N$,

$$D_i T(n)f - D_i T(t)f = \int_t^n D_i AT(s)f \, ds, \;\; 0 < t < n.$$

Using the estimate

$$\begin{aligned} \|D_i AT(s)f\|_\infty &= \|D_i T(s/2) AT(s/2) f\|_\infty \leq \|D_i T(s/2)\|_{\mathcal{L}(C_b(\mathbb{R}^N))} \|AT(s/2)f\|_\infty \\ &\leq \frac{C}{s^{3/2-\alpha}} \|f\|_{D_A(\alpha,\infty)} \end{aligned}$$

we see that we may let $n \to \infty$ to get

$$D_i T(t)f = -\int_t^\infty D_i A T(s) f\, ds, \quad t > 0,$$

and

$$\|D_i T(t)f\|_\infty \le \int_t^\infty \frac{C}{s^{3/2-\alpha}}\, ds \|f\|_{D_A(\alpha,\infty)} = \frac{C}{(1/2-\alpha)t^{1/2-\alpha}} \|f\|_{D_A(\alpha,\infty)}.$$

This estimate is what we need for (3.7) to yield 2α-Hölder continuity of f. For $|x-y| \le 1$ choose $t = |x-y|^2$ to get

$$\begin{aligned} |f(x) - f(y)| &\le 2[[f]]_{D_A(\alpha,\infty)} |x-y|^{2\alpha} + c\|f\|_\infty |x-y|^{2\alpha} \\ &\le C\|f\|_{D_A(\alpha,\infty)} |x-y|^{2\alpha}. \end{aligned}$$

If $|x-y| \ge 1$ then $|f(x) - f(y)| \le 2\|f\|_\infty \le 2\|f\|_{D_A(\alpha,\infty)} |x-y|^{2\alpha}$.

Let us prove (3.6). The embedding $C^{2\alpha+2}(\mathbb{R}^N) \subset D_A(\alpha+1,\infty)$ is an obvious consequence of (3.5). To prove the other embedding we have to show that the functions in $D_A(\alpha+1,\infty)$ have second order derivatives belonging to $C^{2\alpha}(\mathbb{R}^N)$.

Fix any $\lambda > 0$ and any $f \in D_A(\alpha+1,\infty)$. Then $f = R(\lambda, A)g$ where $g := \lambda f - \Delta f \in D_A(\alpha,\infty) = C^{2\alpha}(\mathbb{R}^N)$, and

$$f(x) = \int_0^\infty e^{-\lambda t} (T(t)g)(x) dt, \quad x \in \mathbb{R}^N.$$

We can differentiate twice with respect to x, because for each i, $j = 1, \ldots, N$ the functions $t \mapsto \|e^{-\lambda t} D_i T(t) g\|_\infty$ and $t \mapsto \|e^{-\lambda t} D_{ij} T(t) g\|_\infty$ are integrable in $(0,\infty)$. Indeed, arguing as above we get $\|D_i T(t) g\|_\infty \le c_{2\alpha}[g]_{2\alpha}/t^{1/2-\alpha}$ for every i (see again exercise 2.5.3(8)), so that

$$\|D_{ij} T(t) g\|_\infty = \|D_j T(t/2) D_i T(t/2) g\|_\infty \le \frac{c}{t/2} \frac{c_{2\alpha}}{(t/2)^{1/2-\alpha}} [g]_{2\alpha} = \frac{C}{t^{1-\alpha}} [g]_{2\alpha}. \tag{3.9}$$

Therefore, the integral $\int_0^\infty e^{-\lambda t} T(t) g dt$ is well defined as a $C^2_b(\mathbb{R}^N)$- valued integral, and $f \in C^2_b(\mathbb{R}^N)$. We may go on estimating $[D_{ij} T(t) g]_{2\alpha}$, but we get $[D_{ij} T(t) g]_{2\alpha} \le C[u]_{2\alpha}/t$, and therefore it is not obvious that the integral is well defined as a $C^{2\alpha}$-valued integral. So, we have to follow another way. Since we already know that $D_A(\alpha,\infty) = C^{2\alpha}(\mathbb{R}^N)$, it is sufficient to prove that $D_{ij} f \in D_A(\alpha,\infty)$, i.e. that

$$\sup_{0<\xi\le 1} \|\xi^{1-\alpha} A T(\xi) D_{ij} f\|_\infty < \infty, \quad i,j = 1, \ldots, n.$$

For $0 < \xi \le 1$ it holds

$$\begin{aligned} \|\xi^{1-\alpha} A T(\xi) D_{ij} f\|_\infty &= \left\| \int_0^{+\infty} \xi^{1-\alpha} e^{-\lambda t} A T(\xi + t/2) D_{ij} T(t/2) g\, dt \right\|_\infty \\ \le \int_0^{+\infty} \xi^{1-\alpha} \frac{M_1 C}{(\xi + t/2)(t/2)^{1-\alpha}} dt\, [g]_{2\alpha} &= \int_0^{+\infty} \frac{2M_1 C}{(1+s)s^{1-\alpha}} ds\, [g]_{2\alpha}, \end{aligned} \tag{3.10}$$

where $M_1 = \sup_{t>0} \|tAT(t)\|_{L(C_b(\mathbb{R}^N))}$, and C is the constant in formula (3.9). Therefore, all the second order derivatives of f are in $D_A(\alpha,\infty) = C^{2\alpha}(\mathbb{R}^N)$, their $C^{2\alpha}$ norm is bounded by $C[g]_{2\alpha} \le C(\lambda[f]_{2\alpha} + [\Delta f]_{2\alpha}) \le \max\{\lambda C, C\} \|f\|_{D_A(\alpha+1,\infty)}$, and the statement follows. □

Remark 3.1.8 The case $\alpha = 1/2$ is more delicate. In fact, the inclusion $Lip(\mathbb{R}^N) \subset D_A(1/2, \infty)$ follows as in the first part of the proof, but it is strict. Indeed, it is possible to prove that

$$D_A(1/2, \infty) = \left\{ u \in C_b(\mathbb{R}^N) : \sup_{x \neq y} \frac{|u(x) + u(y) - 2u((x+y)/2)|}{|x-y|} < \infty \right\},$$

and this space is strictly larger than $Lip(\mathbb{R}^N)$ (see [18]).

Example 3.1.7 and corollary 3.1.3 imply that the solution $u(t,x) = (T(t)u_0)(x)$ of the Cauchy problem for the heat equation in $\mathbb{R}^N$,

$$\begin{cases} u_t(t,x) = \Delta u_{xx}(t,x), \;\; t > 0, \; x \in \mathbb{R}^N, \\ \\ u(0,x) = u_0(x), \;\; x \in \mathbb{R}^N, \end{cases}$$

is α-Hölder continuous with respect to t on $[0,T] \times \mathbb{R}^N$ (with Hölder constant independent of x) if and only if the initial datum u_0 belongs to $C^{2\alpha}(\mathbb{R}^N)$. In this case, proposition 3.1.4 implies that $\|u(t,\cdot)\|_{D_A(\alpha,\infty)} \leq C\|u_0\|_{D_A(\alpha,\infty)}$ for $0 \leq t \leq T$, so that u is 2α-Hölder continuous with respect to x as well, with Hölder constant independent of t. We say that u belongs to the parabolic Hölder space $C^{\alpha,2\alpha}([0,T] \times \mathbb{R}^N)$, for all $T > 0$.

Moreover, example 3.1.7 gives us an alternative proof of the classical Schauder Theorem for the Laplacian (see e.g. [7, ch. 6]): if $u \in C^2_b(\mathbb{R}^N)$ and $\Delta u \in C^\theta(\mathbb{R}^N)$ for some $\theta \in (0,1)$, then $u \in C^{2+\theta}(\mathbb{R}^N)$.

Proposition 3.1.4 implies that for every $\theta \in (0,1)$ the operator

$$B : D(B) = D_\Delta(\theta/2 + 1, \infty) = C^{2+\theta}(\mathbb{R}^N) \to D_\Delta(\theta/2, \infty) = C^\theta(\mathbb{R}^N), \quad Bu = \Delta u$$

is sectorial in $C^\theta(\mathbb{R}^N)$.

A characterization of the spaces $D_A(\alpha, \infty)$ for general second order elliptic operators is similar to the above one, but the proof is less elementary since it relies on the deep results of theorem 2.5.2 and on general interpolation techniques.

Theorem 3.1.9 *Let $\alpha \in (0,1)$, $\alpha \neq 1/2$. The following statements hold.*

(i) Let $X = C_b(\mathbb{R}^N)$, and let A be defined by (2.18). Then, $D_A(\alpha, \infty) = C^{2\alpha}(\mathbb{R}^n)$, with equivalence of the norms.

(ii) Let Ω be an open bounded set of $\mathbb{R}^N$ with C^2 boundary, let $X = C(\overline{\Omega})$, and let A be defined by (2.19). Then,

$$D_A(\alpha, \infty) = C^{2\alpha}_0(\overline{\Omega}) = \{ f \in C^{2\alpha}(\overline{\Omega}) : f_{|\partial\Omega} = 0 \},$$

with equivalence of the norms.

(iii) Let Ω be an open bounded set of $\mathbb{R}^N$ with C^2 boundary, let $X = C(\overline{\Omega})$, and let A be defined by (2.20). Then $D_A(\alpha, \infty) = C^{2\alpha}(\overline{\Omega})$ if $0 < \alpha < 1/2$,

$$D_A(\alpha, \infty) = \{ f \in C^{2\alpha}(\overline{\Omega}) : \mathcal{B}f_{|\partial\Omega} = 0 \}$$

if $1/2 < \alpha < 1$, with equivalence of the norms.

Exercises 3.1.10

1. Show that if $\omega < 0$ in definition (1.2.1) then $D_A(\alpha,\infty) = \{x \in X : \ |x|_\alpha = \sup_{t>0} \|t^{1-\alpha}Ae^{tA}x\| < \infty\}$, and that $x \mapsto |x|_\alpha$ is an equivalent norm in $D_A(\alpha,\infty)$ for each $\alpha \in (0,1)$. What about $\omega = 0$?

2. Show that $D_A(\alpha,\infty)$ is a Banach space.

3. Show that the closure of $D(A)$ in $D_A(\alpha,\infty)$ is the subspace of all $x \in X$ such that $\lim_{t\to 0} t^{1-\alpha}Ae^{tA}x = 0$. This implies that, even if $D(A)$ is dense in X, it is not necessarily dense in $D_A(\alpha,\infty)$.

   [Hint: to prove that $e^{tA}x - x$ goes to zero in $D_A(\alpha,\infty)$ provided $t^{1-\alpha}Ae^{tA}x$ goes to zero as $t \to 0$, use formula (1.16) and split the sup over $(0,1]$ in the definition of $[\,\cdot\,]_\alpha$ into the sup over $(0,\varepsilon]$ and over $[\varepsilon,1]$, ε small.]

4. Prove that for every $\theta \in (0,1)$ there is $C = C(\theta) > 0$ such that

   $$\|D_i\varphi\|_\infty \le C(\|\varphi\|_{C^{2+\theta}(\mathbb{R}^n)})^{(1-\theta)/2}(\|\varphi\|_{C^\theta(\mathbb{R}^n)})^{(1+\theta)/2},$$

   $$\|D_{ij}\varphi\|_\infty \le C(\|\varphi\|_{C^{2+\theta}(\mathbb{R}^n)})^{1-\theta/2}(\|\varphi\|_{C^\theta(\mathbb{R}^n)})^{\theta/2},$$

   for every $\varphi \in C^{2+\theta}(\mathbb{R}^N)$, i, $j = 1,\dots,N$.

   [Hint: write $\varphi = \varphi - T(t)\varphi + T(t)\varphi = -\int_0^t T(s)\Delta\varphi\,ds + T(t)\varphi$, $T(t) =$ heat semigroup, and use the estimates $\|D_iT(t)f\|_\infty \le Ct^{-1/2+\theta/2}\|f\|_{C^\theta}$, $\|D_{ij}T(t)f\|_\infty \le Ct^{-1+\theta/2}\|f\|_{C^\theta}$.]

Chapter 4

Non homogeneous problems

Let $A : D(A) \subset X \to X$ be a sectorial operator. In this chapter we study the nonhomogeneous Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t), \quad 0 < t \leq T, \\ \\ u(0) = x, \end{cases} \tag{4.1}$$

where $f : [0, T] \to X$ or $f : [0, \infty) \to X$.

4.2 Strict, classical, and mild solutions

Definition 4.2.1 *Let $f : [0, T] \mapsto X$ be a continuous function, and let $x \in X$. Then:*

(i) $u \in C^1([0, T]; X) \cap C([0, T]; D(A))$ is a strict solution *of (4.1) in $[0, T]$ if $u'(t) = Au(t) + f(t)$ for every $t \in [0, T]$, and $u(0) = x$.*

(ii) $u \in C^1((0, T]; X) \cap C((0, T]; D(A)) \cap C([0, T]; X)$ is a classical solution *of (4.1) in $[0, T]$ if $u'(t) = Au(t) + f(t)$ for every $t \in (0, T]$, and $u(0) = x$.*

If $f : [0, \infty) \to X$, then u is a strict or classical solution of (4.1) if for every $T > 0$ it is a strict or classical solution of (4.1) in $[0, T]$.

Let us see that if (4.1) has a classical (or a strict) solution, then it is given, as in the case of a bounded A, by the *variation of constants formula*

$$u(t) = e^{tA}x + \int_0^t e^{(t-s)A} f(s)ds, \quad 0 \leq t \leq T. \tag{4.2}$$

Whenever the integral in (4.2) does make sense, the function u defined by (4.2) is said to be a *mild solution* of (4.1).

The mild solution satisfies a familiar equality, as the next lemma shows.

Proposition 4.2.2 *Let $f \in C_b((0, T]; X)$, and let $x \in X$. If u is defined by (4.2), then for every $t \in [0, T]$ the integral $\int_0^t u(s)ds$ belongs to $D(A)$, and*

$$u(t) = x + A \int_0^t u(s)ds + \int_0^t f(s)ds, \quad 0 \leq t \leq T. \tag{4.3}$$

Proof. For every $t \in [0,T]$ we have

$$\begin{aligned}\int_0^t u(s)ds &= \int_0^t e^{sA}xds + \int_0^t ds\int_0^s e^{(s-\sigma)A}f(\sigma)d\sigma \\ &= \int_0^t e^{sA}xds + \int_0^t d\sigma\int_\sigma^t e^{(s-\sigma)A}f(\sigma)ds.\end{aligned}$$

By proposition 1.2.6(ii), the integral $\int_0^t u(s)ds$ belongs to $D(A)$, and

$$A\int_0^t u(s)ds = e^{tA}x - x + \int_0^t (e^{(t-\sigma)A}-1)f(\sigma)d\sigma, \;\; 0 \le t \le T,$$

so that (4.3) holds. □

From definition 4.2.1 it is easily seen that if (4.1) has a strict solution, then

$$x \in D(A), \;\; Ax + f(0) = u'(0) \in \overline{D(A)}, \tag{4.4}$$

and if (4.1) has a classical solution, then

$$x \in \overline{D(A)}. \tag{4.5}$$

Proposition 4.2.3 *Let $f \in C((0,T],X)$ be such that $t \mapsto \|f(t)\| \in L^1(0,T)$, and let $x \in \overline{D(A)}$ be given. If u is a classical solution of (4.1), then it is given by formula (4.2).*

Proof. Let u be a classical solution, and fix $t \in (0,T]$. Since $u \in C^1((0,T];X) \cap C((0,T];D(A)) \cap C([0,T];X)$, the function

$$v(s) = e^{(t-s)A}u(s), \;\; 0 \le s \le t,$$

belongs to $C([0,t];X) \cap C^1((0,t),X)$, and

$$\begin{aligned} v(0) &= e^{tA}x, \;\; v(t) = u(t), \\ v'(s) &= -Ae^{(t-s)A}u(s) + e^{(t-s)A}(Au(s)+f(s)) = e^{(t-s)A}f(s), \;\; 0 < s < t.\end{aligned}$$

As a consequence, for $0 < 2\varepsilon < t$ we have

$$v(t-\varepsilon) - v(\varepsilon) = \int_\varepsilon^{t-\varepsilon} e^{(t-s)A}f(s)ds,$$

so that letting $\varepsilon \to 0$ we get

$$v(t) - v(0) = \int_0^t e^{(t-s)A}f(s)ds,$$

and the statement follows. □

Under the assumptions of proposition 4.2.3, the classical solution of (4.1) is unique. In particular, for $f \equiv 0$ and $x \in \overline{D(A)}$, the function

$$t \mapsto u(t) = e^{tA}x, \;\; t \ge 0,$$

is the unique solution of the homogeneous problem (1.1). Of course, proposition 4.2.3 implies also uniqueness of the strict solution.

Therefore, existence of a classical or strict solution of (1.1) is reduced to the problem of regularity of the mild solution. In general, even for $x = 0$ the continuity of f is not sufficient to guarantee that the mild solution is classical. Trying to show that $u(t) \in D(A)$ by estimating $\|Ae^{(t-s)A}f(s)\|$ is useless, because we have $\|Ae^{(t-s)A}f(s)\| \leq C\|f\|_\infty (t-s)^{-1}$ and this is not sufficient to make the integral convergent. More sophisticated arguments, such as in the proof of proposition 1.2.6(ii), do not work. We refer to exercise 4.2.12.1 for a rigorous counterexample.

The continuity of f allows however to show that the mild solution is, at least, Hölder continuous in all intervals $[\varepsilon, T]$ with $\varepsilon > 0$. For the proof we define

$$M_k = \sup_{0<t\leq T+1} \|t^k A^k e^{tA}\|, \quad k = 0, 1, 2.$$

Proposition 4.2.4 *Let $f \in C_b((0,T); X)$. Then, for every $\alpha \in (0,1)$, The function*

$$v(t) = (e^{tA} * f)(t) := \int_0^t e^{(t-s)A} f(s) ds, \quad 0 \leq t \leq T,$$

belongs to $C^\alpha([0,T]; X)$, and there is $C = C(\alpha)$ such that

$$\|v\|_{C^\alpha([0,T];X)} \leq C \sup_{0<s<T} \|f(s)\|. \tag{4.6}$$

Proof. For $0 \leq t \leq T$ we have

$$\|v(t)\| \leq M_0 t \sup_{0\leq s\leq t} \|f(s)\|, \tag{4.7}$$

whereas for $0 \leq s \leq t \leq T$ we have

$$\begin{aligned} v(t) - v(s) &= \int_0^s \left(e^{(t-\sigma)A} - e^{(s-\sigma)A}\right) f(\sigma) d\sigma + \int_s^t e^{(t-\sigma)A} f(\sigma) d\sigma \\ &= \int_0^s d\sigma \int_{s-\sigma}^{t-\sigma} Ae^{\tau A} f(\sigma) d\tau + \int_s^t e^{(t-\sigma)A} f(\sigma) d\sigma, \end{aligned}$$

which implies

$$\begin{aligned} \|v(t) - v(s)\| &\leq M_1 \int_0^s d\sigma \int_{s-\sigma}^{t-\sigma} \frac{1}{\tau} d\tau \|f\|_\infty + M_0(t-s)\|f\|_\infty \\ &\leq M_1 \int_0^s \frac{1}{(s-\sigma)^\alpha} \int_{s-\sigma}^{t-\sigma} \frac{1}{\tau^{1-\alpha}} d\tau \|f\|_\infty + M_0(t-s)\|f\|_\infty \\ &\leq \left(\frac{M_1 T^{1-\alpha}}{\alpha(1-\alpha)}(t-s)^\alpha + M_0(t-s)\right) \|f\|_\infty, \end{aligned} \tag{4.8}$$

so that v is α-Hölder continuous. Estimate (4.6) immediately follows from (4.7) and (4.8). □

The result of proposition 4.2.2 is used in the next lemma, where we give sufficient conditions in order that a mild solution be classical or strict.

Lemma 4.2.5 *Let $f \in C_b((0,T]; X)$, let $x \in \overline{D(A)}$, and let u be the mild solution of (4.1). The following conditions are equivalent.*

(a) $u \in C((0,T]; D(A))$,

(b) $u \in C^1((0,T]; X)$,

(c) u *is a classical solution of (4.1).*

If in addition $f \in C([0,T]; X)$, then the following conditions are equivalent.

(a′) $u \in C([0,T]; D(A))$,

(b′) $u \in C^1([0,T]; X)$,

(c′) *u is a strict solution of (4.1).*

Proof — Of course, (c) is stronger than (a) and (b). Let us show that if either (a) or (b) holds, then u is a classical solution. We already know that u belongs to $C([0,T]; X)$ (see also proposition 4.2.4), and that it satisfies (4.3). Therefore, for every t, h such that t, $t+h \in (0,T]$,

$$\frac{u(t+h)-u(t)}{h} = \frac{1}{h} A \int_t^{t+h} u(s)ds + \frac{1}{h}\int_t^{t+h} f(s)ds. \tag{4.9}$$

Since f is continuous at t, then

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} f(s)ds = f(t). \tag{4.10}$$

Let (a) hold. Then Au is continuous at t, so that

$$\lim_{h\to 0} \frac{1}{h} A \int_t^{t+h} u(s)ds = \lim_{h\to 0} \frac{1}{h}\int_t^{t+h} Au(s)ds = Au(t).$$

By (4.9) and (4.10) we get now that u is differentiable at the point t, with $u'(t) = Au(t) + f(t)$. Since both Au and f are continuous in $(0,T]$, then u' too is continuous, and u is a classical solution.

Let now (b) hold. Since u is continuous at t, then

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} u(s)ds = u(t).$$

On the other hand, by (4.9) and (4.10), there exists the limit

$$\lim_{h\to 0} A\left(\frac{1}{h}\int_t^{t+h} u(s)ds\right) = u'(t) - f(t).$$

Since A is a closed operator, then $u(t)$ belongs to $D(A)$, and $Au(t) = u'(t) - f(t)$. Since both u' and f are continuous in $(0,T]$, then also Au is continuous in $(0,T]$, so that u is a classical solution.

The equivalence of (a'), (b'), (c') may be proved in the same way. □

In the following two theorems we prove that, under some regularity conditions on f the mild solution is strict or classical. In the theorem below we assume time regularity whereas in the next one we assume "space" regularity on f.

Theorem 4.2.6 *Let $0 < \alpha < 1$, $f \in C^\alpha([0,T], X)$, $x \in X$, an let u be the function defined in (4.1). Then u belongs to $C^\alpha([\varepsilon,T], D(A)) \cap C^{1+\alpha}([\varepsilon,T], X)$ for every $\varepsilon \in (0,T)$, and the following statements hold:*

(i) if $x \in \overline{D(A)}$, then u is a classical solution of (4.1);

(ii) if $x \in D(A)$ and $Ax + f(0) \in \overline{D(A)}$, then u is a strict solution of (4.1), and there is $C > 0$ such that

$$\|u\|_{C^1([0,T],X)} + \|u\|_{C([0,T],D(A))} \le C(\|f\|_{C^\alpha([0,T],X)} + \|x\|_{D(A)}). \tag{4.11}$$

(iii) if $x \in D(A)$ and $Ax + f(0) \in D_A(\alpha, \infty)$, then u' and Au belong to $C^\alpha([0,T], X)$, u' belongs to $B([0,T]; D_A(\alpha,\infty))$, and there is C such that

$$\begin{array}{l}\|u\|_{C^{1+\alpha}(X)} + \|Au\|_{C^\alpha(X)} + \|u'\|_{B(D_A(\alpha,\infty))} \\ \leq C(\|f\|_{C^\alpha(X)} + \|x\|_{D(A)} + \|Ax + f(0)\|_{D_A(\alpha,\infty)}).\end{array} \tag{4.12}$$

Proof. We are going to show that if $x \in \overline{D(A)}$ then $u \in C((0,T]; D(A))$, and that if $x \in D(A)$ and $Ax + f(0) \in \overline{D(A)}$ then $u \in C([0,T]; D(A))$. In both cases statements (i) and (ii) will follow from lemma 4.2.5.

Set

$$\begin{cases} u_1(t) = \displaystyle\int_0^t e^{(t-s)A}(f(s) - f(t))ds, \;\; 0 \leq t \leq T, \\ \\ u_2(t) = e^{tA}x + \displaystyle\int_0^t e^{(t-s)A} f(t) ds, \;\; 0 \leq t \leq T. \end{cases} \tag{4.13}$$

so that $u = u_1 + u_2$. Notice that both $u_1(t)$ and $u_2(t)$ belong to $D(A)$ for $t > 0$. Concerning $u_1(t)$, the estimate

$$\|Ae^{(t-s)A}(f(s) - f(t))\| \leq \frac{M_1}{t-s}(t-s)^\alpha [f]_{C^\alpha}$$

implies that the function $e^{(t-s)A}(f(s) - f(t))$ is integrable with values in $D(A)$, whence $u_1(t) \in D(A)$ for every $t \in (0,T]$ (the same holds, of course, for $t = 0$ as well). Concerning $u_2(t)$, we know that $e^{tA}x$ belongs to $D(A)$ for $t > 0$, and that $\int_0^t e^{(t-s)A} f(t) ds$ belongs to $D(A)$ by Proposition 1.2.6(ii). Moreover, we have

$$\begin{cases} (i) \quad Au_1(t) = \displaystyle\int_0^t Ae^{(t-s)A}(f(s) - f(t))ds, \;\; 0 \leq t \leq T, \\ \\ (ii) \quad Au_2(t) = Ae^{tA}x + (e^{tA} - 1)f(t), \;\; 0 < t \leq T. \end{cases} \tag{4.14}$$

If $x \in D(A)$, then equality (4.14)(ii) holds for $t = 0$, too. Let us show that Au_1 is Hölder continuous in $[0,T]$. For $0 \leq s \leq t \leq T$ we have

$$\begin{array}{l} Au_1(t) - Au_1(s) = \\ \displaystyle\int_0^s \Big(Ae^{(t-\sigma)A}(f(\sigma) - f(t)) - Ae^{(s-\sigma)A}(f(\sigma) - f(s)) \Big) d\sigma + \int_s^t Ae^{(t-\sigma)A}(f(\sigma) - f(t)) d\sigma \\ = \displaystyle\int_0^s \left(Ae^{(t-\sigma)A} - Ae^{(s-\sigma)A} \right) (f(\sigma) - f(s)) d\sigma + \int_0^s Ae^{(t-\sigma)A}(f(s) - f(t)) d\sigma \\ = \displaystyle\int_0^s \int_{s-\sigma}^{t-\sigma} A^2 e^{\tau A} d\tau (f(\sigma) - f(s)) d\sigma \\ + (e^{tA} - e^{(t-s)A})(f(s) - f(t)) + \displaystyle\int_s^t Ae^{(t-\sigma)A}(f(\sigma) - f(t)) d\sigma, \end{array} \tag{4.15}$$

so that

$$\begin{array}{rl} \|Au_1(t) - Au_1(s)\| \leq & M_2 \displaystyle\int_0^s (s-\sigma)^\alpha \int_{s-\sigma}^{t-\sigma} \tau^{-2} d\tau \, d\sigma \; [f]_{C^\alpha} \\ & + 2M_0 (t-s)^\alpha [f]_{C^\alpha} + M_1 \displaystyle\int_s^t (t-\sigma)^{\alpha-1} d\sigma \; [f]_{C^\alpha} \\ \leq M_2 \displaystyle\int_0^s d\sigma \int_{s-\sigma}^{t-\sigma} \tau^{\alpha-2} d\tau \; [f]_{C^\alpha} & + (2M_0 + M_1 \alpha^{-1})(t-s)^\alpha [f]_{C^\alpha} \\ & \leq \left(\dfrac{M_2}{\alpha(1-\alpha)} + 2M_0 + \dfrac{M_1}{\alpha} \right) (t-s)^\alpha [f]_{C^\alpha}. \end{array} \tag{4.16}$$

Then, Au_1 is α-Hölder continuous in $[0,T]$. Moreover, it is easily checked that Au_2 is α-Hölder continuous in $[\varepsilon,T]$ for every $\varepsilon\in(0,T)$, and therefore $Au\in C^\alpha([\varepsilon,T];X)$. Since $u\in C^\alpha([\varepsilon,T];X)$ (because $t\mapsto e^{tA}x\in C^\infty((0,T];X)$ and $t\mapsto\int_0^t e^{(t-s)A}f(s)ds\in C^\alpha([0,T];X)$ by Proposition 4.2.4), it follows that $u\in C^\alpha([\varepsilon,T];D(A))$, and $u\in C((0,T];D(A))$ follows from the arbitrariness of ε.

Concerning the behaviour as $t\to 0$, if $x\in\overline{D(A)}$, then $t\mapsto e^{tA}x\in C([0,T],X)$ and then $u\in C([0,T],X)$, see proposition 4.2.4.

If $x\in D(A)$, we may write $Au_2(t)$ in the form

$$Au_2(t)=e^{tA}(Ax+f(0))+e^{tA}(f(t)-f(0))-f(t),\ \ 0\le t\le T. \tag{4.17}$$

If $Ax+f(0)\in\overline{D(A)}$, then $\lim_{t\to 0}Au_2(t)=Ax$, hence Au_2 is continuous at $t=0$, and $u=u_1+u_2$ belongs to $C([0,T];D(A))$.

If $Ax+f(0)\in D_A(\alpha,\infty)$, we already know that $t\mapsto e^{tA}(Ax+f(0))\in C^\alpha([0,T],X)$, with C^α norm estimated by const. $\|Ax+f(0)\|_{D_A(\alpha,\infty)}$. Moreover $f\in C^\alpha([0,T],X)$ by assumption, so we have to show only that $t\mapsto e^{tA}(f(t)-f(0))$ is α-Hölder continuous.

For $0\le s\le t\le T$ we have

$$\begin{array}{l}\|e^{tA}(f(t)-f(0))-e^{sA}(f(s)-f(0))\|\le\|(e^{tA}-e^{sA})(f(s)-f(0))\|+\|e^{tA}(f(t)-f(s))\|\\ \le s^\alpha\|A\int_s^t e^{\sigma A}d\sigma\|_{\mathcal{L}(X)}[f]_{C^\alpha}+M_0(t-s)^\alpha[f]_{C^\alpha}\\ \le\left(\dfrac{M_1}{\alpha}+M_0\right)(t-s)^\alpha[f]_{C^\alpha},\end{array} \tag{4.18}$$

so that Au_2 is Hölder continuous as well, and the estimate

$$\|u\|_{C^{1+\alpha}([0,T];X)}+\|Au\|_{C^\alpha([0,T];X)}\le c(\|f\|_{C^\alpha([0,T],X)}+\|x\|_{D(A)}+\|Ax+f(0)\|_{D_A(\alpha,\infty)})$$

easily follows.

Let us now estimate $[u'(t)]_{D_A(\alpha,\infty)}$. For $0\le t\le T$ we have

$$u'(t)=\int_0^t Ae^{(t-s)A}(f(s)-f(t))ds+e^{tA}(Ax+f(0))+e^{tA}(f(t)-f(0)),$$

so that for $0<\xi\le 1$ we deduce

$$\begin{array}{rcl}\|\xi^{1-\alpha}Ae^{\xi A}u'(t)\| & \le & \left\|\xi^{1-\alpha}\int_0^t A^2e^{(t+\xi-s)A}(f(s)-f(t))ds\right\|\\ & & +\|\xi^{1-\alpha}Ae^{(t+\xi)A}(Ax+f(0))\|+\|\xi^{1-\alpha}Ae^{(t+\xi)A}(f(t)-f(0))\|\\ & \le & M_2\xi^{1-\alpha}\int_0^t(t-s)^\alpha(t+\xi-s)^{-2}ds\ [f]_{C^\alpha}\\ & & +M_0[Ax+f(0)]_{D_A(\alpha,\infty)}+M_1\xi^{1-\alpha}(t+\xi)^{-1}t^\alpha\ [f]_{C^\alpha}\\ & \le & M_2\int_0^\infty\sigma^\alpha(\sigma+1)^{-2}d\sigma[f]_{C^\alpha}+M_0[Ax+f(0)]_{D_A(\alpha,\infty)}+M_1[f]_{C^\alpha}\end{array} \tag{4.19}$$

Then, $[u'(t)]_{D_A(\alpha,\infty)}$ is bounded in $[0,T]$, and the proof is complete. □

Remark 4.2.7 *From the proof of theorem 4.2.6 it follows that the condition* $Ax+f(0)\in D_A(\alpha,\infty)$ *is necessary for* $Au\in C^\alpha([0,T];X)$. *Once this condition is satisfied, it is preserved through the whole interval* $[0,T]$, *in the sense that* $Au(t)+f(t)=u'(t)$ *belongs to* $D_A(\alpha,\infty)$ *for each* $t\in[0,T]$.

In the proof of the next theorem we use the constants

$$M_{k,\alpha}:=\sup_{0<t\le T+1}\|t^{k-\alpha}A^ke^{tA}\|_{\mathcal{L}(D_A(\alpha,\infty),X)}<\infty,\ \ k=1,2. \tag{4.20}$$

Theorem 4.2.8 *Let $0 < \alpha < 1$, and let $f \in C([0,T];X) \cap B([0,T];D_A(\alpha,\infty))$. Then, the function*

$$v(t) = (e^{tA} \star f)(t) = \int_0^t e^{(t-s)A} f(s)ds, \;\; 0 \le t \le T,$$

belongs to $C([0,T];D(A)) \cap C^1([0,T];X)$, and it is the strict solution of

$$v'(t) = Av(t) + f(t), \;\; 0 < t \le T, \;\; v(0) = 0. \tag{4.21}$$

Moreover, v' and Av belong to $B([0,T];D_A(\alpha,\infty))$, Av belongs to $C^\alpha([0,T];X)$, and there is C such that

$$\|v'\|_{B(D_A(\alpha,\infty))} + \|Av\|_{B(D_A(\alpha,\infty))} + \|Av\|_{C^\alpha(X)} \le C\|f\|_{B(D_A(\alpha,\infty))}. \tag{4.22}$$

Proof. Let us prove that v is a strict solution of (4.21), and that (4.22) holds. For $0 \le t \le T$, $v(t)$ belongs to $D(A)$, and

$$\|Av(t)\| \le M_{1,\alpha} \int_0^t (t-s)^{\alpha-1} ds \|f\|_{B(D_A(\alpha,\infty))} = \frac{T^\alpha M_{1,\alpha}}{\alpha} \|f\|_{B(D_A(\alpha,\infty))}. \tag{4.23}$$

Moreover, for $0 < \xi \le 1$ we have

$$\begin{aligned} \|\xi^{1-\alpha} A e^{\xi A} Av(t)\| &= \xi^{1-\alpha} \left\| \int_0^t A^2 e^{(t+\xi-s)A} f(s) ds \right\| \\ &\le M_{2,\alpha} \xi^{1-\alpha} \int_0^t (t+\xi-s)^{\alpha-2} ds \|f\|_{B(D_A(\alpha,\infty))} \le \frac{M_{2,\alpha}}{1-\alpha} \|f\|_{B(D_A(\alpha,\infty))}, \end{aligned} \tag{4.24}$$

so that Av is bounded with values in $D_A(\alpha,\infty)$. Let us prove that Av is Hölder continuous with values in X: for $0 \le s \le t \le T$ we have

$$\begin{aligned} \|Av(t) - Av(s)\| &\le \left\| A \int_0^s \left(e^{(t-\sigma)A} - e^{(s-\sigma)A} \right) f(\sigma) d\sigma \right\| + \left\| A \int_s^t e^{(t-\sigma)A} f(\sigma) d\sigma \right\| \\ &\le M_{2,\alpha} \int_0^s d\sigma \int_{s-\sigma}^{t-\sigma} \tau^{\alpha-2} d\tau \|f\|_{B(D_A(\alpha,\infty))} \\ &\quad + M_{1,\alpha} \int_s^t (t-\sigma)^{\alpha-1} d\sigma \|f\|_{B(D_A(\alpha,\infty))} \\ &\le \left(\frac{M_{2,\alpha}}{\alpha(1-\alpha)} + \frac{M_{1,\alpha}}{\alpha} \right) (t-s)^\alpha \|f\|_{B(D_A(\alpha,\infty))}, \end{aligned} \tag{4.25}$$

hence Av is α-Hölder continuous in $[0,T]$. Estimate (4.22) follows from (4.23), (4.24), (4.25).

The differentiability of v and the equality $v'(t) = Av(t) + f(t)$ follow from Lemma 4.2.5. □

Corollary 4.2.9 *Let $0 < \alpha < 1$, $x \in X$, $f \in C([0,T];X) \cap B([0,T];D_A(\alpha,\infty))$ be given, and let u be given by (4.2). Then, $u \in C^1((0,T];X) \cap C((0,T];D(A))$, and $u \in B([\varepsilon,T];D_A(\alpha+1,\infty))$ for every $\varepsilon \in (0,T)$. Moreover, the following statements hold:*

(i) If $x \in \overline{D(A)}$, then u is the classical solution of (4.1);

(ii) If $x \in D(A)$, $Ax \in \overline{D(A)}$, then u is the strict solution of (4.1);

(iii) If $x \in D_A(\alpha+1,\infty)$, then u' and Au belong to $B([0,T];D_A(\alpha,\infty)) \cap C([0,T];X)$, Au belongs to $C^\alpha([0,T];X)$, and there is C such that

$$\begin{aligned} &\|u'\|_{B(D_A(\alpha,\infty))} + \|Au\|_{B(D_A(\alpha,\infty))} + \|Au\|_{C^\alpha([0,T];X)} \\ &\le C(\|f\|_{B(D_A(\alpha,\infty))} + \|x\|_{D_A(\alpha,\infty)}). \end{aligned} \tag{4.26}$$

Proof. Let us write $u(t) = e^{tA}x + (e^{tA} \star f)(t)$. If $x \in \overline{D(A)}$, the function $t \mapsto e^{tA}x$ is the classical solution of $w' = Aw, \quad t > 0, \quad w(0) = x$. If $x \in D(A)$ and $Ax \in \overline{D(A)}$ it is in fact a strict solution; if $x \in D_A(\alpha+1,\infty)$ then it is a strict solution and it belongs also to $C^1([0,T];X) \cap B([0,T];D_A(\alpha+1,\infty))$. The claim then follows from theorem 4.2.8. □

We recall that for $0 < \theta < 1$ the parabolic Hölder space $C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)$ is the space of the continuous functions f such that

$$\|f\|_{C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)} := \|f\|_\infty + \sup_{x\in\mathbb{R}}[f(\cdot,x)]_{C^{\theta/2}([0,T])} + \sup_{t\in[0,T]}[f(t,\cdot)]_{C^\theta(\mathbb{R})} < \infty,$$

and $C^{1+\theta/2,2+\theta}([0,T]\times\mathbb{R}^N)$ is the space of the functions u such that u_t, and $D_{ij}u$ exist for all i, $j = 1,\dots N$ and belong to $C^{\theta/2,\theta}([0,T]\times\mathbb{R})$. The norm is

$$\|u\|_{C^{1+\theta/2,2+\theta}([0,T]\times\mathbb{R}^N)} := \|u\|_\infty + \sum_{i=1}^N \|D_iu\|_\infty$$
$$+\|u_t\|_{C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)} + \sum_{i,j=1}^N \|D_{ij}u\|_{C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)}.$$

Note that $f \in C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)$ if and only if $t \mapsto f(t,\cdot)$ belongs to $C^{\theta/2}([0,T];C_b(\mathbb{R}^N))$ $\cap\, B([0,T];C^\theta(\mathbb{R}^N))$.

Corollary 4.2.10 (Ladyzhenskaja – Solonnikov – Ural'ceva) *Let $0 < \theta < 1$, $T > 0$ and let $u_0 \in C^{2+\theta}(\mathbb{R}^N)$, $f \in C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)$. Then the initial value problem*

$$\begin{cases} u_t(t,x) = u_{xx}(t,x) + f(t,x), \quad 0 < t \le T, \ x \in \mathbb{R}^N, \\ \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^N, \end{cases} \tag{4.27}$$

has a unique solution $u \in C^{1+\theta/2,2+\theta}([0,T]\times\mathbb{R}^N)$, and there is $C > 0$, independent of u_0 and f, such that

$$\|u\|_{C^{1+\theta/2,2+\theta}([0,T]\times\mathbb{R}^N)} \le C(\|u_0\|_{C^{2+\theta}(\mathbb{R}^N)} + \|f\|_{C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)}).$$

Proof. Set $X = C_b(\mathbb{R}^N)$, $A : D(A) \mapsto X$, $A\varphi = \Delta\varphi$, $T(t) =$ heat semigroup. The function $t \mapsto f(t,\cdot)$ belongs to $C^{\theta/2}([0,T];X) \cap B([0,T];D_A(\theta/2,\infty))$, thanks to the characterization of example 3.1.7. The initial datum u_0 is in $D(A)$, and both Au_0, $f(0,\cdot)$ are in $D_A(\theta/2,\infty)$. Then we may apply both theorems 4.2.6 and 4.2.8 with $\alpha = \theta/2$. They imply that the function u given by the variation of constants formula (4.2) is the unique strict solution to problem (4.1), with initial datum u_0 and with $f(t) = f(t,\cdot)$. Therefore, the function

$$u(t,x) := u(t)(x) = (T(t)u_0)(x) + \int_0^t (T(t-s)f(s,\cdot)(x)ds,$$

is the unique bounded solution to (4.27) with bounded u_t. Moreover, theorem 4.2.6 implies that $u' \in C^{\theta/2}([0,T];C_b(\mathbb{R}^N)) \cap B([0,T];C^\theta(\mathbb{R}^N))$, so that $u_t \in C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)$, with norm bounded by $C(\|u_0\|_{C^{2+\theta}(\mathbb{R}^n)} + \|f\|_{C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)})$ for some $C > 0$. Theorem 4.2.8 implies that u is bounded with values in $D_A(\theta/2+1,\infty)$, so that $u(t,\cdot) \in C^{2+\theta}(\mathbb{R}^N)$ for each t, and $\sup_{0\le t\le T}\|u(t,\cdot)\|_{C^{2+\theta}(\mathbb{R}^N)} \le C(\|u_0\|_{C^{2+\theta}(\mathbb{R}^n)} + \|f\|_{C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)})$ for some $C > 0$.

To finish the proof it remains to show that each second order space derivative $D_{ij}u$ is $\theta/2$-Hölder continuous with respect to t. To this aim we use the interpolatory inequality

$$\|D_{ij}\varphi\|_\infty \le C(\|\varphi\|_{C^{2+\theta}(\mathbb{R}^n)})^{1-\theta/2}(\|\varphi\|_{C^\theta(\mathbb{R}^n)})^{\theta/2},$$

that holds for every $\varphi \in C^{2+\theta}(\mathbb{R}^N)$, $i,\ j = 1, \dots, N$. See exercises 3.1.10. Applying it to the function $\varphi = u(t,\cdot) - u(s,\cdot)$ we get

$$\begin{aligned}&\|D_{ij}u(t,\cdot) - D_{ij}u(s,\cdot)\|_\infty \leq C(\|u(t,\cdot) - u(s,\cdot)\|_{C^{2+\theta}(\mathbb{R}^n)})^{1-\theta/2}(\|u(t,\cdot) - u(s,\cdot)\|_{C^\theta(\mathbb{R}^n)})^{\theta/2}\\ &\leq C(2\sup_{0\leq t\leq T}\|u(t,\cdot)\|_{C^{2+\theta}(\mathbb{R}^n)})^{1-\theta/2}(|t-s|\sup_{0\leq t\leq T}\|u_t(t,\cdot)\|_{C^\theta(\mathbb{R}^n)})^{\theta/2}\\ &\leq C'|t-s|^{\theta/2}(\|u_0\|_{C^{2+\theta}(\mathbb{R}^n)} + \|f\|_{C^{\theta/2,\theta}([0,T]\times\mathbb{R}^N)}),\end{aligned}$$

and the statement follows. □

Remark 4.2.11 If we have a Cauchy problem in an interval $[a,b] \neq [0,T]$,

$$\begin{cases} v'(t) = Au(t) + g(t), & a < t \leq b, \\ \\ v(a) = y, \end{cases} \tag{4.28}$$

we obtain results similar to the case $[a,b] = [0,T]$, by the changement of time variable $\tau = T(t-a)/(b-a)$. The details are left as (easy) exercises. We just write down the variation of constants formula for v,

$$v(t) = e^{(t-a)A}y + \int_a^t e^{(t-s)A}g(s)ds, \quad a \leq t \leq b.$$

Exercises 4.2.12

1. Let $\varphi : (0,T) \times \mathbb{R}^N \mapsto \mathbb{R}$, $u_0 : \mathbb{R}^N \mapsto \mathbb{R}$ be continuous and bounded, and let $T(t)$ be the heat semigroup. Show that the function

   $$u(t,x) = (T(t)u_0)(x) + \int_0^t (T(t-s)\varphi(s,\cdot))(x)ds$$

   belongs to $C([0,T] \times \mathbb{R}^N; \mathbb{R})$.

2. Use estimates (4.20) and the technique of proposition 4.2.4 to prove that for each $f \in C_b((0,T);X)$, the function $v = (e^{tA} * f)$ belongs to $C^{1-\alpha}([0,T]; D_A(\alpha,\infty))$ for every $\alpha \in (0,1)$, with norm bounded by $C(\alpha)\sup_{0<t<T}\|f(t)\|$.

3. Let $A : D(A) \mapsto X$ be a sectorial operator, and let $0 < \alpha < 1$, $a < b \in \mathbb{R}$. Prove that if a function u belongs to $C^{1+\alpha}([a,b];X) \cap C^\alpha([a,b];D(A))$ then u' is bounded in $[a,b]$ with values in $D_A(\alpha,\infty)$.

   [Hint: set $u_0 = u(a)$, $f(t) = u'(t) - Au(t)$, and use theorem Th:4.2.3(iii)].

4. Consider the sectorial operators A_p in the sequence spaces ℓ^p, $1 \leq p < \infty$ given by

   $$D(A_p) = \{(x_n) \in \ell^p : (nx_n) \in \ell^p\}, \qquad A_p(x_n) = -(nx_n) \text{ for } (x_n) \in D(A_p)$$

   and assume that for every $f \in C([0,T];\ell^p)$ the mild solution v of 1.1 corresponding to the initial value $x = 0$ is a strict solution.

   (i) Use the closed graph theorem to show that the linear operator

   $$f \mapsto S(t)f = \int_0^t T(t-s)f(s)ds : C([0,1];\ell^p) \to C([0,1]; D(A_p))$$

   is bounded.

(ii) Let (e_n) be the canonical basis of ℓ^p and consider a nonzero continuous function $g : [0,\infty) \to [0,1]$ with support contained in $[1/2, 1]$. Let $f_n(t) = g(2^n(1-t))e_{2^n}$; then $f_n \in C([0,1];\ell^p)$, $\|f_n\|_\infty \le 1$. Moreover, setting $h_N = f_1 + \cdots + f_N$, we have also $h_N \in C([0,1];\ell_p)$, $\|h_N\|_\infty \le 1$, since the functions f_n have disjoint supports. Show that $S(1)f_n = c2^{-n}e_{2^n}$ where $c = \int_0^\infty e^{-s}g(s)ds$, hence $\|S(1)h_N\|_{D(A_p)} \ge cN^{1/p}$. This implies that $S(1)$ is unbounded, contradicting (i).

(iii) What happens for $p = \infty$?

Chapter 5

Asymptotic behavior in linear problems

5.1 Behavior of e^{tA}

One of the most useful properties of the analytic semigroups is the so called *spectrum determining condition*: roughly speaking, the asymptotic behavior (as $t \to +\infty$) of e^{tA}, and, more generally, of $A^n e^{tA}$, is determined by the spectral properties of A.

Define the *spectral bound* of any sectorial operator A by

$$s(A) = \sup\{\text{Re}\lambda : \lambda \in \sigma(A)\} \tag{5.1}$$

Clearly $s(A) \leq \omega$, where ω is the number in definition 1.2.1.

Proposition 5.1.1 *For every $n \in \mathbb{N} \cup \{0\}$ and $\varepsilon > 0$ there exist $M_{n,\varepsilon} > 0$ such that*

$$\|t^n A^n e^{tA}\|_{L(X)} \leq M_{n,\varepsilon} e^{(\omega_A + \varepsilon)t}, \quad t > 0. \tag{5.2}$$

Proof. For $0 < t \leq 1$, estimates (5.2) are an easy consequence of (1.14). If $t \geq 1$ and $\omega_A + \varepsilon \geq \omega$, (5.2) is still a consequence of (1.14). Let us consider the case in which $t \geq 1$ and $s(A) + \varepsilon < \omega$. Since $\rho(A) \supset S_{\theta,\omega} \cup \{\lambda \in \mathbb{C} : \text{Re}\,\lambda > s(A)\}$, setting $a = (\omega - s(A) - \varepsilon)|\cos\theta|^{-1}$, $b = (\omega - s(A) - \varepsilon)|\tan\theta|$, the path

$$\begin{aligned} \Gamma_\varepsilon &= \{\lambda \in \mathbb{C} : \lambda = \xi e^{-i\theta} + \omega, \ \xi \geq a\} \cup \{\lambda \in \mathbb{C} : \lambda = \xi e^{i\theta} + \omega, \ \xi \geq a\} \\ &\cup \{\lambda \in \mathbb{C} : \text{Re } \lambda = \omega_A + \varepsilon, \ |\text{Im } \lambda| \leq b\} \end{aligned}$$

is contained in $\rho(A)$, and $\|R(\lambda, A)\|_{L(X)} \leq M_\varepsilon |\lambda - s(A)|^{-1}$ on Γ_ε, for some $M_\varepsilon > 0$. Since for every t the function $\lambda \to e^{\lambda t} R(\lambda, A)$ is holomorphic in $\rho(A)$, the path $\omega + \gamma_{r,\eta}$ may be replaced by Γ_ε, obtaining for each $t \geq 1$,

$$\begin{aligned} \|e^{tA}\| &= \left\| \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} e^{t\lambda} R(\lambda, A) d\lambda \right\| \leq \frac{M_\varepsilon}{\pi} \int_a^{+\infty} \frac{e^{(\omega + \xi\cos\theta)t}}{|\xi e^{i\theta} + \omega - s(A)|} d\xi \\ &+ \frac{M_\varepsilon}{2\pi} \int_{-b}^{b} \frac{e^{(s(A)+\varepsilon)t}}{|iy + \varepsilon|} dy \leq \frac{M_\varepsilon}{\pi} \left(\frac{1}{b|\cos\theta|} + \frac{b}{\varepsilon} \right) e^{(s(A)+\varepsilon)t}. \end{aligned}$$

Estimate (5.2) follows from $n = 0$. Arguing in the same way, for $t \geq 1$ we get

$$\begin{aligned} \|Ae^{tA}\| &= \left\| \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} e^{t\lambda} \lambda R(\lambda, A) d\lambda \right\| \\ &\leq \frac{M_\varepsilon}{2\pi} \sup_{\lambda \in \Gamma_\varepsilon} |\lambda(\lambda - s(A))|^{-1} \left(2 \int_a^{+\infty} e^{(\omega + \xi \cos\theta)t} d\xi + \int_{-b}^{b} e^{(\omega_A + \varepsilon)t} dy \right) \\ &\leq \frac{M_\varepsilon}{\pi} (|\cos\theta|^{-1} + b) e^{(\omega_A + \varepsilon)t} \leq \widetilde{M}_\varepsilon e^{(\omega_A + 2\varepsilon)t} t^{-1}. \end{aligned}$$

Since ε is arbitrary, (5.2) follows also for $n = 1$.

From the equality $A^n e^{tA} = (Ae^{\frac{t}{n}A})^n$ we get, for $n \geq 2$,

$$\|A^n e^{tA}\|_{L(X)} \leq (M_{1,\varepsilon} n t^{-1} e^{\frac{t}{n}(s(A)+\varepsilon)})^n \leq (M_{1,\varepsilon} e)^n n!\, t^{-n} e^{(s(A)+\varepsilon)t},$$

and (5.2) is proved. □

We remark that in the case $s(A) = \omega = 0$, estimates (1.14) are better than (5.2) for t large.

We consider now the problem of the boundedness of the function $t \mapsto e^{tA}x$ for t in $[0, +\infty)$. From proposition 5.1.1 it follows that if $s(A) < 0$, then such a function is bounded for every $x \in X$. In the case in which $s(A) \geq 0$, we investigate whether it is possible to characterize the elements x such that $e^{tA}x$ is bounded in $[0, +\infty)$. We shall see that this is possible in the case where the spectrum of A does not intersect the imaginary axis.

5.2 Behavior of e^{tA} for a hyperbolic A

Let us assume that

$$\sigma(A) \cap i\mathbb{R} = \emptyset. \tag{5.3}$$

In this case A is said to be *hyperbolic*. Set $\sigma(A) = \sigma_- \cup \sigma_+$, where

$$\sigma_- = \sigma(A) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}, \quad \sigma_+ = \sigma(A) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}. \tag{5.4}$$

Since σ_-, σ_+ are closed we have

$$-\omega_- = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma_-\} < 0, \quad \omega_+ = \inf\{\operatorname{Re}\lambda : \lambda \in \sigma_+\} > 0. \tag{5.5}$$

σ_- and σ_+ may be also void: in this case we set $\omega_- = +\infty$, $\omega_+ = +\infty$. Let P be the operator defined by

$$P = \frac{1}{2\pi i} \int_{\gamma_+} R(\lambda, A) d\lambda, \tag{5.6}$$

where γ_+ is a closed regular curve contained in $\rho(A)$, surrounding σ_+, oriented counterclockwise, with index 1 with respect to each point of σ_+, and with index 0 with respect to each point of σ_-. P is called *spectral projection* relevant to σ_+.

Proposition 5.2.1 *The following statements hold.*

(i) P is a projection, that is $P^2 = P$. Moreover $P \in L(X, D(A^n))$ for every $n \in \mathbb{N}$.

(ii) For each $t \geq 0$ we have

$$e^{tA}P = Pe^{tA} = \frac{1}{2\pi i} \int_{\gamma_+} e^{\lambda t} R(\lambda, A) d\lambda.$$

Consequently, $e^{tA}(P(X)) \subset P(X)$, $e^{tA}((I-P)(X)) \subset (I-P)(X)$.

(iii) Setting

$$e^{tA}x = \frac{1}{2\pi i}\int_{\gamma_+} e^{\lambda t}R(\lambda, A)x d\lambda, \;\; x \in P(X), \; t < 0,$$

we have

$$e^{tA}e^{sA}x = e^{(t+s)A}x, \;\; \forall x \in P(X), \; t, s \in \mathbb{R},$$

$$e^{tA}x \in D(A^n) \;\; \forall x \in P(X), \; n \in \mathbb{N},$$

$$\frac{d^n}{dt^n}e^{tA}x = A^n e^{tA}x, \;\; t \in \mathbb{R}, \; x \in P(X).$$

(iv) For every $\omega \in [0, \omega_+)$ there exists $N_\omega > 0$ such that for every $x \in P(X)$ we have

$$\|e^{tA}x\| + \|Ae^{tA}x\| + \|A^2e^{tA}x\| \le N_\omega e^{\omega t}\|x\|, \;\; t \le 0.$$

(v) For each $\omega \in [0, \omega_-)$ there exists $M_\omega > 0$ such that for every $x \in (I - P)(X)$ we have

$$\|e^{tA}x\| + \|tAe^{tA}x\| + \|t^2A^2e^{tA}x\| \le M_\omega e^{-\omega t}\|x\|, \;\; t \ge 0.$$

Proof. (i) Let γ_+, γ'_+ be regular curves contained in $\rho(A)$ surrounding σ_+, with index 1 with respect to each point of σ_+, and such that γ_+ is contained in the bounded connected component of $\mathbb{C} \setminus \gamma'_+$. Then we have

$$\begin{aligned}
P^2 &= \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma'_+} R(\xi, A)d\xi \int_{\gamma_+} R(\lambda, A)d\lambda \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma'_+\times\gamma_+} [R(\lambda, A) - R(\xi, A)](\xi - \lambda)^{-1}d\xi d\lambda \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_+} R(\lambda, A)d\lambda \int_{\gamma'_+} (\xi - \lambda)^{-1}d\xi \\
&\quad - \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma'_+} R(\xi, A)d\xi \int_{\gamma_+} (\xi - \lambda)^{-1}d\lambda \\
&= P.
\end{aligned}$$

The proof of (ii) is similar and it is left as an exercise.

(iii) Since the path γ_+ is bounded and the function under integral is continuous with values in $D(A)$, the integral defining $e^{tA}x$, for $t \le 0$ and $x \in P(X)$, has values in $D(A)$. Moreover we have

$$Ae^{tA}x = \frac{1}{2\pi i}\int_{\gamma_+} e^{\lambda t}(\lambda R(\lambda, A)x - x)d\lambda = \frac{1}{2\pi i}\int_{\gamma_+} e^{\lambda t}\lambda R(\lambda, A)x d\lambda,$$

$$\frac{d}{dt}e^{tA}x = \frac{1}{2\pi i}\int_{\gamma_+} \lambda e^{\lambda t}R(\lambda, A)x d\lambda = Ae^{tA}x.$$

One shows by recurrence that $e^{tA}x \in D(A^n)$ for every n, and that

$$\frac{d^n}{dt^n}e^{tA}x = \frac{1}{2\pi i}\int_{\gamma_+} \lambda^n e^{\lambda t}R(\lambda, A)x d\lambda.$$

(iv) Since $\omega \in [0, \omega_+)$, we choose γ_+ such that $\inf_{\lambda\in\gamma_+} \mathrm{Re}\,\lambda = \omega$. Then we have

$$\begin{aligned} \|A^n e^{tA} x\| &\leq \frac{1}{2\pi} \left| \int_{\gamma_+} |\lambda|^n |e^{\lambda t}| \, \|R(\lambda, A)\| \, \|x\| d\lambda \right| \\ &\leq c_n \sup_{\lambda\in\gamma_+} |e^{\lambda t}| \, \|x\| = c_n e^{\omega t} \|x\|. \end{aligned}$$

(v) We have

$$e^{tA}(I - P) = \frac{1}{2\pi} \left(\int_{\gamma_{r,\eta}} - \int_{\gamma_+} \right) e^{\lambda t} R(\lambda, A) d\lambda = \int_{\gamma_-} e^{\lambda t} R(\lambda, A) d\lambda,$$

with $\gamma_- = \{\lambda \in \mathbb{C} : \lambda = -\omega + re^{\pm i\theta}, \ r \geq 0\}$, oriented as usual, $\theta > \pi/2$ suitable. The estimates may be obtained as in the proof of theorem 1.2.3(iii) and of proposition 5.1.1, and they are left as an exercise. □

Corollary 5.2.2 *For $x \in X$ we have*

$$\sup_{t\geq 0} \|e^{tA} x\| < \infty \Longleftrightarrow Px = 0.$$

Proof — Write every $x \in X$ as $x = Px + (I - P)x$, so that $e^{tA}x = e^{tA}Px + e^{tA}(I - P)x$. The norm of the second addendum decays exponentially to 0 as $t \to +\infty$. The norm of the first one is unbounded if $Px \neq 0$. Indeed, $Px = e^{-tA}e^{tA}Px$, so that $\|Px\| \leq \|e^{-tA}\|_{L(P(X))} \|e^{tA}Px\| \leq N_\omega e^{-\omega t} \|e^{tA}Px\|$ with $\omega > 0$, which implies that $\|e^{tA}Px\| \geq e^{\omega t}\|Px\|/N_\omega$. Therefore $t \mapsto e^{tA}x$ is bounded in $\mathbb{R}_+$ if and only if $Px = 0$. □

Example 5.2.3 Let us consider again examples 2.1.1 and 2.1.2, choosing as X a space of continuous functions.

In the case of example 2.1.1, we have $X = C_b(\mathbb{R})$, $A : D(A) = C_b^2(\mathbb{R}) \mapsto X$, $Au = u''$, $\rho(A) = \mathbb{C} \setminus (-\infty, 0]$, $\|\lambda R(\lambda, A)\| \leq (\cos\theta/2)^{-1}$, with $\theta = \arg\lambda$. In this case $\omega = s(A) = 0$, and estimates (5.2) are worse than (1.14) for large t. It is convenient to use (1.14), which give

$$\|e^{tA}\| \leq M_0, \ \ \|t^k A^k e^{tA}\| \leq M_k, \ \ k \in \mathbb{N}, \ t > 0.$$

Therefore for every initial datum u_0, $e^{tA}u_0$ is bounded, and the k-th derivative with respect to time, the $2k$-th derivative with respect to x decay as $t \to \infty$ at least like t^{-k}, in the sup norm.

Let us consider now the problem

$$\begin{cases} u_t(t,x) = u_{xx}(t,x) + \alpha u(t,x), \ \ t > 0, \ 0 \leq x \leq \pi, \\ u(0,x) = u_0(x), \ \ 0 \leq x \leq \pi, \\ u(t,0) = u(t,\pi) = 0, \ \ t \geq 0, \end{cases} \tag{5.7}$$

with $\alpha \in \mathbb{R}$. Choose $X = C([0,\pi])$, $A : D(A) = \{f \in C^2([0,\pi]) : f(0) = f(\pi) = 0\} \mapsto X$, $Au = u'' + \alpha u$. Then the spectrum of A consists of the sequence of eigenvalues

$$\lambda_n = -n^2 + \alpha, \ \ n \in \mathbb{N}.$$

In particular, if $\alpha < 1$ the spectrum is contained in the halfplane $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\}$, and by proposition 5.1.1 the solution $u(t, \cdot) = e^{tA}u_0$ of (5.7) and all its derivatives decay exponentially as $t \to +\infty$, for any initial datum u_0.

If $\alpha = 1$, assumption (1.9) holds with $\omega = 0$. This is not immediate; one has to study the explicit expression of $R(\lambda, A)$ (which coincides with $R(\lambda - 1, B)$ where $B : D(A) \mapsto x$, $Bf = f''$) near $\lambda = 0$, see example 2.1.2). We use then theorem 1.2.3(iii), which implies that for every initial datum u_0 the solution is bounded.

If $\alpha > 1$, there are elements of the spectrum of A with positive real part. In the case where $\alpha \neq n^2$ for every $n \in \mathbb{N}$ (say $n^2 < \alpha < (n+1)^2$) assumption (5.3) is satisfied. By corollary 5.2.2, the initial data u_0 such that the solution is bounded are those which satisfy $Pu_0 = 0$. The projection P may be written as

$$P = \sum_{k=1}^{n} P_k,$$

where $P_k = \frac{1}{2\pi i}\int_{C(\lambda_k,\varepsilon)} R(\lambda, A)d\lambda$, and the numbers $\lambda_k = -k^2 + \alpha$, $k = 1, \ldots, n$, are the eigenvalues of A with positive real part.

It is possible to show that

$$(P_k f)(x) = \frac{2}{\pi}\int_0^{\pi} \sin ky\, f(y)dy \sin kx, \quad x \in [0, \pi]. \tag{5.8}$$

Consequently, the solution if (5.7) is bounded in $[0, +\infty)$ if and only if

$$\int_0^{\pi} \sin ky\, u_0(y)dy = 0, \quad k = 1, \ldots, n.$$

Exercises 5.2.4

1. Let α, $\beta \in \mathbb{R}$, and let A be the realization of the second order derivative in $C([0, 1])$, with domain $\{f \in C^2([0, 1]) : \alpha f(i) + \beta f'(i) = 0, \ i = 0, 1\}$. Find $s(A)$.

2. Let A satisfy (5.3), and let $T > 0$, $f : [-T, 0] \mapsto P(X)$ be a continuous function, let $x \in P(X)$. Prove that the backward problem

$$\begin{cases} u'(t) = Au(t) + f(t), & -T \leq t \leq 0 \\ \\ u(0) = x, \end{cases}$$

has a unique strict solution in the interval $[0, T]$ with values in $P(X)$, given by the variation of constants formula

$$u(t) = e^{tA}x + \int_0^t e^{(t-s)A} f(s)ds, \quad -T \leq t \leq 0.$$

3. Let A be a sectorial operator such that $\sigma(A) = \sigma_1 \cup \sigma_2$, where σ_1 is compact, σ_2 is closed, and $\sigma_1 \cap \sigma_2 = \emptyset$. Define P by

$$P = \frac{1}{2\pi i}\int_{\gamma} R(\lambda, A)d\lambda,$$

where γ is any regular closed curve in $\rho(A)$, around σ_1, with index 1 with respect to each point in σ_1 and with index 0 with respect to each point in σ_2.

Prove that the part A_1 of A in $P(X)$ is a bounded operator, and that the group generated by A_1 may be expressed as

$$e^{tA_1} = \frac{1}{2\pi i}\int_{\gamma} e^{\lambda t} R(\lambda, A)d\lambda.$$

5.3 Bounded solutions in unbounded intervals

5.3.1 Bounded solutions in $[0, +\infty)$

In this section we consider the problem

$$\begin{cases} u'(t) = Au(t) + f(t), & t > 0, \\ \\ u(0) = u_0, \end{cases} \tag{5.9}$$

where $f : [0, +\infty) \mapsto X$ is a continuous function and $x \in \overline{D(A)}$. We assume throughout that A is hyperbolic, i.e. (5.3) holds, and we define $\sigma_-(A)$, $\sigma_+(A)$ and $-\omega_-$, ω_+ as in section 5.2.

Let P be the projection defined by (5.6). Fix once and for all a positive number ω such that

$$-\omega_- < -\omega < \omega < \omega_+,$$

and let M_ω, N_ω the constants given by proposition 5.2.1(iv)(v).

Given $f \in C_b([0, +\infty); X)$, $u_0 \in X$, we set

$$u_1(t) = e^{tA}(I - P)u_0 + \int_0^t e^{(t-s)A}(I - P)f(s)ds, \;\; t \geq 0,$$

$$u_2(t) = -\int_t^{+\infty} e^{(t-s)A} P f(s)ds, \;\; t \geq 0.$$

Lemma 5.3.1 *The following statements hold.*

(i) For every $f \in C_b([0, +\infty); X)$ and $u_0 \in \overline{D(A)}$ the function u_1 is in $C_b([0, +\infty); X)$, and

$$\|u_1\|_\infty \leq C_1(\|u_0\| + \|f\|_\infty). \tag{5.10}$$

If in addition $f \in C^\alpha([0, +\infty); X)$, $u_0 \in D(A)$, $Au_0 + f(0) \in \overline{D(A)}$, then u_1', Au_1 belong to $C_b([0, +\infty); X)$, and

$$\|u_1\|_\infty + \|u_1'\|_\infty + \|Au_1\|_\infty \leq C_{1,\alpha}(\|u_0\| + \|Au_0\| + \|f\|_{C^\alpha}). \tag{5.11}$$

(ii) For each $f \in C_b([0, +\infty); X)$, $u_2 \in C_b([0, +\infty); D(A))$, moreover u_2 is differentiable, $u_2' \in C_b([0, +\infty); X)$, and

$$\|u_2\|_\infty + \|u_2'\|_\infty + \|Au_2\|_\infty \leq C_2 \|f\|_{C^\alpha}. \tag{5.12}$$

Proof — (i) For every $t \geq 0$ we have

$$\begin{aligned} \|u_1(t)\| &\leq M_\omega e^{-\omega t}\|(I - P)u_0\| + \int_0^t M_\omega e^{-\omega(t-s)}ds \sup_{0 \leq s \leq t} \|f(s)\| \\ &\leq M_\omega \|(I - P)\| \left(\|u_0\| + \frac{1}{\omega}\|f\|_\infty \right). \end{aligned}$$

If $u_0 \in D(A)$ then $(I-P)u_0 \in D(A)$; if $f \in C^\alpha([0,+\infty);X)$ then for every $t \geq 0$ we have

$$\begin{aligned}
\|Au_1(t)\| &\leq M_\omega e^{-\omega t}\|(I-P)Au_0\| + \left\|A\int_0^t e^{(t-s)A}(I-P)(f(s)-f(t))ds\right\| \\
&\quad + \left\|A\int_0^t e^{sA}(I-P)f(t)ds\right\| \\
&\leq M_\omega\|(I-P)Au_0\| + M_\omega\int_0^t \frac{e^{-\omega(t-s)}}{(t-s)^{1-\alpha}}ds[(I-P)f]_{C^\alpha} \\
&\quad + \|(e^{tA}-I)(I-P)f(t)\| \\
&\leq \|(I-P)\|\left(M_\omega(\|Au_0\| + \frac{\Gamma(\alpha)}{\omega^\alpha}[f]_{C^\alpha}\right) + (M_\omega+1)\|f\|_\infty).
\end{aligned}$$

(ii) For every $t \geq 0$ we have

$$\|u_2(t)\| \leq N_\omega \int_t^\infty e^{\omega(t-s)}ds \sup_{s\geq 0}\|Pf(s)\| = \frac{N_\omega}{\omega}\|P\|\,\|f\|_\infty).$$

Similarly, $\|Au_2(t)\| \leq \omega^{-1}N_\omega\|P\|\,\|f\|_\infty$. Moreover

$$u_2'(t) = Pf(t) - \int_t^{+\infty} Ae^{(t-s)A}Pf(s)ds = Au_2(t) + Pf(t), \quad t \geq 0,$$

so that

$$\sup_{t\geq 0}\|u_2'(t)\| + \sup_{t\geq 0}\|Au_2(t)\| \leq \left(\frac{3N_\omega}{\omega} + 1\right)\|P\|\,\|f\|_\infty.$$

□

From lemma 5.3.1 we get easily a necessary and sufficient condition on the data u_0, f for problem (5.9) have a X-bounded solution in $[0,+\infty)$.

Proposition 5.3.2 *Let $f \in C_b([0,+\infty);X)$, $u_0 \in \overline{D(A)}$. Then the mild solution u of (5.9) belongs to $C_b([0,+\infty);X)$ if and only if*

$$Pu_0 = -\int_0^{+\infty} e^{-sA}Pf(s)ds. \tag{5.13}$$

If (5.13) holds we have

$$u(t) = e^{tA}(I-P)u_0 + \int_0^t e^{(t-s)A}(I-P)f(s)ds - \int_t^{+\infty} e^{(t-s)A}Pf(s)ds, \quad t \geq 0. \tag{5.14}$$

If in addition $f \in C^\alpha([0,+\infty);X)$, $u_0 \in D(A)$, $Au_0 + f(0) \in \overline{D(A)}$, then u belongs to $C_b([0,+\infty);D(A))$.

Proof — For every $t \geq 0$ we have

$$\begin{aligned}
u(t) &= (I-P)u(t) + Pu(t) \\
&= e^{tA}(I-P)u_0 + \int_0^t e^{(t-s)A}(I-P)f(s)ds + e^{tA}Pu_0 + \int_0^t e^{(t-s)A}Pf(s)ds \\
&= u_1(t) + e^{tA}Pu_0 + \left(\int_0^{+\infty} - \int_t^{+\infty}\right)e^{(t-s)A}Pf(s)ds \\
&= u_1(t) + u_2(t) + e^{tA}\left(Pu_0 + \int_0^{+\infty} e^{-sA}Pf(s)ds\right).
\end{aligned}$$

The functions u_1 and u_2 are bounded thanks to lemma 5.3.1, hence u is bounded if and only if $t \mapsto e^{tA}\left(Pu_0 + \int_0^{+\infty} e^{-sA}Pf(s)ds\right)$ is bounded. On the other hand $y = Pu_0 + \int_0^{+\infty} e^{-sA}Pf(s)ds$ is an element of $P(X)$. Therefore $e^{tA}y$ is bounded if and only if $y = 0$, namely (5.13) holds.

In the case where (5.13) holds, then $u = u_1 + u_2$, that is (5.14) holds. The remaining part of the proposition follows from lemma 5.3.1. □

5.3.2 Bounded solutions in $(-\infty, 0]$

In this section we study backward solutions of

$$\begin{cases} v'(t) = Av(t) + g(t), & t \le 0, \\ v(0) = v_0, \end{cases} \tag{5.15}$$

where $g : (-\infty, 0] \mapsto X$ is a continuous and bounded function, $v_0 \in \overline{D(A)}$. We assume again that A is hyperbolic.

Problem (5.15) is in general ill-posed. We shall see in fact that to find a solution we will have to assume rather restrictive conditions on the data. On the other hand, such conditions will ensure nice regularity properties of the solutions.

Given $g \in C_b((-\infty, 0]; X)$, $v_0 \in X$, we set

$$v_1(t) = \int_{-\infty}^{t} e^{(t-s)A}(I - P)g(s)ds, \quad t \le 0,$$

$$v_2(t) = e^{tA}Pv_0 + \int_0^t e^{(t-s)A}Pg(s)ds, \quad t \le 0.$$

Lemma 5.3.3 *The following statements hold.*

(i) For every $g \in C_b((-\infty, 0]; X)$ the function v_1 belongs to $C_b((-\infty, 0]; X)$, and moreover

$$\|v_1\|_\infty \le C\|g\|_\infty. \tag{5.16}$$

If in addition $g \in C^\alpha((-\infty, 0]; X)$ for some $\alpha \in (0, 1)$, then $v_1 \in C^\alpha((-\infty, 0]; D(A))$, moreover v_1 is differentiable, $v_1' \in C^\alpha((-\infty, 0]; X)$, and we have

$$\|v_1'\|_{C^\alpha} + \|Av_1\|_{C^\alpha} \le C\|g\|_{C^\alpha}. \tag{5.17}$$

(ii) For every $g \in C_b((-\infty, 0]; X)$ and for every $v_0 \in X$, the function v_2 belongs to $C_b((-\infty, 0]; D(A))$; moreover v_2 is differentiable, $v_2' = Av_2 + Pg$, and

$$\|v_2\|_\infty + \|v_2'\|_\infty + \|Av_2\|_\infty \le C(\|v_0\| + \|g\|_\infty). \tag{5.18}$$

Proof — (i) For each $t \le 0$ we have

$$\|v_1(t)\| \le M_\omega \int_{-\infty}^{0} e^{-\omega(t-s)}ds \sup_{s \le 0} \|(I - P)g(s)\| \le \frac{M_\omega}{\omega}\|I - P\| \, \|g\|_\infty.$$

If $g \in C^\alpha((-\infty,0];X)$ then $v_1(t) \in D(A)$, and we have

$$\begin{aligned}
\|Av_1(t)\| &\leq \left\|\int_{-\infty}^0 Ae^{(t-s)A}(I-P)(g(s)-g(t))ds\right\| \\
&\quad + \left\|\int_{-\infty}^0 Ae^{(t-s)A}(I-P)g(t)ds\right\| \\
&\leq M_\omega \int_{-\infty}^t \frac{e^{-\omega(t-s)}}{(t-s)^{1-\alpha}}[(I-P)g]_{C^\alpha} + \left\|A\int_0^{+\infty} e^{\sigma A}d\sigma(I-P)g(t)\right\| \\
&\leq M_\omega \frac{\Gamma(\alpha)}{\omega^\alpha}\|I-P\|\,[g]_{C^\alpha} + \|-(I-P)g(t)\| \\
&\leq \|I-P\|\left(M_\omega\frac{\Gamma(\alpha)}{\omega^\alpha} + \|g\|_\infty\right).
\end{aligned}$$

The proof of $Av_1 \in C^\alpha((-\infty,0];X)$ is similar to this one and to the one of theorem 4.2.6, and it is left as an exercise.

Let us prove (ii). For every $t \leq 0$ we have

$$\begin{aligned}
\|v_2(t)\| &\leq N_\omega e^{\omega t}\|Pv_0\| + N_\omega \left|\int_0^t e^{\omega(t-s)ds}\right| \sup_{s\leq 0}\|Pg(s)\| \\
&\leq N_\omega\left(\|Pv_0\| + \frac{1}{\omega}\|P\|\,\|g\|_\infty\right).
\end{aligned}$$

Similarly we get $\|Av_2(t)\| \leq N_\omega(\|Pv_0\| + \omega^{-1}\|P\|\,\|g\|_\infty)$. Since $v_2' = Av_2 + Pg$, estimate (5.18) follows. □

Lemma 5.3.3 allows us to give a necessary and sufficient condition on the data g, u_0 for problem (5.15) have a X-bounded solution in $(-\infty,0]$.

A function $v \in C((-\infty,0];X)$ is said to be a mild solution of (5.15) in $(-\infty,0]$ if $v(0) = v_0$ and for each $a < 0$ we have

$$v(t) = e^{(t-a)A}v(a) + \int_a^t e^{(t-s)A}g(s)ds, \quad a \leq t \leq 0. \tag{5.19}$$

In other words, v is a mild solution of (5.15) if and only if for every $a < 0$, setting $y = v(a)$, v is a mild solution of the problem

$$\begin{cases} v'(t) = Av(t) + g(t), & a < t \leq 0, \\ v(a) = y, \end{cases} \tag{5.20}$$

and moreover $v(0) = v_0$.

Proposition 5.3.4 *Let $g \in C_b((-\infty,0];X)$, $v_0 \in X$. Then problem (5.15) has a mild solution $v \in C_b((-\infty,0];X)$ if and only if*

$$(I-P)v_0 = \int_{-\infty}^0 e^{-sA}(I-P)g(s)ds. \tag{5.21}$$

If (5.21) holds, the bounded solution is unique and it is given by

$$v(t) = e^{tA}Pv_0 + \int_0^t e^{(t-s)A}Pg(s)ds + \int_{-\infty}^t e^{(t-s)A}(I-P)g(s)ds, \quad t \leq 0. \tag{5.22}$$

If in addition $g \in C^\alpha((-\infty,0];X)$ for some $\alpha \in (0,1)$, then v is a strict solution and it belongs to $C^\alpha((-\infty,0];D(A))$, v' belongs to $C^\alpha((-\infty,0];X)$.

Proof — Assume that (5.15) has a bounded mild solution v. Then for every $a<0$ and for every $t\in[a,0]$ we have

$$
\begin{aligned}
v(t) &= (I-P)v(t)+Pv(t)\\
&= e^{(t-a)A}(I-P)v(a)+\int_a^t e^{(t-s)A}(I-P)g(s)ds+Pv(t)\\
&= e^{(t-a)A}(I-P)v(a)+\left(\int_{-\infty}^t-\int_{-\infty}^a\right)e^{(t-s)A}(I-P)g(s)ds+Pv(t)\\
&= e^{(t-a)A}\left((I-P)v(a)+\int_{-\infty}^a e^{(a-s)A}(I-P)g(s)ds\right)+v_1(t)+Pv(t)\\
&= e^{(t-a)A}((I-P)v(a)+v_1(a))+v_1(t)+Pv(t).
\end{aligned}
$$

Thanks to lemma 5.3.3, $\sup_{a\le 0}\|v_1(a)\|<\infty$, moreover by assumption $\sup_{a\le 0}\|(I-P)v(a)\|<\infty$. Letting $a\to-\infty$ we get

$$v(t)=v_1(t)+Pv(t),\ \ t\le 0.$$

On the other hand, Pv is a mild (indeed, strict) solution of problem

$$\begin{cases} w'(t)=Aw(t)+Pg(t), & a<t\le 0,\\ w(a)=Pv(a), \end{cases}$$

and since $Pv(0)=Pv_0$, we have for $t\le 0$,

$$Pv(t)=e^{tA}Pv_0+\int_0^t e^{(t-s)A}Pg(s)ds=v_2(t),$$

so that $v(t)=v_1(t)+v_2(t)$, and (5.22) holds. Therefore, $(I-P)v(t)=v_1(t)$, and for $t=0$ we get (5.21).

Conversely, by lemma 5.3.3 the function v defined by (5.22) belongs to $C_b((\infty,0];X)$. One checks easily that for every $a<0$ it is a mild solution of (5.20), and if (5.21) holds we have $v(0)=Pv_0+\int_{-\infty}^0 e^{-sA}(I-P)g(s)ds=Pv_0+(I-P)v_0=v_0$.

The last statement follows again from lemma 5.3.3. □

5.3.3 Bounded solutions in R

Here we study existence and properties of bounded solutions in $\mathbb{R}$ of the equation

$$z'(t)=Az(t)+h(t),\ \ t\in\mathbb{R},\tag{5.23}$$

where $h:\mathbb{R}\mapsto X$ is continuous and bounded. We shall assume again that A is hyperbolic.

A function $z\in C_b(\mathbb{R};X)$ is said to be a mild solution of (5.23) in $\mathbb{R}$ if for every $a\in\mathbb{R}$ we have

$$z(t)=e^{(t-a)A}z(a)+\int_a^t e^{(t-s)A}h(s)ds,\ \ t\ge a,\tag{5.24}$$

that is if for every $a\in\mathbb{R}$, setting $z(a)=\overline{z}$, z is a mild solution of

$$\begin{cases} z'(t)=Av(t)+h(t), & t>a,\\ z(a)=\overline{z}. \end{cases}\tag{5.25}$$

Proposition 5.3.5 *For every $h \in C_b(\mathbb{R};X)$ problem (5.23) has a unique mild solution $z \in C_b(\mathbb{R};X)$, given by*

$$z(t) = \int_{-\infty}^{t} e^{(t-s)A}(I-P)h(s)ds - \int_{t}^{\infty} e^{(t-s)A}Ph(s)ds, \ \ t \in \mathbb{R}. \tag{5.26}$$

If in addition $h \in C^\alpha(\mathbb{R};X)$ for some $\alpha \in (0,1)$, then z is a strict solution and it belongs to $C^\alpha(\mathbb{R};D(A))$.

Proof — Let z be a mild solution belonging to $C_b(\mathbb{R};X)$, and let $z(0) = z_0$. By proposition 5.3.2,

$$Pz_0 = -\int_0^{+\infty} e^{-sA}Ph(s)ds,$$

and by proposition 5.3.4,

$$(I-P)z_0 = \int_{-\infty}^{0} e^{-sA}(I-P)h(s)ds.$$

Due again to proposition 5.3.2 for $t \geq 0$ we have

$$\begin{aligned} z(t) &= e^{tA}\int_{-\infty}^{0} e^{-sA}(I-P)h(s)ds \\ &\quad + \int_0^t e^{(t-s)A}(I-P)h(s)ds - \int_t^{+\infty} e^{(t-s)A}Ph(s)ds \\ &= \int_{-\infty}^{t} e^{(t-s)A}(I-P)h(s)ds - \int_t^{+\infty} e^{(t-s)A}Ph(s)ds. \end{aligned}$$

Moreover due to proposition 5.3.4 for $t \leq 0$ we have

$$\begin{aligned} z(t) &= e^{tA}\left(-\int_0^{\infty} e^{-sA}Ph(s)ds\right) \\ &\quad + \int_0^t e^{(t-s)A}Ph(s)ds + \int_{-\infty}^{t} e^{(t-s)A}(I-P)h(s)ds \\ &= -\int_t^{+\infty} e^{(t-s)A}Ph(s)ds + \int_{-\infty}^{t} e^{(t-s)A}(I-P)h(s)ds, \end{aligned}$$

so that (5.26) holds. On the other hand, by lemmas 5.3.1 and 5.3.3, the function z given by (5.26) belongs to $C_b(\mathbb{R};X)$, and one can easily check that it is a mild solution. If in addition $h \in C^\alpha(\mathbb{R};X)$, then $z \in C^\alpha(\mathbb{R};X)$, due again to lemmas 5.3.1 and 5.3.3. □

Remark 5.3.6 It is not hard to verify that

(i) if h is constant, then z is constant;

(ii) if $\lim_{t\to+\infty} h(t) = h_\infty$ (respectively, $\lim_{t\to-\infty} h(t) = h_{-\infty}$) then

$$\lim_{t\to+\infty} z(t) = \int_0^{+\infty} e^{sA}(I-P)h_\infty ds - \int_{-\infty}^{0} e^{sA}Ph_\infty ds$$

(respectively, the same but $+\infty$ replaced by $-\infty$);

(iii) if h is T-periodic, then z is T-periodic.

5.4 Solutions with exponential growth and exponential decay

Assumption (5.3) is replaced now by

$$\sigma(A) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = \omega\} = \emptyset, \tag{5.27}$$

for some $\omega \in \mathbb{R}$. Note that (5.27) is satisfied by every $\omega > s(A)$. If I is any of the sets $(-\infty, 0]$, $[0, +\infty)$, $\mathbb{R}$, we set

$$C_\omega(I; X) = \{f : I \mapsto X \text{ continuous} |\, \|f\|_{C_\omega} = \sup_{t\in I} \|e^{-\omega t} f(t)\| < \infty\},$$

and for $\alpha \in (0, 1)$

$$C^\alpha_\omega(I; X) = \{f : I \mapsto X |\, t \mapsto e^{-\omega t} f(t) \in C^\alpha(I; X)\},$$

$$\|f\|_{C^\alpha_\omega} = \sup_{t\in I} \|e^{-\omega t} f(t)\| + \sup_{t,s\in I,\, t\neq s} \frac{\|e^{-\omega t} f(t) - e^{-\omega s} f(s)\|}{|t - s|^\alpha}.$$

Let $f \in C_\omega([0, +\infty); X)$, $g \in C_\omega((-\infty, 0]; X)$, $h \in C_\omega(\mathbb{R}; X)$. One checks easily that problems (5.9), (5.15), (5.23) have mild solutions $u \in C_\omega([0, +\infty); X)$, $v \in C_\omega((-\infty, 0]; X)$, $z \in C_\omega(\mathbb{R}; X)$ if and only if the problems

$$\begin{cases} \tilde{u}'(t) = (A - \omega I)\tilde{u}(t) + e^{-\omega t} f(t), \quad t > 0, \\ \\ u(0) = u_0, \end{cases} \tag{5.28}$$

$$\begin{cases} \tilde{v}'(t) = (A - \omega I)\tilde{v}(t) + e^{-\omega t} g(t), \quad t \le 0, \\ \\ v(0) = v_0, \end{cases} \tag{5.29}$$

$$\tilde{z}'(t) = (A - \omega I)\tilde{z}(t) + e^{-\omega t} h(t), \quad t \in \mathbb{R}, \tag{5.30}$$

have mild solutions $\tilde{u} \in C_b([0, +\infty); X)$, $\tilde{v} \in C_b((-\infty, 0]; X)$, $\tilde{z} \in C(\mathbb{R}; X)$, and in this case we have $u(t) = e^{\omega t}\tilde{u}(t)$, $v(t) = e^{\omega t}\tilde{v}(t)$, $z(t) = e^{\omega t}\tilde{z}(t)$. On the other hand the operator $\tilde{A} = A - \omega I : D(A) \mapsto X$ is sectorial and hyperbolic, hence all the results of the previous section may be applied to problems (5.28), (5.29), (5.30). Note that the projection P is associated to the operator $\tilde{A}$, so that

$$P = \frac{1}{2\pi i} \int_{\gamma_+} R(\lambda, A - \omega I) d\lambda = \frac{1}{2\pi i} \int_{\gamma_+ + \omega} R(z, A) dz, \tag{5.31}$$

where the path $\gamma_+ + \omega$ surrounds $\sigma^\omega_+ = \{\lambda \in \sigma(A) : \ \operatorname{Re}\lambda > \omega\}$ and is contained in the halfplane $\{\operatorname{Re}\lambda > \omega\}$. Set moreover $\sigma^\omega_- = \{\lambda \in \sigma(A) : \ \operatorname{Re}\lambda < \omega\}$. Note that if $\omega > s(A)$ then $P = 0$.

Applying the results of the previous section we get the following theorems.

Theorem 5.4.1 *Under assumption (5.27) let P be defined by (5.31). The following statements hold:*

(i) If $f \in C_\omega([0, +\infty); X)$ and $u_0 \in \overline{D(A)}$, the mild solution u of problem (5.9) belongs to $C_\omega([0, +\infty); X)$ if and only if

$$P u_0 = -\int_0^{+\infty} e^{-s(A-\omega I)} e^{-\omega s} P f(s) ds,$$

that is [1]

$$Pu_0 = -\int_0^{+\infty} e^{-sA}Pf(s)ds.$$

In this case u is given by (5.14), and there exists $C_1 = C_1(\omega)$ such that

$$\|u\|_{C_\omega([0,+\infty);X)} \leq C_1(\|u_0\| + \|f\|_{C_\omega([0,+\infty);X)}).$$

If in addition $f \in C^\alpha_\omega([0,+\infty);X)$ for some $\alpha \in (0,1)$, $u_0 \in D(A)$, $Au_0 + f(0) \in \overline{D(A)}$, then $u \in C_\omega([0,+\infty);D(A))$, and there exists $C_1' = C_1'(\omega,\alpha)$ such that

$$\|u\|_{C_\omega([0,+\infty);D(A))} \leq C_1(\|u_0\|_{D(A)} + \|f\|_{C^\alpha_\omega([0,+\infty);X)}).$$

(ii) If $g \in C_\omega((-\infty,0];X)$ and $v_0 \in X$, problem (5.15) has a mild solution $v \in C_\omega((-\infty,0]$ $;X)$ if and only if (5.21) holds. In this case the solution is unique in $C_\omega((-\infty,0];X)$ and it is given by (5.22). There is $C_2 = C_2(\omega)$ such that

$$\|v\|_{C_\omega((-\infty,0];X)} \leq C(\|v_0\| + \|g\|_{C_\omega((-\infty,0];X)}).$$

If in addition $g \in C^\alpha_\omega((-\infty,0];X)$ for some $\alpha \in (0,1)$, then $v \in C^\alpha_\omega((-\infty,0];D(A))$ and there exists $C = C(\omega,\alpha)$ such that

$$\|v\|_{C^\alpha_\omega((-\infty,0];D(A))} \leq C_2(\|v_0\| + \|g\|_{C^\alpha_\omega((-\infty,0];X)}).$$

(iii) If $h \in C_\omega(\mathbb{R};X)$, problem (5.23) has a unique mild solution $z \in C_\omega(\mathbb{R};X)$, given by (5.26), and there is $C_3 = C_3(\omega)$ such that

$$\|z\|_{C_\omega(\mathbb{R};X)} \leq C_3\|h\|_{C_\omega(\mathbb{R};X)}.$$

If in addition $h \in C^\alpha_\omega(\mathbb{R};X)$ for some $\alpha \in (0,1)$, then $z \in C^\alpha_\omega(\mathbb{R};D(A))$ and there is $C_4 = C_4(\omega,\alpha)$ such that

$$\|x\|_{C^\alpha_\omega(\mathbb{R};D(A))} \leq C_4\|h\|_{C^\alpha_\omega(\mathbb{R};X)}.$$

Remark 5.4.2 The definition (5.3) of a hyperbolic operator needs that X be a complex Banach space, and the proofs of the properties of P, Pe^{tA} etc., rely on properties of Banach space valued holomorphic functions.

If X is a real Banach space, we have to use the complexification of X as in remark 1.2.17. If $A : D(A) \mapsto X$ is a linear operator such that the complexification $\widetilde{A}$ is sectorial in $\widetilde{X}$, the projecion P maps X into itself. It is convenient to choose as γ_+ a circumference $C = \{\omega' + re^{i\eta} : \eta \in [0,2\pi]\}$ with center ω' on the real axis. For each $x \in X$ we have

$$Px = \frac{1}{2\pi}\int_0^{2\pi} re^{i\eta}R(re^{i\eta},A)x\,d\eta$$

$$= \frac{r}{2\pi}\int_0^{\pi} \left(e^{i\eta}R(re^{i\eta},A) - e^{-i\eta}R(re^{-i\eta},A)x\right)d\eta,$$

and the imaginary part of the function under the integral is zero. Therefore, $P(X) \subset X$,and consequently $(I-P)(X) \subset X$. Consequently, the results of the last two sections remain true even if X is a real Banach space.

[1] Note that since σ^ω_+ is bounded, $e^{tA}P$ is well defined also for $t < 0$, and the results of Proposition 5.2.1 hold, with obvious modifications.

Example 5.4.3 Consider the nonhomogeneous heat equation

$$\begin{cases} u_t(t,x) = u_{xx}(t,x) + f(t,x), \quad t > 0, \ 0 \le x \le \pi, \\ \\ u(0,x) = u_0(x), \quad 0 \le x \le \pi, \\ \\ u(t,0) = u(t,\pi) = 0, \quad t \ge 0, \end{cases} \tag{5.32}$$

where $f : [0,+\infty) \times [0,\pi] \mapsto \mathbb{R}$ is continuous, u_0 is continuous and vanishes at 0, π. We choose as usual $X = C([0,\pi])$, $A : D(A) = \{f \in C^2([0,\pi]) : f(0) = f(\pi) = 0\} \mapsto X$, $Au = u''$. Since $s(A) = -1$, A is hyperbolic, and in this case $P = 0$. Proposition 5.3.2 implies that for every continuous and bounded f and for every $u_0 \in C([0,\pi])$ such that $u_0(0) = u_0(\pi) = 0$, the solution of (5.32) is bounded.

As far as exponentially decaying solutions are concerned, we use theorem 5.4.1(i). Fixed $\omega \ne n^2$ for each $n \in \mathbb{N}$, f continuous and such that

$$\sup_{t \ge 0,\, 0 \le x \le \pi} |e^{\omega t} f(t,x)| < \infty$$

the solution u of (5.32) satisfies

$$\sup_{t \ge 0,\, 0 \le x \le \pi} |e^{\omega t} u(t,x)| < \infty$$

if and only if (5.13) holds. This is equivalent to (see example 5.2.3)

$$\int_0^\pi u_0(x) \sin kx \, dx = -\int_0^{+\infty} e^{k^2 s} \int_0^\pi f(s,x) \sin kx \, dx \, ds,$$

for every natural number k such that $k^2 < \omega$. (We remark that since $A \sin kx = -k^2 \sin kx$ we have $e^{tA} \sin kx = e^{-tk^2}$, for every $t \in \mathbb{R}$).

Let us consider now the backward problem

$$\begin{cases} v_t(t,x) = v_{xx}(t,x) + g(t,x), \quad t < 0, \ 0 \le x \le \pi, \\ \\ v(0,x) = v_0(x), \quad 0 \le x \le \pi, \\ \\ v(t,0) = v(t,\pi) = 0, \quad t \le 0, \end{cases} \tag{5.33}$$

to which we apply proposition 5.3.4. Since $P = 0$, if $g : (-\infty,0] \times [0,\pi] \mapsto \mathbb{R}$ is continuous and bounded, there is only a final datum v_0 such that the solution is bounded, and it is given by (see formula (5.21))

$$v_0(x) = \int_{-\infty}^0 e^{-sA} g(s,\cdot) ds(x), \quad 0 \le x \le \pi.$$

Thanks to theorem 5.4.1(i), a similar conclusion holds if g is continuous and it decays exponentially,

$$\sup_{t \le 0,\, 0 \le x \le \pi} |e^{-\omega t} g(t,x)| < \infty$$

with $\omega > 0$.

Let us consider the problem on $\mathbb{R}$

$$\begin{cases} z_t(t,x) = z_{xx}(t,x) + h(t,x), \quad t \in \mathbb{R}, \ 0 \le x \le \pi, \\ \\ z(t,0) = z(t,\pi) = 0, \quad t \in \mathbb{R}. \end{cases} \tag{5.34}$$

Thanks to proposition 5.3.5, for every continuous and bounded $h : \mathbb{R} \times [0, \pi] \mapsto \mathbb{R}$ problem (5.34) has a unique bounded solution given by

$$z(t, x) = \int_{-\infty}^{t} e^{(t-s)A} h(s, \cdot) ds(x), \quad t \in \mathbb{R}, \ 0 \le x \le \pi.$$

The considerations of remark 5.3.6 hold: in particular, if h is T-periodic with respect to time, then z is T-periodic too; if h is independent of time also z is independent of time, and we have

$$z(t, x) = \int_{-\infty}^{t} e^{(t-s)A} h(\cdot) ds(x) = (-A^{-1}h)(x).$$

The explicit expression of $A^{-1}h$ may be easily computed by solving the ordinary differential equation $f'' = h$, $f(0) = f(\pi) = 0$.

Chapter 6

Nonlinear problems

6.1 Local existence, uniqueness, regularity

Consider the initial value problem

$$\begin{cases} u'(t) = Au(t) + F(t, u(t)), \quad t > 0, \\ \\ u(0) = u_0, \end{cases} \tag{6.1}$$

where $A : D(A) \subset X \mapsto X$ is a sectorial operator and $F : [0, T] \times X \mapsto X$ is a regular function (at least, continuous with respect to (t, u) and locally Lipschitz continuous with respect to u).

As in the case of linear problems, a function u defined in an interval $I = [0, \tau)$ or $I = [0, \tau]$, with $\tau \le T$, is said to be a *strict solution* of problem (6.1) in I if it is continuous with values in $D(A)$ and differentiable with values in X in the interval I, and it satisfies (6.1). It is said to be a *classical solution* if it is continuous with values in $D(A)$ and differentiable with values in X in the interval $I \setminus \{0\}$, it is continuous in I with values in X, and it satisfies (6.1). It is said to be a *mild solution* if it is continuous with values in X in $I \setminus \{0\}$ and it satisfies

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}F(s, u(s))ds, \quad t \in I. \tag{6.2}$$

Thanks to proposition 4.2.3 every strict solution satisfies (6.2), and every classical solution u such that $t \mapsto \|F(t, u(t))\| \in L^1(0, \varepsilon)$ for some $\varepsilon > 0$ satisfies (6.2). It is natural to solve (6.2) using a fixed point theorem to find a mild solution, and then to show that under appropriate assumptions the mild solution is classical or strict.

We assume that $F : [0, T] \times X \mapsto X$ is continuous, and for every $R > 0$ there is $L > 0$ such that

$$\|F(t, x) - F(t, y)\| \le L\|x - y\|, \quad \forall t \in [0, T], \quad \forall x, y \in B(0, R). \tag{6.3}$$

Theorem 6.1.1 *Let $F : [0, T] \times X \mapsto X$ be a continuous function satisfying (6.3). Then for every $\overline{u} \in X$ there exist r, $\delta > 0$, $K > 0$ such that for $\|u_0 - \overline{u}\| \le r$ problem (6.1) has a unique mild solution $u = u(\cdot; u_0) \in C_b((0, \delta]; X)$. u belongs to $C([0, \delta]; X)$ if and only if $u_0 \in \overline{D(A)}$.*

Moreover for u_0, $u_1 \in B(\overline{u}, r)$ we have

$$\|u(t; u_0) - u(t; u_1)\| \le K\|u_0 - u_1\|, \quad 0 \le t \le \delta. \tag{6.4}$$

Proof. Let M_0 such that $\|e^{tA}\|_{L(X)} \leq M_0$ for $0 \leq t \leq T$. Fix $R > 0$ such that $R \geq 8M_0\|\overline{u}\|$, so that if $\|u_0 - \overline{u}\| \leq r = R/8M_0$ we have

$$\sup_{0 \leq t \leq T} \|e^{tA}u_0\| \leq R/4.$$

Let moreover $L > 0$ be such that

$$\|F(t,v) - F(t,w)\| \leq L\|v - w\| \text{ for } 0 \leq t \leq T, \ v, w \in B(0,R).$$

We look for a mild solution belonging to the metric space Y defined by

$$Y = \{u \in C_b((0,\delta];X) : \|u(t)\| \leq R \ \forall t \in (0,\delta]\},$$

where $\delta \in (0,T]$ has to be chosen properly. Y is the closed ball $B(0,R)$ in the space $C_b((0,\delta];X)$, and for every $v \in Y$ the function $t \mapsto F(\cdot, v(\cdot))$ belongs to $C_b((0,\delta];X)$. We define a nonlinear operator Γ on Y,

$$\Gamma(v)(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}F(s,v(s))ds, \ \ 0 \leq t \leq \delta. \tag{6.5}$$

Clearly, a function $v \in Y$ is a mild solution of (6.1) in $[0,\delta]$ if and only if it is a fixed point of Γ.

We shall show that Γ is a contraction and maps Y into itself provided δ is sufficiently small.

Let v_1, $v_2 \in Y$. We have

$$\begin{aligned} &\|\Gamma(v_1) - \Gamma(v_2)\|_{C_b([0,\delta];X)} \leq \delta M_0 \|F(\cdot, v_1(\cdot)) - F(\cdot, v_2(\cdot))\|_{C_b((0,\delta];X)} \\ &\leq \delta M_0 L \|v_1 - v_2\|_{C_b((0,\delta;X)}. \end{aligned} \tag{6.6}$$

Therefore, if

$$\delta \leq \delta_0 = (2M_0L)^{-1},$$

Γ is a contraction with constant $1/2$ in Y. Moreover for every $v \in Y$ and $t \in [0,\delta]$, with $\delta \leq \delta_0$, we have

$$\begin{aligned} \|\Gamma(v)\|_{C_b((0,\delta];X)} &\leq \|\Gamma(v) - \Gamma(0)\|_{C_b((0,\delta];X)} + \|\Gamma(0)\|_{C((0,\delta];X)} \\ &\leq R/2 + \|e^{tA}u_0\|_{C_b((0,\delta];X)} + M_0\delta\|F(\cdot,0)\|_{C_b((0,\delta];X)} \\ &\leq R/2 + R/4 + M_0\delta\|F(\cdot,0)\|_{C_b((0,\delta];X)}. \end{aligned} \tag{6.7}$$

Therefore if $\delta \leq \delta_0$ is such that

$$M_0\delta\|F(\cdot,0)\|_{C_b((0,\delta];X)} \leq R/4,$$

then Γ maps Y into itself, so that it has a unique fixed point in Y.

Concerning the continuity of u up to $t = 0$, we remark that the function $t \mapsto u(t) - e^{tA}u_0$ belongs to $C([0,\delta];X)$, whereas $t \mapsto e^{tA}u_0$ belongs to $C([0,\delta];X)$ if and only if $u_0 \in \overline{D(A)}$. Therefore, $u \in C([0,\delta];X)$ if and only if $u_0 \in \overline{D(A)}$.

Let us prove the statement about the dependence on the initial data. Let u_0, u_1 belong to $B(\overline{u}, r)$. Since Γ is a contraction with constant $1/2$ in Y and both $u(\cdot;u_0)$, $u(\cdot;u_1)$ belong to Y, we have

$$\|u(\cdot;u_0) - u(\cdot;u_1)\|_{C_b((0,\delta];X)} \leq 2\|e^{tA}(u_0 - u_1)\|_{C_b((0,\delta];X)} \leq 2M_0\|u_0 - u_1\|,$$

so that (6.4) holds, with $K = 2M_0$.

Let us prove uniqueness: if u_1, $u_2 \in C_b((0,\delta];X)$ are solutions of (6.1), we define

$$t_0 = \sup\{t \in [0,\delta] : u_1(s) = u_2(s) \text{ for } 0 \le s \le t\}, \tag{6.8}$$

and we set $y = u_1(t_0) = u_2(t_0)$. If $t_0 < \delta$, the problem

$$v'(t) = Av(t) + F(t, v(t)), \;\; t > t_0, \quad v(t_0) = y, \tag{6.9}$$

has a unique solution in the set

$$Y' = \{u \in C_b((t_0, t_0+\varepsilon];X) : \;\; \|u(t)\| \le R' \; \forall t \in (t_0, t_0+\varepsilon]\},$$

provided R' is sufficiently large and ε is sufficiently small. Since u_1 and u_2 are bounded, there exists R' such that $\|u_i(t)\| \le R'$ for $t_0 \le t \le \delta$, $i = 1, 2$. On the other hand, u_1 and u_2 are different mild solutions of (6.9) in $[t_0, t_0+\varepsilon]$ for every $\varepsilon \in (0, \delta - t_0]$. This is a contradiction, hence $t_0 = \delta$ and the mild solution of (6.1) is unique in $C_b((0,\delta];X)$. □

Remark 6.1.2 *In the proof of theorem 6.1.1 we have shown uniqueness of the mild solution in $[0,\delta]$, but the same argument works in any interval contained in $[0,T]$.*

6.2 The maximally defined solution

Now we can construct a maximally defined solution as follows. Set

$$\begin{cases} \tau(u_0) = \sup\{a > 0 : \text{problem (6.1) has a mild solution } u_a \text{ in } [0,a]\} \\ \\ u(t;u_0) = u_a(t), \;\; \text{if } t \le a. \end{cases}$$

$u(t;u_0)$ is well defined thanks to remark 6.1.2 in the interval

$$I(u_0) = \cup\{[0,a] : \text{ problem (6.1) has a mild solution } u_a \text{ in } [0,a]\},$$

and we have $\tau(u_0) = \sup I(u_0)$. Moreover, if $\tau(u_0) < T$, then $\tau(u_0)$ does not belong to $I(u_0)$ because otherwise the solution could be extended to a bigger interval, contradicting the definition of $\tau(u_0)$. See exercise 6.2.4.2.

Let us prove now regularity and existence in the large results.

Proposition 6.2.1 *Let F satisfy (6.3). Then for every $u_0 \in X$, the mild solution u of problem (6.1) is bounded with values in $D_A(\theta,\infty)$ in the interval $[\varepsilon, \tau(u_0) - \varepsilon]$, for each $\theta \in (0,1)$ and $\varepsilon \in (0, \tau(u_0)/2)$.*

Assume in addition that there is $\alpha \in (0,1)$ such that for every $R > 0$ we have

$$\|F(t,x) - F(s,x)\| \le C(R)(t-s)^\alpha, \;\; 0 \le s \le t \le T, \; \|x\| \le R. \tag{6.10}$$

Then, for every $u_0 \in X$, $u \in C^\alpha([\varepsilon, \tau(u_0) - \varepsilon]; D(A)) \cap C^{1+\alpha}([\varepsilon, \tau(u_0) - \varepsilon]; X)$ for every $\varepsilon \in (0, \tau(u_0)/2)$. Moreover the following statements hold.

(i) If $u_0 \in \overline{D(A)}$ then $u(\cdot;u_0)$ is a classical solution of (6.1).

(ii) If $u_0 \in D(A)$ and $Au_0 + F(0,u_0) \in \overline{D(A)}$ then $u(\cdot;u_0)$ is a strict solution of (6.1).

Proof. The function $t \mapsto e^{tA}u_0$ belongs to $C((0,+\infty);D(A))$ so that its restriction to $[\varepsilon, \tau(u_0) - \varepsilon]$ is bounded with values in each $D_A(\theta,\infty)$. The function

$$t \mapsto v(t) = \int_0^t e^{(t-s)A} F(s, u(s))ds$$

is bounded with values in $D_A(\theta,\infty)$ because $\|e^{(t-s)A}\|_{\mathcal{L}(X,D_A(\theta,\infty))} \le C(t-s)^{\theta-1}$, so that $\|v(t)\|_{D_A(\theta,\infty)} \le$ const. $\sup_{0<s\le\tau(u_0)-\varepsilon} \|F(s,u(s))\|$.

Assume now that (6.10) holds; let $a < \tau(u_0)$ and $0 < \varepsilon < a$. Since $t \mapsto F(t,u(t))$ belongs to $C_b((0,a];X)$, proposition 4.2.4 implies that the function v defined above belongs to $C^\alpha([0,a];X)$. Moreover, $t \mapsto e^{tA}u_0$ belongs to $C^\infty([\varepsilon,a];X)$. Summing up, we find that u belongs to $C^\alpha([\varepsilon,a];X)$. Assumptions (6.3) and (6.10) imply that the function $t \mapsto F(t,u(t))$ belongs to $C^\alpha([\varepsilon,a];X)$. Recalling that u satisfies

$$u(t) = e^{(t-\varepsilon)A}u(\varepsilon) + \int_\varepsilon^t e^{(t-s)A}F(s,u(s))ds, \;\; \varepsilon \le t \le a, \tag{6.11}$$

we may apply theorem 4.2.6 in the interval $[\varepsilon,a]$ (see remark 4.2.11), to obtain that u belongs to $C^\alpha([2\varepsilon,a];D(A)) \cap C^{1+\alpha}([2\varepsilon,a];X)$ for each $\varepsilon \in (0,a/2)$, and

$$u'(t) = Au(t) + F(t,u(t)), \;\; \varepsilon < t \le a.$$

Since a and ε are arbitrary, then $u \in C^\alpha([\varepsilon,\tau(u_0)-\varepsilon];D(A)) \cap C^{1+\alpha}([\varepsilon,\tau(u_0)-\varepsilon];X)$ for each $\varepsilon \in (0,\tau(u_0)/2)$. If $u_0 \in \overline{D(A)}$, then $t \mapsto e^{tA}u_0$ is continuous up to 0, and statement (i) follows.

Let us prove (ii). We know already that the function $t \mapsto u(t) - e^{tA}u_0$ is α-Hölder continuous up to $t=0$ with values in X. Since $u_0 \in D(A) \subset D_A(\alpha,\infty)$, the same is true for $t \mapsto e^{tA}u_0$. Therefore u is α-Hölder continuous up to $t=0$ with values in X, so that $t \mapsto F(t,u(t))$ is α-Hölder continuous in $[0,a]$ with values in X. Statement (ii) follows now from 4.2.6(ii). □

Proposition 6.2.2 *Assume that F satisfies (6.3). Let u_0 be such that $I(u_0) \subset [0,T]$, $I(u_0) \ne [0,T]$. Then $t \mapsto \|u(t)\|$ is unbounded in $I(u_0)$.*

Proof. Assume by contradiction that u is bounded and set $\tau = \tau(u_0)$. Then $t \mapsto F(t,u(t;u_0))$ is continuous and bounded with values in X in the interval $(0,\tau)$. Since u satisfies the variation of constants formula (6.2), it may be continuously extended at $t=\tau$, in such a way that the extension is Hölder continuous in every interval $[\varepsilon,\tau]$, with $0 < \varepsilon < \tau$. Indeed, $t \mapsto e^{tA}u_0$ is well defined and analytic in the whole $(0,+\infty)$, and the function $v = e^{tA} * F(\cdot,u(\cdot))$ belongs to $C^\alpha([0,\tau];X)$ for each $\alpha \in (0,1)$ because of proposition 4.2.4.

Moreover, $u(\tau) \in \overline{D(A)}$. By theorem 6.1.1, the problem

$$v'(t) = Av(t) + F(t,v(t)), \;\; t \ge \tau, \quad v(\tau) = u(\tau),$$

has a unique mild solution $v \in C([\tau,\tau+\delta];X)$ for some $\delta > 0$. The function w defined by $w(t) = u(t)$ for $0 \le t < \tau$, $w(t) = v(t)$ for $\tau \le t \le \tau+\delta$, is a mild solution of (6.1) in $[0,\tau+\delta]$. This is in contradiction with the definition of τ. Therefore, $u(\cdot;u_0)$ cannot be bounded. □

The result of proposition 6.2.2 is used to prove existence in the large when we have an *a priori* estimate on the norm of $u(t)$. Such *a priori* estimate is easily available for each u_0 if f grows not more than linearly as $\|x\| \to \infty$.

Proposition 6.2.3 *Assume that there is $C > 0$ such that*

$$\|F(t,x)\| \le C(1+\|x\|) \;\; \forall x \in X, \; t \in [0,T]. \tag{6.12}$$

Let $u : I(u_0) \mapsto X$ be the mild solution to (6.1). Then u is bounded in $I(u_0)$ with values in X.

Proof. For each $t \in I$ we have

$$\|u(t)\| \leq M_0\|u_0\| + M_0 C \int_0^t (1 + \|u(s)\|)ds = M_0\|u_0\| + M_0 C \left(t + \int_0^t \|u(s)\|ds \right).$$

Applying the Gronwall lemma to the real valued function $t \mapsto \|u(t)\|$ we get

$$\|u(t)\| \leq e^{M_0 C t}(M_0\|u_0\| + M_0 CT), \;\; t \in I(u_0),$$

and the statement follows. □

We remark that (6.12) is satisfied if F is globally Lipschitz continuous with respect to x, with Lipschitz constant independent of t.

Exercises 6.2.4

1. Prove that

    (a) if F satisfies (6.3) and $u \in C_b((0,\delta];X)$ with $0 < \delta \leq T$ then the composition $\varphi(t) = f(t, u(t))$ belongs to $C_b((0,\delta];X)$,

    (b) if F satisfies (6.10) and $u \in C^\alpha([a,b];X)$ with $0 \leq a < b \leq T$ then the composition $\varphi(t) = f(t,u(t))$ belongs to $C^\alpha([a,b];X)$.

    These properties have been used in the proofs of theorem 6.1.1 and of proposition 6.2.1.

2. Prove that if u is a mild solution to (6.1) in an interval $[0, t_0]$ and v is a mild solution to

$$\begin{cases} v'(t) = Av(t) + F(t, v(t)), \;\; t_0 < t < t_1, \\ \\ v(t_0) = u(t_0), \end{cases}$$

    then the function z defined by $z(t) = u(t)$ for $0 \leq t \leq t_0$, $z(t) = v(t)$ for $t_0 \leq t \leq t_1$, is a mild solution to (6.1) in the interval $[0, t_1]$.

3. Under the assumptions of theorem 6.1.1, for $t_0 \in [0,T)$ let $u(t;t_0,x) : [t_0, \tau(t_0,x)) \mapsto X$ the maximally defined solution to problem $u' = Au + f(t,u)$, $t > t_0$, $u(t_0) = x$. Prove that for each $a \in (0, \tau(0,u_0))$ we have $\tau(u(a;u_0)) = \tau(0,u_0) - a$, and for $t \in [a, \tau(0,u_0))$ we have $u(t; u(a,u_0)) = u(a+t; 0, u_0)$.

4. Under the assumptions of theorem 6.1.1, prove that the maximally defined solution to (6.1) depends locally Lipschitz continuously on the initial datum, i.e. for each u_0 and for each $b \in (0, \tau(u_0))$ there are $r > 0$, $K > 0$ such that if $\|u_0 - u_1\| \leq r$ then $\tau(u_1) \geq b$ and $\|u(t;u_0) - u(t;u_1)\| \leq K\|u_0 - u_1\|$ for each $t \in [0,b]$.

    (Hint: cover the orbit $\{u(t;u_0) : 0 \leq t \leq b\}$ by a finite number of balls such as in the statement of theorem 6.1.1).

6.3 Reaction – diffusion equations and systems

Let us consider a differential system in $[0,T] \times \mathbb{R}^n$. Let $d_1, \ldots, d_m > 0$ and let D be the diagonal matrix $D = diag(d_1, \ldots, d_m)$. Consider the problem

$$u_t = D\Delta u + \varphi(t,x,u), \;\; t > 0, \; x \in \mathbb{R}^n; \quad u(0,x) = u_0(x), \;\; x \in \mathbb{R}^n, \tag{6.13}$$

where $u = (u_1, \ldots, u_m)$ is unknown, and the regular function $\varphi : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^m$, the bounded and continuous $u_0 : \mathbb{R}^n \mapsto \mathbb{R}^m$ are data.

This type of problems are often encountered as mathematical models in chemistry and biology. The part $D\Delta u$ in the system is called diffusion part, the numbers d_i are called diffusion coefficients, $\varphi(t,x,u)$ is called reaction part. Detailed treatments of these problems may be found in the books of Rothe [13], Smoller [14], Pao [11], and in other ones.

Set

$$X = C_b(\mathbb{R}^n;\mathbb{R}^m).$$

The linear operator A defined by

$$\begin{cases} D(A) = \{u \in W^{2,p}_{loc}(\mathbb{R}^n;\mathbb{R}^m)\ \forall p \geq 1 :\ u,\ \Delta u \in X\}, \\ \\ A : D(A) \mapsto X,\ \ Au = D\Delta u, \end{cases}$$

is sectorial in X, see exercise 1.2.18.3, and

$$\overline{D(A)} = BUC(\mathbb{R}^n;\mathbb{R}^m).$$

Assume that φ is continuous, and there exists $\theta \in (0,1)$ such that for every $r > 0$

$$|\varphi(t,x,u) - \varphi(s,x,v)|_{\mathbb{R}^m} \leq K((t-s)^\theta + |u-v|_{\mathbb{R}^m}), \tag{6.14}$$

for $0 \leq s < t \leq T$, $x \in \mathbb{R}^n$, u, $v \in \mathbb{R}^m$, $|v|_{\mathbb{R}^m} + |u|_{\mathbb{R}^m} \leq r$, with $K = K(r)$. Then, setting

$$F(t,u)(x) = \varphi(t,x,u(x)),\ \ 0 \leq t \leq T,\ x \in \mathbb{R}^n, u \in X,$$

the function $F : [0,T] \times X \mapsto X$ is continuous, and it satisfies (6.3). The local existence and uniqueness theorem 6.1.1 implies that there exists a unique mild solution $t \mapsto u(t) \in C_b((0,\delta];X)$ di (6.1). Moreover, since F satisfies (6.10) too, by proposition 6.2.1 u, u', Au are continuous in $(0,\delta]$ with values in X. Then the function $(t,x) \mapsto u(t,x) := u(t)(x)$ is continuous and bounded in $[0,\delta] \times \mathbb{R}^n$ (why is it continuous up to $t = 0$? Compare with exercise 4.2.12.1), it is differentiable in $(0,\delta] \times \mathbb{R}^n$, it has second order space derivatives $D_{ij}u(t,\cdot) \in L^p_{loc}(\mathbb{R}^n;\mathbb{R}^m)$, Δu is continuous in $(0,\delta] \times \mathbb{R}^n$, and u satisfies (6.13).

If in addition $u_0 \in BUC(\mathbb{R}^n;\mathbb{R}^m)$, then $u(t,x) \to u_0(x)$ as $t \to 0$, uniformly for x in $\mathbb{R}^n$. Moreover u is the unique solution to (6.13) in the class of functions v such that $t \mapsto v(t,\cdot)$ belongs to $C^1((0,\delta];C_b(\mathbb{R}^n;\mathbb{R}^m)) \cap C([0,\delta];C_b(\mathbb{R}^n;\mathbb{R}^m))$.

For each initial datum u_0 the solution may be extended to a maximal time interval $I(u_0)$. proposition 6.2.2 implies that if u is bounded in $I(u_0) \times \mathbb{R}^n$ then $I(u_0) = [0,T]$.

A sufficient condition for u to be bounded is given by proposition 6.2.3:

$$|\varphi(t,x,u)|_{\mathbb{R}^m} \leq C(1 + |u|_{\mathbb{R}^m})\ \ \forall t \in [0,T],\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m. \tag{6.15}$$

Indeed, in this case the nonlinear function

$$F : [0,T] \times X \mapsto X,\ \ F(t,u)(x) = \varphi(t,x,u(x))$$

satisfies (6.12).

Similar results hold for reaction – diffusion systems in $[0,T] \times \overline{\Omega}$, where Ω is a bounded open set in $\mathbb{R}^n$ with C^2 boundary.

The simplest case is a single equation,

$$\begin{cases} u_t = \Delta u + \varphi(t,x,u),\ \ t > 0,\ x \in \overline{\Omega}, \\ \\ u(0,x) = u_0(x),\ \ x \in \overline{\Omega}, \end{cases} \tag{6.16}$$

with Dirichlet boundary condition,

$$u(t,x) = 0,\ \ t > 0,\ x \in \partial\Omega, \tag{6.17}$$

or Neumann boundary condition,

$$\frac{\partial u(t,x)}{\partial \nu} = 0, \;\; t > 0, \; x \in \partial\Omega. \tag{6.18}$$

$\varphi : [0,T] \times \overline{\Omega} \times \mathbb{R} \mapsto \mathbb{R}$ is a regular function satisfying (6.14); $u_0 : \overline{\Omega} \mapsto \mathbb{R}$ is continuous and satisfies the compatibility condition $u_0(x) = 0$ for $x \in \partial\Omega$ in the case of the Dirichlet boundary condition.

Again, we set our problem in the space $X = C(\overline{\Omega})$, getting a unique classical solution in a maximal time interval. Arguing as before, we see that if there is $C > 0$ such that

$$|\varphi(t,x,u)| \le C(1+|u|) \;\; \forall t \in [0,T], \; x \in \overline{\Omega}, \; u \in \mathbb{R}$$

then for each initial datum u_0 the solution exists globally. But this assumption is rather restrictive, and it is not satisfied in many mathematical models. In the next subsection we shall see a more general assumption that yields existence in the large.

In this section, up to now we have chosen to work with real valued functions just because in most mathematical models the unknown u is real valued. But we could replace $C_b(\mathbb{R}^n, \mathbb{R}^m)$ and $C(\overline{\Omega};\mathbb{R})$ by $C_b(\mathbb{R}^n;\mathbb{C}^m)$ and $C(\overline{\Omega};\mathbb{C})$ as well without any modification in the proofs, getting the same results in the case of complex valued data. On the contrary, the results of the next subsection hold only for real valued functions.

6.3.1 The maximum principle

Using the well known properties of the first and second order derivatives of real valued functions at relative maximum or minimum points it is possible to find estimates on the solutions to several first or second order partial differential equations. Such techniques are called *maximum principles*.

To begin with, we give a sufficient condition for the solution of (6.16) – (6.17) or of (6.16) – (6.18) to be bounded (and hence, to exist in the large).

Proposition 6.3.1 *Let Ω be a bounded open set in $\mathbb{R}^n$ with C^2 boundary, and let $\varphi : [0,T] \times \overline{\Omega} \times \mathbb{R} \mapsto \mathbb{R}$ be a continuous function satisfying $|\varphi(t,x,u) - \varphi(s,x,v)| \le K((t-s)^\theta + |u-v|)$, for $0 \le s < t \le T$, $x \in \overline{\Omega}$, u, $v \in \mathbb{R}$, $|v| + |u| \le r$, with $K = K(r)$. Assume moreover that*

$$u\varphi(t,x,u) \le C(1+u^2), \;\; 0 \le t \le T, \; x \in \overline{\Omega}, \; u \in \mathbb{R}. \tag{6.19}$$

Then for each initial datum u_0 the solution to (6.16) – (6.17) or to (6.16) – (6.18) satisfies

$$\sup_{t \in I(u_0),\, x \in \overline{\Omega}} |u(t,x)| < +\infty.$$

Proof. Fix $\lambda > C$, $a < \tau(u_0)$ and set

$$v(t,x) = u(t,x)e^{-\lambda t}, \;\; 0 \le t \le a, \; x \in \overline{\Omega}.$$

The function v satisfies

$$v_t(t,x) = \Delta v(t,x) + \varphi(t,x,e^{\lambda t}v(t,x))e^{-\lambda t} - \lambda v(t,x), \;\; 0 < t \le a, \; x \in \overline{\Omega},$$

it satisfies the same boundary condition of u, and $v(0,\cdot) = u_0$. Since v is continuous, there exists (t_0,x_0) such that $v(t_0,x_0) = \pm\|v\|_{C([0,a]\times\overline{\Omega})}$. (t_0,x_0) is either a point of positive maximum of of negative minimum for v. Assume for instance that (t_0,x_0) is a maximum point. If $t_0 = 0$ we have obviously $\|v\|_\infty \le \|u_0\|_\infty$. If $t_0 > 0$ and $x_0 \in \Omega$ we rewrite the differential equation at (t_0, x_0) and we multiply both sides by $v(t_0, x_0) = \|v\|_\infty$: since $v_t(t_0, x_0) \geq 0$, $\Delta v(t_0, x_0) \leq 0$ we get

$$\lambda \|v\|_\infty^2 \leq C(1 + |e^{\lambda t_0} v(t_0, x_0)|^2) e^{-2\lambda t_0} = C(1 + e^{2\lambda t_0} \|v\|_\infty^2) e^{-2\lambda t_0}$$

so that

$$\|v\|_\infty^2 \leq \frac{C}{\lambda - C}.$$

Let us consider the case $t_0 > 0$, $x_0 \in \partial\Omega$. If u satisfies the Dirichlet boundary condition, then $v(t_0, x_0) = 0$. If u satisfies the Neumann boundary condition, we have $D_i v(t_0, x_0) = 0$ for each i and we go on as in the case $x_0 \in \Omega$.

If (t_0, x_0) is a minimum point the proof is similar. So, we have

$$\|v\|_\infty \leq \max\{\|u_0\|_\infty, \sqrt{C/(\lambda - C)}\}$$

so that

$$\|u\|_{L^\infty([0,a]\times\overline{\Omega})} \leq e^{\lambda T} \max\{\|u_0\|_\infty, \sqrt{C/(\lambda - C)}\}$$

and the statement follows. □

In the proof of proposition 6.3.1 we used a property of the functions in $v \in D(A)$, where A is the realization of the Laplacian with Dirichlet or Neumann boundary condition in $C(\overline{\Omega})$: if $x \in \Omega$ is a relative maximum point for u, then $\Delta u \leq 0$. This is obvious if $v \in C^2(\Omega)$, it has to be proved if v is not twice differentiable pointwise.

Lemma 6.3.2 *Let $x_0 \in \mathbb{R}^n$, $r > 0$, and let $v : B(x_0, r) \mapsto \mathbb{R}$ be a continuous function. Assume that $v \in W^{2,p}(B(x_0, r))$ for each $p \in [1, +\infty)$, that Δv is continuous, and that x_0 is a maximum (respectively, minimum) point for v. Then $\Delta v(x_0) \leq 0$ (respectively, $\Delta v(x_0) \geq 0$).*

Proof. Possibly replacing v by $v + c$ we may assume $v(x) \geq 0$ for $|x - x_0| \leq r$. Let $\theta : \mathbb{R}^n \mapsto \mathbb{R}$ be a smooth function with support contained in $B(x_0, r)$, such that $0 \leq \theta(x) \leq 1$ for each x, and $\theta(x_0) > \theta(x)$ for $x \neq x_0$. Define

$$\widetilde{v}(x) \begin{cases} = & v(x)\theta(x), \quad x \in B(x_0, r), \\ \\ = & 0, \quad x \in \mathbb{R}^n \setminus B(x_0, r). \end{cases}$$

Then $\widetilde{v}(x_0)$ is the maximum of $\widetilde{v}$, and it is attained only at $x = x_0$. Moreover, $\widetilde{v}$ and $\Delta\widetilde{v}$ are uniformly continuous and bounded in the whole $\mathbb{R}^n$, so that there is a sequence $(\widetilde{v}_n)_{n\in\mathbb{N}} \subset C^2(\mathbb{R}^n)$ such that $\widetilde{v}_n \to \widetilde{u}$, $\Delta\widetilde{v}_n \to \Delta\widetilde{v}$ (for instance, we can take $\widetilde{v}_n = T(1/n)\widetilde{v}$ where $T(t)$ is the heat semigroup defined in (2.5)). Since x_0 is the unique maximum point of $\widetilde{v}$, there is a sequence x_n going to x_0 such as x_n is a relative maximum point of v_n, for each n. Since $\widetilde{v}_n \in C^2$, we have $\Delta\widetilde{v}_n(x_n) \leq 0$. Letting $n \to \infty$ we get $\Delta v(x_0) \leq 0$.

If x_0 is a minimum point the proof is similar. □

Similar arguments may be used also in some systems. For instance, let us consider

$$\begin{cases} \mathbf{u}_t(t, x) = \Delta\mathbf{u}(t, x) + f(\mathbf{u}(t, x)), \quad t > 0, \ x \in \overline{\Omega}, \\ \\ \mathbf{u}(t, x) = 0, \quad t > 0, \ x \in \overline{\Omega}, \\ \\ \mathbf{u}(0, x) = \mathbf{u_0}(x), \quad x \in \overline{\Omega}, \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^n$ with C^2 boundary, and $f : \mathbb{R}^k \mapsto \mathbb{R}^k$ is a locally Lipschitz continuous function such that

$$\langle \mathbf{x}, f(\mathbf{x}) \rangle \leq C(1 + |\mathbf{x}|^2), \quad \mathbf{x} \in \mathbb{R}^k \tag{6.20}$$

As in the case of a single equation, it is convenient to fix $a \in (0, \tau(\mathbf{u_0}))$ and to introduce the function $\mathbf{v} : [0, a] \times \overline{\Omega} \mapsto \mathbb{R}$, $\mathbf{v}(t, x) = \mathbf{u}(t, x)e^{-\lambda t}$ with $\lambda > C$, that satisfies

$$\begin{cases} \mathbf{v}_t(t,x) = \Delta\mathbf{v}(t,x) + f(e^{\lambda t}\mathbf{v}(t,x))e^{-\lambda t} - \lambda\mathbf{v}(t,x), \;\; t>0, \; x \in \overline{\Omega}, \\ \\ \mathbf{v}(t,x) = 0, \;\; t > 0, x \in \overline{\Omega}, \end{cases}$$

Instead of $|\mathbf{v}|$ it is better to work with $\varphi(t, x) = |\mathbf{v}(t, x)|^2 = \sum_{i=1}^k v_i(t, x)^2$, which is more regular. Let us remark that $\varphi_t = 2\langle \mathbf{v}_t, \mathbf{v}\rangle$, $D_j\varphi = 2\langle D_j\mathbf{v}, \mathbf{v}\rangle$, $\Delta\varphi = 2\sum_{i=1}^k |Dv_i|^2 + 2\langle \mathbf{v}, \Delta\mathbf{v}\rangle$.

If $(t_0, x_0) \in (0, a] \times \Omega$ is a positive maximum point for φ (i.e. for $|\mathbf{v}|$) we have $\varphi_t(t_0, x_0) \geq 0$, $\Delta\varphi(t_0, x_0) \leq 0$ and hence $\langle \mathbf{v}(t_0, x_0), \Delta\mathbf{v}(t_0, x_0)\rangle \leq 0$. Writing the differential system at (t_0, x_0) and taking the scalar product by $\mathbf{v}(t_0, x_0)$ we get

$$\begin{aligned} & 0 \leq \langle \mathbf{v}_t(t_0,x_0), \mathbf{v}(t_0,x_0)\rangle \\ & = \langle \Delta\mathbf{v}(t_0,x_0), \mathbf{v}(t_0,x_0)\rangle + \\ & + \langle f(e^{\lambda t_0}\mathbf{v}(t_0,x_0)), \mathbf{v}(t_0,x_0)e^{-\lambda t_0}\rangle - \lambda|\mathbf{v}(t_0,x_0)|^2 \\ & \leq C(1 + |\mathbf{v}(t_0,x_0)|^2) - \lambda|\mathbf{v}(t_0,x_0)|^2 \end{aligned}$$

so that $\|\mathbf{v}\|_\infty^2 \leq C/(\lambda - C)$. Therefore, $\|\mathbf{v}\|_\infty \leq \max\{\|u_0\|_\infty, \sqrt{C/(\lambda - C)}\}$, which implies that $\|\mathbf{u}\|_{L^\infty([0,a]\times\overline{\Omega})} \leq e^{\lambda T}\max\{\|\mathbf{u_0}\|_\infty, \sqrt{C/(\lambda - C)}\}$, the same result as in the scalar case. Consequently, $\mathbf{u}$ esists in the large.

The problem of existence in the large for reaction – diffusion systems is still a research subject.

Let us remark that (6.15) is a growth condition at infinity, while (6.19) is an algebraic condition and it is not a growth condition. For instance, it is satisfied by $\varphi(t, x, u) = \lambda u - u^{2k+1}$ for each $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. The sign $-$ is important: for instance, in the problem

$$\begin{cases} u_t = \Delta u + |u|^{1+\varepsilon}, \;\; t > 0, \; x \in \overline{\Omega}, \\ \\ u(0,x) = \overline{u}, \;\; x \in \overline{\Omega}, \\ \\ \dfrac{\partial u}{\partial \nu}(t,x) = 0, \;\; t \geq 0, \; x \in \partial\Omega, \end{cases} \tag{6.21}$$

with $\varepsilon > 0$ and constant initial datum $\overline{u}$, the solution is independent of x and it coincides with the solution to the ordinary differential equation

$$\begin{cases} \xi'(t) = |\xi(t)|^{1+\varepsilon}, \;\; t > 0, \\ \\ \xi(0) = \overline{u}, \end{cases}$$

which blows up in finite time if $\overline{u} > 0$.

The maximum principle is used also to prove qualitative properties of the solutions, for instance to prove that the solutions are nonnegative for nonnegative initial data, or nonpositive for nonpositive initial data. Let us give an example.

$$\begin{cases} u_t = u_{xx} + \lambda u - \rho u^2, \;\; t \geq 0, \; 0 \leq x \leq \pi, \\ \\ u(t,0) = u(t,\pi) = 0, \;\; t \geq 0, \\ \\ u(0,x) - u_0(x), \;\; 0 \leq x \leq \pi. \end{cases} \tag{6.22}$$

Here λ, $\rho > 0$. Let us prove that if $u_0(x) \leq 0$ (respectively, $u_0(x) \geq 0$) for each $x \in [0, \pi]$ then $u(t, x) \leq 0$ for each $t \in [0, \tau(u_0))$, $x \in [0, \pi]$.

First, we consider the case $u_0 \leq 0$ in $[0, \pi]$. Fixed any $a \in (0, \tau(u_0))$, let us prove that $u(t, x) \leq 0$ in $[0, a] \times [0, \pi]$. Assume by contradiction that $u(t, x) > 0$ for some (t, x), then the same is true for $v(t, x) := e^{-\lambda t} u(t, x)$. Since $[0, a] \times [0, \pi]$ is compact, v has a maximum point (t_0, x_0) in $[0, a] \times [0, \pi]$, with $v(t_0, x_0) > 0$. This is impossible if $t_0 = 0$, or $x_0 = 0$, or $x_0 = \pi$; therefore $(t_0, x_0) \in (0, a] \times (0, \pi)$, and

$$0 \leq v_t(t_0, x_0) = v_{xx}(t_0, x_0) - \rho(v(t_0, x_0))^2 e^{\lambda t_0} < 0,$$

which is impossible. Then $u(t, x) \leq 0$ for each $t \in I(u_0)$, $x \in [0, \pi]$.

The case $u_0(x) \geq 0$ is a bit more complicated. Fix $\mu > \lambda$. Since u is continuous, there exists $a > 0$ such that $\|u(t, \cdot) - u_0\|_\infty < (\mu - \lambda)/\rho$ per $0 \leq t \leq a$. In particular,

$$u(t, x) \geq -\frac{\mu - \lambda}{\rho}, \quad 0 \leq t \leq a, \ 0 \leq x \leq \pi.$$

Let us consider again the function $v(t, x) := e^{-\mu t} u(t, x)$. We want to show that $v \geq 0$ in $[0, a] \times [0, \pi]$. Assume by contradiction that the minimum of v in $[0, a] \times [0, \pi]$ is strictly negative. If (t_0, x_0) is a minimum point then $t_0 \neq 0$, $x_0 \neq 0$, $x_0 \neq \pi$. Therefore $(t_0, x_0) \in (0, a] \times (0, \pi)$, and

$$0 \geq v_t(t_0, x_0) = v_{xx}(t_0, x_0) + (\lambda - \mu) v(t_0, x_0) - \rho(v(t_0, x_0))^2 e^{\mu t_0}$$

$$\geq (\lambda - \mu) v(t_0, x_0) - \rho(v(t_0, x_0))^2 e^{\mu t_0}$$

so that, dividing by $v(t_0, x_0) < 0$,

$$u(t_0, x_0) = v(t_0, x_0) e^{\mu t_0} \leq -\frac{\mu - \lambda}{\rho},$$

a contradiction. Consequently v, and hence u, has nonnegative values in $[0, a] \times [0, \pi]$.

Set now $\mathcal{I} = \{a \in (0, \tau(u_0)) : \ u(t, x) \geq 0 \text{ in } [0, a] \times [0, \pi]\}$. We have proved above that $\mathcal{I}$ is not empty. Moreover, $\sup \mathcal{I} = \tau(u_0)$. Indeed, if this is not true we may repeat the above procedure with $a_0 := \sup \mathcal{I}$ instead of 0; we find another interval $[a_0, a_0 + \delta]$ in which the solution is nonnegative, and this is a contradiction because of the definition of a_0.

Let us see a system from combustion theory. Here u and v are a concentration and a temperature, respectively, both normalized and rescaled. The numbers $\mathcal{L}$, ε, q are positive parameters. Ω is a bounded open set in $\mathbb{R}^n$ with C^2 boundary.

$$\begin{cases} u_t(t, x) = \mathcal{L} \Delta u(t, x) - \varepsilon u f(v), \quad t > 0, \ x \in \overline{\Omega}, \\ \\ v_t(t, x) = \Delta u(t, x) + q u f(v), \quad t > 0, \ x \in \overline{\Omega}, \\ \\ \dfrac{\partial u}{\partial \nu}(t, x) = 0, \quad v = 1, \quad t > 0, \ x \in \overline{\Omega}, \\ \\ u(0, x)(x) = u_0(x), \quad v(0, x) = v_0(x), \quad x \in \overline{\Omega}, \end{cases} \tag{6.23}$$

f is the Arrhenius function

$$f(v) = e^{-h/v},$$

with $h > 0$. The initial data u_0 and v_0 are continuous nonnegative functions, with $u_0 \equiv 1$ at $\partial\Omega$. Replacing the unknowns (u, v) by $(u, v - 1)$, problem (6.23) may be reduced to a standard problem with zero Dirichlet boundary condition, which we locally solve using the above techniques.

The physically meaningful solutions are such that u, $v \geq 0$. Using the maximum principle we can prove that for nonnegative initial data we get nonnegative solutions.

Let us consider u: if, by contradiction, there is $a > 0$ such that the restriction of u to $[0, a] \times \overline{\Omega}$ has negative minimum, at a minimum point (t_0, x_0) we have $t_0 > 0$, $x_0 \in \Omega$ and

$$0 \geq u_t(t_0, x_0) = \mathcal{L}\Delta u - \varepsilon u(t_0, x_0) f(v(t_0, x_0)) > 0,$$

a contradiction. Therefore u cannot have negative values.

To study the sign of v it is again convenient to introduce the function $z(t, x) := e^{-\lambda t} v(t, x)$ with $\lambda > 0$. If there is $a > 0$ such that the restriction of z to $[0, a] \times \overline{\Omega}$ has negative minimum, at a minimum point (t_0, x_0) we have $t_0 > 0$, $x_0 \in \Omega$ and

$$0 \geq z_t(t_0, x_0) = \Delta z(t_0, x_0) - \lambda z(t_0, x_0) + q u(t_0, x_0) f(z(t_0, x_0) e^{\lambda t_0}) e^{-\lambda t_0} > 0,$$

again a contradiction. Therefore, v too cannot have negative values.

Exercises 6.3.3

1. *Let Ω be an open set in $\mathbb{R}^n$ with C^1 boundary, and let $x_0 \in \partial\Omega$ be a relative maximum point for a C^1 function $v : \overline{\Omega} \mapsto \mathbb{R}$. Prove that if the normal derivative of v vanishes at x_0 then all the partial derivatives of v vanish at x_0.*

   If $\partial\Omega$ and v are C^2, prove that we have also $\Delta v(x_0) \leq 0$.

   These properties have been used in the proof of proposition 6.19.

2. *Prove that for each continuous nonnegative initial function u_0 such that $u_0(0) = u_0(\pi) = 0$, the solution to (6.22) exists in the large.*

Chapter 7

Behavior near stationary solutions

Let $A : D(A) \subset X \mapsto X$ be a sectorial operator, and let $F : X \mapsto X$ be continuously differentiable in a neighborhood of 0, satisfying (6.3) and such that

$$F(0) = 0, \quad F'(0) = 0. \tag{7.1}$$

We shall study the stability of the null solution of

$$u'(t) = Au(t) + F(u(t)), \quad t > 0. \tag{7.2}$$

Thanks to theorem 6.1.1, for every initial datum $u_0 \in \overline{D(A)}$ the initial value problem for equation (7.2) has a unique classical solution $u(\cdot, u_0) : [0, \tau(u_0)) \mapsto X$. The assumption $F(0) = 0$ implies that equation (7.2) has the zero solution. The assumption $F'(0) = 0$ is not restrictive: if $F'(0) \neq 0$ we replace A by $A + F'(0)$ and $F(u)$ by $G(u) = F(u) - F'(0)u$ whose Fréchet derivative vanishes at 0.

Definition 7.0.4 *The null solution of (7.2) is said to be* stable *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$u_0 \in \overline{D(A)}, \quad \|u_0\| \leq \delta \Longrightarrow \tau(u_0) = +\infty, \quad \|u(t; u_0)\| \leq \varepsilon \ \forall t \geq 0.$$

The null solution of (7.2) is said to be asymptotically stable *if it is stable and moreover there exists $\delta > 0$ such that if $\|u_0\| \leq \delta$ then $\lim_{t\to+\infty} u(t; u_0) = 0$.*

The null solution of (7.2) is said to be unstable *if it is not stable.*

7.1 Linearized stability

The main assumption is

$$s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} < 0. \tag{7.3}$$

Theorem 7.1.1 *Let (7.3) hold. Then for every $\omega \in [0, -s(A))$ there exist $M = M(\omega)$, $r = r(\omega) > 0$ such that if $u_0 \in \overline{D(A)}$, $\|u_0\| \leq r$, we have $\tau(u_0) = \infty$ and*

$$\|u(t; u_0)\| \leq Me^{-\omega t}\|u_0\|, \quad t \geq 0. \tag{7.4}$$

Therefore, the null solution is asymptotically stable. Moreover, for every $a > 0$ we have

$$\sup_{t\geq a} \|e^{\omega t}u(t; u_0)\|_{D(A)} < \infty. \tag{7.5}$$

If in addition $u_0 \in D(A)$, $Au_0 + F(u_0) \in \overline{D(A)}$, then

$$\sup_{t\geq 0} \|e^{\omega t}u(t; u_0)\|_{D(A)} < \infty. \tag{7.6}$$

Proof — Let $\rho > 0$ such that

$$K(\rho) = \sup_{\|x\|\le\rho} \|F'(x)\|_{L(X)} < \infty.$$

Since F' is continuous and $F'(0) = 0$, we have

$$\lim_{\rho\to 0} K(\rho) = 0.$$

Let Y be the closed ball centered at 0 with radius ρ in $C_{-\omega}([0,+\infty);X)$, namely

$$Y = \{u \in C_{-\omega}([0,+\infty);X) : \sup_{t\ge 0} \|e^{\omega t}u(t)\| \le \rho\}.$$

We look for the solution (6.1) as a fixed point of the operator Γ defined on Y by

$$(\Gamma u)(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}F(u(s))ds, \quad t \ge 0.$$

If $u \in Y$ then

$$\begin{aligned} \|F(u(t))\| = \|F(u(t)) - F(0)\| = \left\| \int_0^1 F'(\sigma u(t))u(t)d\sigma \right\| \\ \le K(\rho)\|u(t)\| \le K(\rho)\rho e^{-\omega t}, \quad t \ge 0, \end{aligned} \tag{7.7}$$

so that $F(u(\cdot)) \in C_{-\omega}([0,+\infty);X)$. Moreover $\sigma(A) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \ge -\omega\} = \emptyset$, so that we may use theorem 5.4.1(i) (with ω replaced now by $-\omega$): we find that $\Gamma u \in C_{-\omega}([0,+\infty);X)$, and moreover there exists $C_1 = C_1(-\omega)$ such that

$$\|\Gamma u\|_{C_{-\omega}([0,+\infty);X)} \le C_1 \left(\|u_0\| + \|F(u(\cdot))\|_{C_{-\omega}([0,+\infty);X)}\right). \tag{7.8}$$

If ρ is so small that

$$K(\rho) \le \frac{1}{2C_1},$$

and

$$\|u_0\| \le r = \frac{\rho}{2C_1},$$

then $\Gamma u \in Y$. Moreover, for u_1, $u_2 \in Y$ we have

$$\|\Gamma u_1 - \Gamma u_2\|_{C_{-\omega}([0,+\infty);X)} \le C_1\|F(u_1(\cdot)) - F(u_2(\cdot))\|_{C_{-\omega}([0,+\infty);X)},$$

where

$$\begin{aligned} \|F(u_1(t)) - F(u_2(t))\| = \left\| \int_0^1 F'(\sigma u_1(t) + (1-\sigma)u_2(t))(u_1(t) - u_2(t))d\sigma \right\| \\ \le K(\rho)\|u_1(t) - u_2(t)\|. \end{aligned}$$

It follows that

$$\|\Gamma u_1 - \Gamma u_2\|_{C_{-\omega}([0,+\infty);X)} \le \frac{1}{2}\|u_1 - u_2\|_{C_{-\omega}([0,+\infty);X)},$$

so that Γ is a contraction with constant $1/2$. Consequently there exists a unique fixed point of Γ in Y, which is the solution of (6.1). Moreover from (7.7), (7.8) we get

$$\|u\|_{C_{-\omega}} = \|\Gamma u\|_{C_{-\omega}} \le C_1(\|u_0\| + K(\rho)\|u\|_{C_{-\omega}}) \le C_1\|u_0\| + \frac{1}{2}\|u\|_{C_{-\omega}}$$

which implies (7.4), with $M(\omega) = 2C_1(-\omega)$. As far as (7.5) is concerned, since $F(u(\cdot)) \in C_{-\omega}([0, +\infty); X)$ we find

$$u_1(t) = \int_0^t e^{(t-s)A} F(u(s))ds \in C^\alpha_{-\omega}([0, \infty); X), \quad \forall \alpha \in (0, 1),$$

moreover $u_2(t) = e^{tA}u_0 \in C^\alpha_{-\omega}([a, \infty); X)$ for every $a > 0$; consequently $u = u_1 + u_2 \in C^\alpha_{-\omega}([a, \infty); X)$ for every $a > 0$. Moreover by theorem 6.1.1 $u(a) \in D(A)$, and $Au(a) + F(u(a)) = u'(a) \in \overline{D(A)}$. From proposition 5.3.2 it follows that $u \in C_{-\omega}([a, +\infty); D(A))$, namely (7.5) holds. The last statement, as well as (7.6), follow from these considerations and from theorem 6.1.1. □

7.1.1 Linearized instability

Assume now that

$$\begin{cases} \sigma_+(A) = \sigma(A) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} \neq \emptyset, \\ \\ \inf\{\operatorname{Re}\lambda : \lambda \in \sigma_+(A)\} = \omega_+ > 0. \end{cases} \tag{7.9}$$

Then it is possible to prove an instability result for the null solution. We shall use the projection P defined by

$$P = \frac{1}{2\pi i} \int_{\gamma_+} R(\lambda, A) d\lambda,$$

γ_+ being any regular path with range in Re $\lambda > 0$, with index 1 with respect to each $\lambda \in \sigma_+(A)$.

Theorem 7.1.2 *If (7.9) holds, the null solution of (7.2) is unstable. Specifically, there exists $r_+ > 0$ such that for every $x \in P(X)$ satisfying $\|x\| \le r_+$, the problem*

$$\begin{cases} v'(t) = Av(t) + F(v(t)), \quad t \le 0, \\ \\ Pv(0) = x, \end{cases} \tag{7.10}$$

has a backward solution v such that $\lim_{t\to-\infty} v(t) = 0$. (Taking $x_n = v(-n)$, we have $x_n \to 0$ but since $u(t; x_n) = v(t-n)$ we have $\sup_{t\in I(x_n)} \|u(t; x_n)\| \ge \sup \|v(t)\|$, independent of n, so that 0 is unstable).

Proof — Let $\omega \in (0, \omega_+)$, and let $\rho_+ > 0$ be such that

$$\sup_{\|x\|\le\rho_+} \|F'(x)\|_{L(X)} \le \frac{1}{2C_2(\omega)},$$

where $C_2(\omega)$ is given by theorem 5.4.1(ii). Let Y_+ be the closed ball centered at 0 with radius ρ_+ in $C_\omega((-\infty, 0]; X)$. We look for a solution to (7.10) as a fixed point of the operator Γ_+ defined on Y_+ by

$$(\Gamma_+ v)(t) = e^{tA}x + \int_0^t e^{(t-s)A} PF(v(s))ds + \int_{-\infty}^t e^{(t-s)A}(I - P)F(v(s))ds, \quad t \le 0.$$

If $v \in Y_+$, then $F(v(\cdot)) \in C_\omega((-\infty, 0]; X)$; moreover $\sigma(A) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = \omega\} = \emptyset$, so that we may use theorem 5.4.1(ii), which implies $\Gamma_+ v \in C_\omega((-\infty, 0]; X)$, and

$$\|\Gamma_+ v\|_{C_\omega((-\infty,0];X)} \le C_2 \left(\|x\| + \|F(v(\cdot))\|_{C_\omega((-\infty,0];X)}\right).$$

The rest of the proof is quite similar to the proof of theorem 7.1.1 and it is left as an exercise. □

7.1.2 The saddle point property

If A is hyperbolic we may show a saddle point property, constructing the so called stable and unstable manifolds. We shall consider the forward problem (6.1) and the backward problem

$$\begin{cases} v'(t) = Av(t) + F(v(t)), \;\; t \le 0, \\ \\ v(0) = v_0. \end{cases} \tag{7.11}$$

Theorem 7.1.3 *Assume that*

$$\sigma(A) \cap i\mathbb{R} = \emptyset, \;\; \sigma_+(A) \neq \emptyset.$$

Set

$$-\omega_- = \sup\{\text{Re}\,\lambda : \; \lambda \in \sigma(A), \; \text{Re}\,\lambda < 0\},$$

$$\omega_+ = \inf\{\text{Re}\,\lambda : \; \lambda \in \sigma(A), \; \text{Re}\,\lambda > 0\},$$

and fix $\omega \in [0, \min\{\omega_+, \omega_-\})$. Then there exist r, $\rho > 0$ and two continuous functions

$$h : \{x_+ \in P(X) : \; \|x_+\| \le r\} \mapsto \overline{D(A)},$$

$$k : \{x_- \in (I-P)(X) : \; \|x_-\| \le r\} \mapsto \overline{D(A)},$$

such that setting

$$\mathcal{V}_I = \mathcal{V}_I(\omega) = \{h(x_+) : x_+ \in P(X), \; \|x_+\| \le r\},$$

$$\mathcal{V}_S = \mathcal{V}_S(\omega) = \{k(x_-) : x_- \in (I-P)(X) \cap \overline{D(A)}, \; \|x_-\| \le r\},$$

the following statements hold.

(i) For every $u_0 \in \mathcal{V}_S$ the classical solution u of (6.1) exists in the large, it belongs to $C_{-\omega}([0,+\infty);X)$, and $\|u\|_{C_{-\omega}} \le \rho$. Conversely, if $u_0 \in \overline{D(A)}$ is such that $\|(I-P)u_0\| \le r$ and the solution of (6.1) exists in the large, belongs to $C_{-\omega}([0,+\infty),X)$, and its norm is $\le \rho$, then $u_0 \in \mathcal{V}_S$.

(ii) For every $v_0 \in \mathcal{V}_I$ the problem (7.11) has a solution $v \in C_\omega((-\infty,0];X)$, such that $\|v\|_{C_\omega} \le \rho$. Conversely, if v_0 is such that $\|Pv_0\| \le r$ and the problem (7.11) has a solution belonging to $C_\omega((-\infty,0];X)$, with norm $\le \rho$, then $v_0 \in \mathcal{V}_I$.

Proof — Let us prove (i). Let $\rho_- > 0$ be such that

$$\sup_{\|x\| \le \rho_-} \|F'(x)\|_{L(X)} \le \frac{1}{2C_1(-\omega)},$$

where C_1 is given by theorem 5.4.1. Set $Y = B(0,\rho_-) \subset C_{-\omega}([0,+\infty);X)$. For each $u \in Y$, $F(u) \in C_{-\omega}([0,+\infty);X)$. Since $\sigma(A) \cap \{\lambda \in \mathbb{C} : \text{Re}\,\lambda = -\omega\} = \emptyset$, all the solutions of (6.1) belonging to Y may be represented as

$$u(t) = e^{tA}x_- + \int_0^t e^{(t-s)A}(I-P)F(u(s))ds - \int_t^{+\infty} e^{(t-s)A}PF(u(s))ds, \;\; t \ge 0,$$

with any $x_- \in (I-P)(X) \cap \overline{D(A)}$. So, fix $x_- \in (I-P)(X) \cap \overline{D(A)}$ with $\|x_-\| \le r_-$ where $r_- > 0$ has to be chosen, and look for a fixed point of the operator Γ_- defined on Y_- by

$$(\Gamma_- u)(t) = e^{tA}x_- + \int_0^t e^{(t-s)A}(I-P)F(u(s))ds - \int_t^{+\infty} e^{(t-s)A}PF(u(s))ds.$$

Arguing as in the proof of theorem 7.1.1 one sees that Γ_- is a contraction with constant $1/2$, and that if

$$r_- = \frac{\rho_-}{2C_1(-\omega)}$$

then Γ_- maps Y into itself, so that it has a unique fixed point $u_- \in Y$, such that

$$\|u_-\|_{C_{-\omega}([0,+\infty);X)} \le 2C_1(-\omega)\|x_-\|. \tag{7.12}$$

Moreover, the function

$$(\overline{D(A)} \cap (I-P)(X) \cap B(0,r_-)) \times Y \mapsto C_{-\omega}([0,+\infty);X); \quad (x_-,u) \mapsto \Gamma_- u$$

is continuous, so that the fixed point of Γ depends continuously on x_- thanks to the contraction theorem depending on a parameter. Moreover the function

$$\begin{cases} k : (I-P)(X) \cap \overline{D(A)} \cap B(0,r_-) \mapsto \overline{D(A)}, \\ \\ k(x) = u_-(0), \end{cases}$$

is continuous. The solution of (6.1) with initial datum $u_0 = u_-(0)$ coincides with u_-, so that it belongs to $C_{-\omega}([0,+\infty);X)$ and its norm is $\le \rho_-$.

Let now $u_0 \in (I-P)(X) \cap \overline{D(A)}$ be such that $\|(I-P)u_0\| \le r_-$, and that the solution of (6.1) belongs to $C_{-\omega}([0,+\infty);X)$ and has norm $\le \rho_-$. Then, since $F(u(\cdot)) \in C_{-\omega}([0,+\infty);X)$, by theorem 5.4.1(i) we have, for $t \ge 0$,

$$u(t) = e^{tA}(I-P)u_0 + \int_0^t e^{(t-s)A}(I-P)F(u(s))ds - \int_t^\infty e^{(t-s)A}F(u(s))ds,$$

so that u is a fixed point of the operator Γ_- if we choose $x_- = (I-P)u_0$. Since there exists a unique fixed point of Γ_- with norm $\le \rho_-$, then $u_0 = k((I-P)u_0)$, namely $u_0 \in \mathcal{V}_S$. Statement (i) is proved.

The proof of statement (ii) is quite similar: one follows the proof of theorem 7.1.2 and one sets

$$\begin{cases} h : P(X) \cap B(0,r_+) \mapsto \overline{D(A)}, \\ \\ h(x) = v(0), \end{cases}$$

where v is the fixed point of the operator Γ_+ in Y_+, which exists if $r_+ = \rho_+/2C_2(\omega)$.

We take finally $r = \min\{r_-, r_+\}$, $\rho = \min\{\rho_-, \rho_+\}$. $\square$

Remark 7.1.4 The stable manifold $\mathcal{V}_S$ (respectively, the unstable manifold $\mathcal{V}_I$) is tangent at the origin to $(I-P)(X)$ (respectively, to $P(X)$), in the sense that k (respectively, h) is Fréchet differentiable at 0 with derivative $k'(0) = I_{|(I-P)(X)}$ (respectively, $h'(0) = I_{|P(X)}$). Indeed, since by (7.12), $\|u_-\|_{C_{-\omega}} \le 2C_1\|x_-\|$, then we have

$$\|F(u_-(\cdot))\|_{C_{-\omega}} \le \sup_{\|x\|\le\rho_-} \|F'(x)\|_{L(X)} 2C_1\|x_-\| = K(\rho_-)\|x_-\|.$$

Consequently

$$\|k(x_-) - x_-\| = \|u_-(0) - (I-P)u_-(0)\| = \|Pu_-(0)\|$$

$$= \left\| \int_0^{+\infty} e^{-sA}PF(u_-(s))ds \right\| \le C_1\|F(u_-(\cdot))\|_{C_{-\omega}} \le C_1K(\rho_-)\|x_-\|.$$

Given $\varepsilon > 0$, let $\rho_1 > 0$ be such that $C_1K(\rho_1) < \varepsilon$; for every $x_- \in (I-P)(X) \cap \overline{D(A)}$ with $\|x_-\| \le \rho_1/2C_1$ we have $\|k(x_-) - x_-\|/\|x_-\| \le \varepsilon$.

The proof of the statement concerning the function h is similar.

Remark 7.1.5 The proof of theorem 7.1.2 works also for $\omega = 0$, and this implies that if $u : [0, +\infty) \mapsto X$ is a solution of (6.1) with $\sup_{t\geq 0} \|u(t)\|$ sufficiently small, then in fact u decays exponentially to 0, and $u_0 \in \mathcal{V}_S(\omega)$ with $\omega > 0$. Indeed, if $\sup_{t\geq 0} \|u(t)\|$ is small, then also $(I - P)u_0$ is small, and hence u is the fixed point of the operator Γ relevant to the case $\omega = 0$, with $x_- = (I - P)u_0$. On the other hand, for the same choice of x_-, Γ has also a fixed point in $C_{-\omega}([0, +\infty); X)$, and the two fixed points coincide.

Similarly, since $\omega > 0$, if $v : (-\infty, 0] \mapsto X$ is a backward solution of (6.1) and $\sup_{t\leq 0} \|v(t)\|$ is sufficiently small, then v decays exponentially to 0 as $t \to -\infty$, and $v(0) \in \mathcal{V}_I(\omega)$.

Remark 7.1.6 In the case $\omega > 0$, $\mathcal{V}_S$ and $\mathcal{V}_I$ enjoy the following invariance property: if $u_0 \in \mathcal{V}_S$ (respectively, $u_0 \in \mathcal{V}_I$), then there exists t_0 such that $u(t; u_0) \in \mathcal{V}_S$ for every $t \geq t_0$ (respectively, for every $t \leq t_0$).

Indeed, we know already that if $u_0 \in \mathcal{V}_S$ then $u(\cdot; u_0)$ concides with the fixed point u of the operator Γ_- relevant to the initial datum $x_- = (I - P)u_0$. In particular, for $t \geq t_0 \geq 0$,

$$\begin{aligned} u(t) &= e^{(t-t_0)A}(I-P)u(t_0) + \int_{t_0}^{t} e^{(t-\sigma)A}(I-P)F(u(\sigma))d\sigma \\ &\quad - \int_{t}^{+\infty} e^{(t-\sigma)A} PF(u(\sigma))d\sigma, \end{aligned}$$

so that, setting $t = t_0 + s$, for $s > 0$ we obtain, by the changement of variable $\sigma = \tau + t_0$ in the integrals,

$$\begin{aligned} u(s+t_0) &= e^{sA}(I-P)u(t_0) + \int_{0}^{s} e^{(s-\tau)A}(I-P)F(u(t_0+\tau))d\tau \\ &\quad - \int_{s}^{+\infty} e^{(s-\tau)A} PF(u(t_0+\tau))d\tau, \end{aligned}$$

namely the function $v(s) = u(s+t_0)$ is a fixed point of the operator Γ relevant to the initial datum $y = (I - P)u(t_0)$. It follows that $k\ ((I - P)u(t_0)) = u(t_0)$, that is $u(t_0)$ belongs to the range of k. Moreover, since u decays exponentially, if t_0 is sufficiently large then $\|(I - P)u(t_0; u_0)\| \leq r_-$, so that $u(t_0; u_0) \in \mathcal{V}_S$. Similar arguments hold if $\mathcal{V}_S$ is replaced by $\mathcal{V}_I$.

Up to now we assumed $F(0) = 0$, so that the problem (7.2) has the stationary (= independent of time) solution $u(t) \equiv 0$. Concerning the stability of other possible stationary solutions, that is of the $\overline{u} \in D(A)$ such that

$$A\overline{u} + F(\overline{u}) = 0,$$

we reduce to the case of the null stationary solution by defining a new unknown

$$v(t) = u(t) - \overline{u},$$

and studying the problem

$$v'(t) = Av(t) + F(v(t) + \overline{u}) + A\overline{u},$$

which has the stationary solution $v \equiv 0$.

7.2 Examples

7.2.1 A Cauchy-Dirichlet problem

Let Ω be a bounded open set in $\mathbb{R}^n$ with C^2 boundary $\partial\Omega$, and let $u_0 \in C(\overline{\Omega})$ vanish on the boundary, let $f : \mathbb{R} \mapsto \mathbb{R}$ be continuously differentiable and such that $f(0) = 0$. We study the stability of the null solution of

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + f(u(t,x)), \;\; t > 0, \; x \in \overline{\Omega} \\ \\ u(t,x) = 0, \;\; t > 0, x \in \overline{\Omega}. \end{cases} \tag{7.13}$$

The local existence and uniqueness theorem 6.1.1 may be applied to the initial value problem for equation (7.13),

$$u(0,x) = u_0(x), \;\; x \in \overline{\Omega}, \tag{7.14}$$

choosing as usual $X = C(\overline{\Omega})$. The function

$$F : X \mapsto X, \;\; (F(\varphi))(x) = f(\varphi(x)),$$

is continuously differentiable, and

$$F(0) = 0, \;\; (f'(0)\varphi)(x) = f'(0)\varphi(x), \;\; \forall \varphi \in X.$$

Then, set

$$\begin{cases} A : D(A) = \{\varphi \in \bigcap_{p\geq 1} W^{2,p}(\Omega) : \; \Delta\varphi \in C(\overline{\Omega}), \; \varphi_{|\partial\Omega = 0}\} \mapsto X, \\ \\ A\phi = \Delta\varphi + F'(0)\varphi. \end{cases}$$

A is a sectorial operator, and the spectrum of A consists of a sequence of real eigenvalues which tends to $-\infty$, given by

$$\mu_n = -\lambda_n + f'(0), \;\; n \in \mathbb{N},$$

$\{-\lambda_n\}_{n\in\mathbb{N}}$ being the sequence of the eigenvalues of Δ with Dirichlet boundary condition.

The assumption that $u_0 \in C(\overline{\Omega})$ vanishes on the boundary implies that $u_0 \in \overline{D(A)}$. Theorem 6.1.1 guarantees the existence of a unique local solution $u : [0, \tau(u_0)) \mapsto X$ of the abstract problem (6.1). Setting as usual

$$u(t,x) := u(t)(x), \;\; t \in [0, \tau(u_0)), \; x \in \overline{\Omega},$$

the function u is continuous in $[0, \tau(u_0)) \times \overline{\Omega}$, continuously differentiable with respect to time for $t > 0$, and it satisfies (7.13), (7.14).

Concerning the stability of the null solution, theorem 7.1.1 implies that if $\sup_{\lambda\in\sigma(A)} \mathrm{Re}\,\lambda < 0$, that is, if

$$f'(0) < \lambda_1,$$

($-\lambda_1$ being the first eigenvalue of Δ), then the null solution of (7.13) is exponentially stable: for every $\omega \in (0, \lambda_1 - f'(0))$ there exist r, $C > 0$ such that if $\|u_0\|_\infty \leq r$, then

$$\tau(u_0) = +\infty, \;\; |u(t,x)| \leq Ce^{-\omega t}\|u_0\|_\infty \; \forall t \geq 0, \; x \in \overline{\Omega}.$$

On the contrary, if

$$f'(0) > \lambda_1,$$

then there are elements in the spectrum of A with positive real part. Since they are isolated they satisfy condition (7.9). Theorem 7.1.2 implies that the null solution of

(7.13) is unstable: there exist $\delta > 0$ and initial data u_0 with $\|u_0\|_\infty$ arbitrarily small, but $\sup_{t\geq 0,\, x\in\overline{\Omega}} |u(t,x)| \geq \delta$.

If in addition

$$f'(0) \neq \lambda_n \;\; \forall n \in \mathbb{N},$$

then the assumptions of theorem 7.1.3 hold, so that there exist the stable and the unstable manifolds. The unstable manifold is finite dimensional because it is the graph of a function defined in $P(X)$ which is the space spanned by the finitely many eigenfunctions of Δ corresponding to the eigenvalues $-\lambda_n$ such that $f'(0) - \lambda_n > 0$.

The critical case of stability

$$f'(0) = \lambda_1,$$

where the sup of the real parts of the elements of $\sigma(A)$ is zero, is more difficult and other tools are needed to study it.

7.2.2 A Cauchy-Neumann problem

Similar considerations hold for the problem

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + f(u(t,x)), \;\; t > 0,\; x \in \overline{\Omega} \\ \\ \partial u/\partial\nu(t,x) = 0, \;\; t > 0, x \in \overline{\Omega}, \end{cases} \tag{7.15}$$

where $\nu = \nu(x)$ is the exterior normal vector to $\partial\Omega$ at x. For every continuous initial datum u_0 we write (7.15)-(7.15) in the abstract form (7.2) choosing $X = C(\overline{\Omega})$, $F(\varphi)(x) = f(\varphi(x))$, and

$$\begin{cases} A : D(A) = \{\varphi \in \bigcap_{p\geq 1} W^{2,p}(\Omega) :\; \Delta\varphi \in C(\overline{\Omega}),\, \partial\varphi/\partial\nu = 0\} \mapsto X, \\ \\ A\phi = \Delta\varphi + f'(0)\varphi. \end{cases}$$

A is a sectorial operator by theorem 2.5.2. The spectrum of A consists of a sequence of real eigenvalues which goes to $-\infty$, given again by

$$\mu_n = -\lambda_n + f'(0), \;\; n \in \mathbb{N},$$

$\{-\lambda_n\}_{n\in\mathbb{N}}$ being the ordered sequence of the eigenvalues of Δ with Neumann boundary condition. So, $\lambda_1 = 0$ and $-\lambda_n < 0$ for $n > 1$.

Theorem 6.1.1 guarantees the existence of a unique local solution $u : [0, \tau(u_0)) \mapsto X$ of the abstract problem (6.1). Setting

$$u(t,x) = u(t)(x), \;\; t \in [0, \tau(u_0)),\; x \in \overline{\Omega},$$

the function u is continuous in $[0, \tau(u_0)) \times \overline{\Omega}$, continuously differentiable with respect to time for $t > 0$, and it satisfies (7.15), (7.14).

Concerning the stability of the null solution, theorem 7.1.1 implies that if $\sup_{\lambda\in\sigma(A)}$ $\operatorname{Re}\lambda < 0$, that is, if

$$f'(0) < 0,$$

then the null solution of (7.15) is exponentially stable: for every $\omega \in (0, -f'(0))$ there exist r, $C > 0$ such that if $\|u_0\|_\infty \leq r$, then

$$\tau(u_0) = +\infty, \;\; |u(t,x)| \leq Ce^{-\omega t}\|u_0\|_\infty \; \forall t \geq 0,\; x \in \overline{\Omega}.$$

If

$$f'(0) > 0,$$

then there are elements in the spectrum of A with positive real part. Since they are isolated they satisfy condition (7.9). Theorem 7.1.2 implies that the null solution of (7.15) is unstable.

If in addition

$$f'(0) \neq \lambda_n \ \ \forall n \in \mathbb{N},$$

then the assumptions of theorem 7.1.3 hold, so that there exist the stable and unstable manifolds. Also in this case the unstable manifold is finite dimensional because it is the graph of a function defined in $P(X)$ which is the space spanned by the finitely many eigenfunctions of Δ corresponding to the eigenvalues $-\lambda_n > -f'(0)$.

Appendix A

Vector-valued integration

In this appendix we collect a few basic results on calculus for Banach space valued functions defined in a real interval. These results are assumed to be either known to the reader, or at least not surprising at all, as they follow quite closely the finite-dimensional theory.

Let $I \subset \mathbb{R}$ be an interval, and let X be a Banach space, whose dual is denoted by X', with duality bracket $< x, x' >$. We denote by $C(I;X)$ the vector space of continuous functions $u : I \mapsto X$, by $B(I;X)$ the space of the bounded functions, endowed with the sup-norm

$$\|u\|_\infty = \sup_{t\in I} \|u(t)\|.$$

We also set $C_b(I;X) = C(I;X) \cap B(I;X)$. The definition of the derivative is readily extended to the present situation: a function $f \in C(I;X)$ is differentiable at $t_0 \in I$ if the following limit exists

$$\lim_{t\to t_0} \frac{f(t) - f(t_0)}{t - t_0}.$$

As usual, the limit is denoted by $f'(t_0)$ and is called derivative of f at t_0. In an analogous way we can define right and left derivatives.

For every $k \in \mathbb{N}$ (resp., $k = \infty$), $C^k(I;X)$ denotes the space of X-valued functions with continuous derivatives in I up to the order k (resp., of any order).

Let us define the Riemann integral of an X-valued function on a real interval.

Let $f : [a,b] \to X$ be a bounded function. If there is $x \in X$ such that for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every partition $\mathcal{P} = \{a = t_0 < t_1 < \ldots < t_n = b\}$ of $[a,b]$ with $t_i - t_{i-1} < \delta$ for all i and for any choice of the points $\xi_i \in [t_{i-1}, t_i]$ it follows

$$\Big\|x - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})\Big\| < \varepsilon,$$

we say that f is *integrable* on $[a,b]$ and set

$$\int_a^b f(t)dt = x.$$

generalized integrals of unbounded functions, or on unbounded intervals can be defined as in the real-valued case. From the above definition we obtain immediately the following

Proposition A.1.1 *Let $\alpha, \beta \in \mathbb{C}$, f, g be integrable on $[a,b]$ with values in X.*

(a) $\int_a^b (\alpha f(t) + \beta g(t))dt = \alpha \int_a^b f(t)dt + \beta \int_a^b g(t)dt$;

(b) $|| \int_a^b f(t)dt|| \le \sup_{t\in[a,b]} ||f(t)||(b-a)$;

(c) $< \int_a^b f(t)dt, x' >= \int_a^b < f(t), x' > dt$ *for all* $x' \in X'$;

(d) $||\int_a^b f(t)dt|| \leq \int_a^b ||f(t)||dt$*;*

(e) $A\int_a^b f(t)dt = \int_a^b Af(t)dt$ *for all* $A \in B(X,Y)$*, where* Y *is another Banach space;*

(f) if (f_n) *is a sequence of continuous functions and there is* f *such that*

$$\lim_n \max_{t\in[a,b]} ||f_n(t) - f(t)|| = 0,$$

then $\lim_n \int_a^b f_n(t)dt = \int_a^b f(t)dt$*.*

It is also easy to generalize to the present situation the fundamental theorem of elementary calculus. The proof is the same as for the real-valued case.

Theorem A.1.2 (Calculus Fundamental Theorem) *Let* $f : [a,b] \to X$ *be continuous. Then the integral function*

$$F(t) = \int_a^t f(s)\, ds$$

is differentiable, and $F'(t) = f(t)$ *for every* $t \in [a,b]$*.*

Let us now come to review some basic facts concerning vector-valued functions of a complex variable.

Let Ω be an open subset of $\mathbb{C}$, $f : \Omega \to X$ be a continuous function and $\gamma : [a,b] \to \Omega$ be a C^1-curve. The integral of f along $\{\gamma\}$ is defined by

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Let Ω be an open subset of $\mathbb{C}$ and $f : \Omega \to X$ a continuous function.

Definition A.1.3 f *is holomorphic in* Ω *if for each* $z_0 \in \Omega$ *the limit*

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

exists in the norm of X*.*
f *is weakly holomorphic in* Ω *if the complex-valued functions* $\Omega \ni z \mapsto < f(z), x' >$ *are holomorphic in* Ω *for every* $x' \in X'$*.*

Clearly, any holomorphic function is weakly holomorphic; actually, the converse is also true, as the following theorem shows.

Theorem A.1.4 *Let* $f : \Omega \to X$ *be a weakly holomorphic function. Then* f *is holomorphic.*

Proof. Let $\bar{B}(z_0, r)$ be a closed ball contained in Ω; we prove that for all $z \in B(z_0, r)$ the following Cauchy integral formula holds:

$$f(z) = \frac{1}{2\pi i}\int_{\partial B(z_0,r)} \frac{f(\xi)}{\xi - z} d\xi. \tag{A.1}$$

First of all, we observe that the right hand side is well-defined since f is continuous. Since f is weakly holomorphic in Ω, the complex-valued function $\Omega \ni z \mapsto < f(z), x' >$

is holomorphic in Ω for all $x' \in X'$, and hence the ordinary Cauchy integral formula in $B(z_0, r)$ holds, i.e.,

$$< f(z), x' >= \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{< f(\xi), x' >}{\xi - z} d\xi =< \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(\xi)}{\xi - z} d\xi, x' >;$$

by the arbitrariness of $x' \in X'$, we obtain (A.1). Differentiating under the integral sign, we deduce that f is holomorphic and that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi$$

for all $z \in B(z_0, r)$ and $n \in \mathbb{N}$. □

Definition A.1.5 *Let $f : \Omega \to X$ be a vector-valued function. We say that f has a power series expansion around a point $z_0 \in \Omega$ if there exists $r > 0$ such that $B(z_0, r) \subset \Omega$ and*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{in } B(z_0, r),$$

where $(a_n) \subset X$ and the series is norm-convergent.

Theorem A.1.6 *Let $f : \Omega \to X$ be a vector-valued function; then f is holomorphic if and only if f has a power series expansion around every point of Ω.*

Proof. Assume that f is holomorphic in Ω. Then, if $z_0 \in \Omega$ and $B(z_0, r) \subset \Omega$, Cauchy integral formula (A.1) holds for every $z \in B(z_0, r)$.

Fix $z \in B(z_0, r)$ and observe that the series

$$\sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}} = \frac{1}{\xi - z}$$

converges uniformly for ξ in $\partial B(z_0, r)$, since $\left|\frac{z-z_0}{\xi-z_0}\right| < r^{-1}|z - z_0|$. Consequently, by (A.1) and Proposition A.1.1(f), we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial B(z_0,r)} f(\xi) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}} d\xi \\ &= \sum_{n=0}^{\infty} \Big[\frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \Big] (z - z_0)^n, \end{aligned}$$

the series being norm-convergent.

Suppose, conversely, that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{in } B(z_0, r),$$

where $(a_n) \subset X$ and the series is norm-convergent. Then, for each $x' \in X'$,

$$< f(z), x' >= \sum_{n=0}^{\infty} < a_n, x' > (z - z_0)^n \quad \text{in } B(z_0, r).$$

This means that the complex-valued function $\Omega \ni z \mapsto < f(z), x' >$ is holomorphic in $B(z_0, r)$ for all $x' \in X'$ and hence f is holomorphic by Theorem A.1.4. □

Let us now extend some classical theorems in complex analysis to the case of vector-valued holomorphic functions.

Theorem A.1.7 (Cauchy Theorem) *Let $f : \Omega \to X$ be holomorphic in Ω and let D be a regular domain contained in Ω. Then*

$$\int_{\partial D} f(z)dz = 0.$$

Proof. For each $x' \in X'$ the complex-valued function $\Omega \ni z \mapsto < f(z), x' >$ is holomorphic in Ω and hence

$$0 = \int_{\partial D} < f(z), x' > dz = < \int_{\partial D} f(z)dz, x' > .$$

□

Remark A.1.8 [generalized complex integrals] *As in the case of vector-valued functions defined on a real interval, it is possible to define* generalized complex integrals *in an obvious way. Let $f : \Omega \to X$ be holomorphic, with $\Omega \subset \mathbb{C}$ possibly unbounded. If $I = (a, b)$ is a (possibly unbounded) interval and $\gamma : I \to \mathbb{C}$ is a (piecewise) C^1 curve in Ω, then we set*

$$\int_\gamma f(z)dz = \lim_{\substack{s\to a^+\\ t\to b^-}} \int_s^t f(\gamma(\tau))\gamma'(\tau)d\tau,$$

provided that the limit exists in X. In particular, it is easily seen, as in the elementary case, that if γ' is bounded and for some $c > 0$, $\alpha > 1$ the estimate $\|f(z)\| \le c|z|^{-\alpha}$ holds on γ for large $|z|$, then the integral $\int_\gamma f$ is convergent.

To prove that Laurent expansion holds also for vector-valued holomorphic functions we need the following lemma.

Lemma A.1.9 *Let (a_n) be a sequence in X. Suppose that the power series*

$$\sum_{n=0}^{\infty} < a_n, x' > (z_1 - z_0)^n, \qquad z_1 \ne z_0,$$

converges for all $x' \in X'$. Then the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges in norm for all z with $|z - z_0| < |z_1 - z_0|$.

Proof. We have, for all $x' \in X'$,

$$\lim_n < a_n, x' > (z_1 - z_0)^n = 0;$$

by the uniform boundedness principle, there exists $M > 0$ such that $\|a_n(z_1 - z_0)^n\| \le M$ for all natural n. Putting $q = \left|\frac{z-z_0}{z_1-z_0}\right| < 1$, we have

$$\|a_n(z - z_0)^n\| = \|a_n(z_1 - z_0)^n\|q^n \le Mq^n,$$

and the assertion follows. □

Theorem A.1.10 (Laurent expansion) *Let $f : D = \{z \in \mathbb{C} : r < |z - z_0| < R\} \to X$ be holomorphic. Then, for every $z \in D$*

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n(z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}dz$$

and $C = \{z : |z - z_0| = \varrho\}$, $r < \varrho < R$.

Proof. Since for each $x' \in X'$ the function $D \ni z \mapsto < f(z), x' >$ is holomorphic the usual Laurent expansion holds, that is

$$< f(z), x' >= \sum_{n=-\infty}^{+\infty} a_n(x')(z - z_0)^n$$

where the coefficients $(a_n(x'))$ are given by

$$a_n(x') = \frac{1}{2\pi i} \int_C \frac{< f(z), x' >}{(z - z_0)^{n+1}} dz.$$

By Proposition A.1.1(c), it follows that

$$a_n(x') =< a_n, x' >$$

where the a_n are those indicated in the statement; the assertion then follows from Lemma A.1.9. □

Exercises

A.1 Given a function $u : [a, b] \times [0, 1] \to \mathbb{R}$, set $U(t)(x) = u(t, x)$. Show that $U \in C([a, b]; C([0, 1]))$ if and only if u is continuous, and that $U \in C^1([a, b]; C([0, 1]))$ if and only if u is continuous, differentiable with respect to t and the derivative u_t is continuous.

A.2 Let $f : I \to X$ be a continuous function. Prove that if f admits a continuous right-derivative on I, then it is differentiable in I.

A.3 Let $f : [a, b] \to X$ be a continuous function. Show that f is integrable.

A.4 Prove Proposition A.1.1.

A.5 Show that if $f : (a, b] \to X$ is continuous and $\|f(t)\| \leq g(t)$ for all $t \in (a, b]$, with g integrable in $[a, b]$, then the generalized integral of f on $[a, b]$ is convergent.

A.6 Let I_1, I_2 be two real intervals, and let $g : I_1 \times I_2 \to X$ be continuous, and such that for every $(\lambda, t) \in I_1 \times I_2$ the inequality $\|g(\lambda, t)\| \leq \varphi(t)$ holds, with φ integrable in I_2. Prove that the function

$$G(\lambda) = \int_{I_2} g(\lambda, t) dt, \qquad \lambda \in I_1$$

is continuous in I_1. Show that if g is differentiable with respect to λ, g_λ is continuous and $\|g_\lambda(\lambda, t)\| \leq \psi(t)$ with ψ integrable in I_2, then G is differentiable in I_1 and

$$G'(\lambda) = \int_{I_2} g_\lambda(\lambda, t) dt, \quad \lambda \in I_1.$$

Appendix B

Basic Spectral Theory

In this appendix we collect a few basic results on elementary spectral theory, in order to fix the notation used in the lectures and to give easy references.

Let us denote by $\mathcal{L}(X)$ the Banach algebra of linear and continuous operators $T : X \to X$, endowed with the norm

$$\|T\| = \sup_{x\in X:\,\|x\|=1} \|Tx\| = \sup_{x\in X\setminus\{0\}} \frac{\|Tx\|}{\|x\|}.$$

If $D(L)$ is a vector subspace of X and $L : D(L) \to X$ is linear, we say that L is *closed* if its graph

$$\mathcal{G}_L = \{(x, y) \in X \times X : x \in D(L),\ y = Lx\}$$

is a closed set of $X \times X$. In an equivalent way, L is closed if and only if the following implication holds:

$$\{x_n\} \subset D(L),\ \ x_n \to x,\ \ Lx_n \to y \qquad \Longrightarrow \qquad x \in D(L),\ y = Lx.$$

We say that L is *closable* if there is an (obviously unique) operator $\overline{L}$, whose graph is the closure of $\mathcal{G}_L$. It is readily checked that L is closable if and only if the implication

$$\{x_n\} \subset D(L),\ \ x_n \to 0,\ \ Lx_n \to y \qquad \Longrightarrow \qquad y = 0.$$

holds. If $L : D(L) \subset X \to X$ is a closed operator, we endow $D(L)$ with its *graph norm*

$$\|x\|_{D(L)} = \|x\| + \|Lx\|.$$

$D(L)$ turns out to be a Banach space and $L : D(L) \to X$ is continuous.

Let us prove some useful lemmas.

Lemma B.1.1 *Let X, Y be two Banach spaces, let D be a subspace of X, and let $\{A_n\}_{n\geq 0}$ be a sequence of continuous linear operators from X to Y such that*

$$\|A_n\| \leq M,\ \ \forall n \in \mathbb{N}, \quad \lim_{n\to\infty} A_n x = A_0 x \ \ \forall x \in D.$$

Then

$$\lim_{n\to\infty} A_n x = A_0 x \ \ \forall x \in \overline{D},$$

where $\overline{D}$ is the closure of D in X.

Proof. Let $x \in \overline{D}$ and $\varepsilon > 0$ be given. For $y \in D$ with $\|x - y\| \leq \varepsilon$ and for every $n \in \mathbb{N}$ we have

$$\|A_n x - A_0 x\| \leq \|A_n(x - y)\| + \|A_n y - A_0 y\| + \|A_0(y - x)\|.$$

If n_0 is such that $\|A_n y - A_0 y\| \le \varepsilon$ for every $n > n_0$, we have

$$\|A_n x - A_0 x\| \le M\varepsilon + \varepsilon + \|A_0\|\varepsilon$$

for all $n \ge n_0$. $\square$

Lemma B.1.2 *Let $A : D(A) \subset X \to X$ be a closed operator, I a real interval with endpoints a, b $(-\infty \le a < b \le +\infty)$ and let $f : I \to D(A)$ be such that the functions $t \mapsto f(t)$, $t \mapsto Af(t)$ are integrable on I. Then*

$$\int_a^b f(t)dt \in D(A), \quad A\int_a^b f(t)dt = \int_a^b Af(t)dt.$$

Proof. Assume first that I is compact. Set $x = \int_a^b f(t)dt$, let us choose a sequence $\mathcal{P}_k = \{a = t_0^k < \ldots < t_{n_k}^k = b\}$ of partitions of $[a, b]$ such that $\max_{i=1,\ldots,n_k}(t_i^k - t_{i-1}^k) < 1/k$. Let $\xi_i^k \in [t_i^k, t_{i-1}^k]$ for $i = 0, \ldots, n_k$, and consider

$$S_k = \sum_{i=1}^{n_k} f(\xi_i)(t_i - t_{i-1}).$$

All S_k are in $D(A)$, and

$$AS_k = \sum_{i=1}^{n} Af(\xi_i)(t_i - t_{i-1}).$$

Since both f and Af are integrable, S_k tends to x and AS_k tends to $y = \int_a^b Af(t)dt$, and since A is closed, x belongs to $D(A)$ and $Ax = y$. Let now I be unbounded, say $I = [a, +\infty)$; then, for every $b > a$ the equality

$$A\int_a^b f(t)dt = \int_a^b Af(t)dt$$

holds. By hypothesis

$$\int_a^b Af(t)dt \;\to\; \int_a^\infty Af(t)dt \quad \text{and} \quad \int_a^b f(t)dt \;\to\; \int_a^\infty f(t)dt \quad \text{as } b \to +\infty,$$

hence

$$A\int_a^b f(t)dt \to \int_a^\infty Af(t)dt$$

and, by the closedness of A, the thesis follows. $\square$

Given an operator (not necessarily closed) $A : D(A) \subset X \to X$, define its *adjoint* $A' : D(A') \subset X' \to X'$ through

$$\begin{aligned} &D(A') = \{y \in X' :\ \exists\, z \in X' \text{ such that } \langle Ax, y\rangle = \langle x, z\rangle\ \forall\, x \in D(A)\}, \\ &A'y = z \text{ for } y, z \text{ as above.} \end{aligned}$$

Notice that $(A', D(A'))$ is always a closed operator.

Let us now introduce the notions of resolvent and spectrum of a linear operator.

Definition B.1.3 *Let $A : D(A) \subset X \to X$ be a linear operator. The* resolvent set $\rho(A)$ *and the* spectrum $\sigma(A)$ *of A are defined by*

$$\rho(A) = \{\lambda \in \mathbb{C} : \exists\, (\lambda I - A)^{-1} \in \mathcal{L}(X)\}, \quad \sigma(A) = \mathbb{C}\backslash\rho(A). \tag{B.1}$$

The complex numbers $\lambda \in \sigma(A)$ such that $\lambda I - A$ is not injective are the eigenvalues *of A, and the elements $x \in D(A)$ such that $x \ne 0$, $Ax = \lambda x$ are the* eigenvectors *(or* eigenfunctions*, when X is a function space) of A relative to the eigenvalue λ. The set $\sigma_p(A)$ whose elements are the eigenvalues of A is the* point spectrum *of A.*

If $\lambda \in \rho(A)$, set

$$(\lambda I - A)^{-1} = R(\lambda, A) \tag{B.2}$$

and $R(\lambda, A)$ is the *resolvent operator* or briefly *resolvent.*

It is easily seen (cf Exercise 1 below) that if $\rho(A) \neq \emptyset$ then A is closed.

Let us recall some simple properties of resolvent and spectrum. First of all, it is clear that if $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ are linear operators such that $R(\lambda_0, A) = R(\lambda_0, B)$ for some $\lambda_0 \in \mathbb{C}$, then $D(A) = D(B)$ and $A = B$. In fact,

$$D(A) = \text{Range}\, R(\lambda_0, A) = \text{Range}\, R(\lambda_0, B) = D(B),$$

and for every $x \in D(A) = D(B)$, set $y = \lambda_0 x - Ax$, one has $x = R(\lambda_0, A)y = R(\lambda_0, B)y$, and this, applying $\lambda_0 I - B$, implies $\lambda_0 x - Bx = y$, so that $\lambda_0 x - Ax = \lambda_0 x - Bx$ and therefore $Ax = Bx$.

The following formula, called *resolvent identity*, can be easily verified:

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A), \ \ \forall\ \lambda,\ \mu \in \rho(A). \tag{B.3}$$

In fact, write

$$\begin{aligned} R(\lambda, A) &= [\mu R(\mu, A) - AR(\mu, A)]R(\lambda, A) \\ R(\mu, A) &= [\lambda R(\lambda, A) - AR(\lambda, A)]R(\mu, A) \end{aligned}$$

and subtract the above equations; taking into account that $R(\lambda, A)$ and $R(\mu, A)$ commute, we get (B.3).

The resolvent identity characterizes the resolvent operators, as specified in the following proposition.

Proposition B.1.4 *Let $\Omega \subset \mathbb{C}$ be an open set, and let $\{F(\lambda) : \lambda \in \Omega\} \subset \mathcal{L}(X)$ be linear operators verifying the resolvent identity*

$$F(\lambda) - F(\mu) = (\mu - \lambda)F(\lambda)F(\mu), \ \ \forall\lambda,\ \mu \in \Omega.$$

If for some $\lambda_0 \in \Omega$, the operator $F(\lambda_0)$ is invertible, then there is a linear operator $A : D(A) \subset X \to X$ such that $\rho(A)$ contains Ω, and $R(\lambda, A) = F(\lambda)$ for all $\lambda \in \Omega$.

Proof. Fix $\lambda_0 \in \Omega$, and set

$$D(A) = \text{Range}\ F(\lambda_0), \ \ Ax = \lambda_0 x - F(\lambda_0)^{-1}x \ \ \forall x \in D(A).$$

For $\lambda \in \Omega$ and $y \in X$ the resolvent equation $\lambda x - Ax = y$ is equivalent to $(\lambda - \lambda_0)x + F(\lambda_0)^{-1}x = y$. Applying $F(\lambda)$ we obtain $(\lambda - \lambda_0)F(\lambda)x + F(\lambda)F(\lambda_0)^{-1}x = F(\lambda)y$, and using the resolvent identity it is easily seen that

$$F(\lambda)F(\lambda_0)^{-1} = F(\lambda_0)^{-1}F(\lambda) = (\lambda_0 - \lambda)F(\lambda) + I.$$

Hence, if x is solution of the resolvent equation, then $x = F(\lambda)y$. Let us check that $x = F(\lambda)y$ is actually a solution. In fact, $\lambda_0 F(\lambda)y + F(\lambda_0)^{-1}F(\lambda) = y$, and therefore λ belongs to $\rho(A)$ and the equality $R(\lambda, A) = F(\lambda)$ holds. □

Next, let us show that $\rho(A)$ is an open set.

Proposition B.1.5 *Let λ_0 be in $\rho(A)$. Then, $|\lambda - \lambda_0| < \frac{1}{\|R(\lambda_0, A)\|}$ implies that λ belongs to $\rho(A)$ and the equality*

$$R(\lambda, A) = R(\lambda_0, A)(I + (\lambda - \lambda_0)R(\lambda_0, A))^{-1} \tag{B.4}$$

holds. As a consequence, $\rho(A)$ is open and $\sigma(A)$ is closed.

Proof. In fact,

$$(\lambda - A)(I + (\lambda - \lambda_0)R(\lambda_0, A))(\lambda_0 - A$$

Since $\|(\lambda - \lambda_0)R(\lambda_0, A)\| < 1$, the operator $I + (\lambda - \lambda_0)R(\lambda_0, A)$ is invertible and has a continuous inverse (see Exercise (B.2)). Hence,

$$R(\lambda, A) = R(\lambda_0, A)(I + (\lambda - \lambda_0)R(\lambda_0, A))^{-1}$$

□

Further properties of the resolvent operator are listed in the following proposition.

Proposition B.1.6 *The function $R(\cdot, A)$ is holomorphic in $\rho(A)$ and the equalities*

$$R(\lambda, A) = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_0)^n R^{n+1}(\lambda_0, A) \tag{B.5}$$

$$\frac{d^n R(\lambda, A)}{d\lambda^n}_{|\lambda=\lambda_0} = (-1)^n n! R^{n+1}(\lambda_0, A) \tag{B.6}$$

hold.

Proof. (i) If $|\lambda - \lambda_0| < \frac{1}{\|R(\lambda_0, A)\|}$, from (B.4) we deduce

$$R(\lambda, A) = R(\lambda_0, A) \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_0)^n R(\lambda_0, A)^n = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_0)^n R(\lambda_0, A)^{n+1}$$

and the statement follows. □

Proposition B.1.5 implies also that the resolvent set is the domain of analyticity of the function $\lambda \mapsto R(\lambda, A)$.

Corollary B.1.7 *The domain of analyticity of the function $\lambda \mapsto R(\lambda, A)$ is $\rho(A)$, and the estimate*

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \geq \frac{1}{\text{dist}(\lambda, \sigma(A))}. \tag{B.7}$$

holds.

Proof. It suffices to prove (B.7), because it proves that $R(\cdot, A)$ is unbounded approaching $\sigma(A)$. From Proposition B.1.5 for every $\lambda \in \rho(A)$ we get that if $|z - \lambda| < 1/\|R(\lambda, A)\|_{\mathcal{L}(X)}$ then $z \in \rho(A)$, and then $\text{dist}\,(\lambda, \sigma(A)) \geq 1/\|R(\lambda, A)\|_{\mathcal{L}(X)}$, from which (B.7) follows. □

Let us recall also some spectral properties of bounded operators.

Proposition B.1.8 *Let us consider $T \in \mathcal{L}(X)$; the power series*

$$F(z) = \sum_{k=0}^{\infty} z^k T^k, \quad z \in \mathbb{C}. \tag{B.8}$$

(called Neumann series of $(I - zT)^{-1}$*) is norm-convergent in the disk $\overline{B}(0, 1/r(T))$, where*

$$r(T) = \limsup_{n\to\infty} \sqrt[n]{\|T^n\|}.$$

Moreover, $|z| < 1/r(T)$ implies

$$F(z) = (I - zT)^{-1} \tag{B.9}$$

and $|z| < 1/\|T\|$ implies

$$\|(I - zT)^{-1}\| \leq \frac{1}{1 - |z|\,\|T\|} \tag{B.10}$$

Proof. The convergence of (B.8) in the disk $\overline{B}(0, r(T))$ easily follows from the root criterion applied to the scalar series $\sum_{k=1}^{\infty} \|T^k\| \, |z|^k$. To prove equation (B.9), it suffices to check that if $|z| < 1/r(T)$ then

$$(I - zT)F(z) = F(z)(I - zT) = I$$

Finally, (B.10) follows from the inequality

$$\|F(z)\| \leq \sum_{k=0}^{\infty} |z|^k \|T\|^k = \frac{1}{1 - |z| \, \|T\|}.$$

□

Proposition B.1.9 *Consider $T \in \mathcal{L}(X)$. Then the following properties hold.*

(i) $\sigma(T)$ is contained in the disk $\overline{B}(0, r(T))$ and if $|\lambda| > r(T)$ then $\lambda \in \rho(T)$, and the equality

$$R(\lambda, T) = \sum_{k=0}^{\infty} T^k \lambda^{-k-1}. \tag{B.11}$$

holds. For this reason, $r(T)$ is called spectral radius of T. *Moreover, $|\lambda| > \|T\|$ implies*

$$\|R(\lambda, T)\| \leq \frac{1}{|\lambda| - \|T\|} \tag{B.12}$$

(ii) $\sigma(T)$ is non-empty.

Proof. (i) follows from Proposition B.1.8, noticing that, for $\lambda \neq 0$, $\lambda - T = \lambda(I - (1/\lambda)T)$.
(ii) Suppose by contradiction that $\sigma(T) = \emptyset$. Then, $R(\cdot, T)$ is an entire function, and then for every $x \in X$, $x' \in X'$ the function $\langle R(\cdot, T)x, x' \rangle$ is entire and tends to 0 at infinity and then is constant by Liouville theorem. As a consequence, $R(\lambda, T) = 0$ for all $\lambda \in \mathbb{C}$, which is absurd. □

Exercises

B.1 Show that if $A : D(A) \subset X \to X$ has non-empty resolvent set, then A is closed.

B.2 Show that if $A \in \mathcal{L}(X)$ and $\|A\| < 1$ then $I + A$ is invertible, and

$$(I + A)^{-1} = \sum_{k=0}^{\infty} (-1)^k A^k.$$

B.3 Show that for every $\alpha \in \mathbb{C}$ the equalities $\sigma(\alpha A) = \alpha\sigma(A)$, $\sigma(\alpha I - A) = \alpha - \sigma(A)$ hold. Prove also that if $0 \in \rho(A)$ then $\sigma(A^{-1}) \setminus \{0\} = 1/\sigma(A)$, and that $\rho(A + \alpha I) = \rho(A) + \alpha$, $R(\lambda, A + \alpha I) = R(\lambda - \alpha, A)$ for all $\lambda \in \rho(A) + \alpha$.

B.4 Let $\varphi : [a, b] \to \mathbb{C}$ be a continuous function, and consider the multiplication operator $A : C([a, b]; \mathbb{C}) \to C([a, b]; \mathbb{C})$, $(Af)(x) = f(x)\varphi(x)$. Compute the spectrum of A. In which cases are there eigenvalues in $\sigma(A)$?

Solve the same problems with $L^p((a, b); \mathbb{C})$, $p \geq 1$, in place of $C([a, b]; \mathbb{C})$.

B.5 Let $C_b(\mathbb{R})$ be the space of bounded and continuous functions on $\mathbb{R}$, endowed with the sup-norm, and let A be the operator defined by

$$D(A) = C_b^1(\mathbb{R}) = \{f \in C_b(\mathbb{R}) : \; \exists f' \in C_b(\mathbb{R})\} \to C_b(\mathbb{R}), \quad Af = f'.$$

Compute $\sigma(A)$ and $R(\lambda, A)$, for $\lambda \in \rho(A)$. Which are the eigenvalues of A?

B.6 Let $P \in \mathcal{L}(X)$ be a projection, i.e., $P^2 = P$. Compute $\sigma(A)$, find the eigenvalues and compute $R(\lambda, P)$ for $\lambda \in \rho(P)$.

B.7 Consider the space $X = C([0,\pi])$ and the operators $D(A_1) = \{f \in C^2([0,\pi]) : f(0) = f(\pi) = 0\}$, $A_1 f = f''$, $D(A_2) = \{f \in C^2([0,\pi]) : f'(0) = f'(\pi) = 0\}$, $A_2 f = f''$. Compute $\sigma(A_1)$, $\sigma(A_2)$ and $R(\lambda, A_1)$, $R(\lambda, A_2)$ for $\lambda \in \rho(A_1)$ and $\lambda \in \rho(A_2)$, respectively.

B.8 Let $X = C([0,1])$, and consider the operators A, B, C on X defined by

$$\begin{array}{rcl} D(A) & = & C^1([0,1]): \ Au = u', \\ D(B) & = & \{u \in C^1([0,1]): \ u(0) = 0\}, \ Bu = u', \\ D(C) & = & \{u \in C^1([0,1]); \ u(0) = u(1)\}, \ Cu = u'. \end{array}$$

Show that

$$\begin{array}{l} \rho(A) = \emptyset, \ \ \sigma(A) = \mathbb{C}. \\ \rho(B) = \mathbb{C}, \ \ \sigma(B) = \emptyset, \ \ (R(\lambda, B)f)(\xi) = -\displaystyle\int_0^\xi e^{\lambda(\xi-\eta)} f(\eta) d\eta, \ \ 0 \le \xi \le 1. \\ \rho(C) = \mathbb{C} \setminus \{2k\pi i: \ k \in \mathbb{Z}\}, \ \ \sigma(C) = \{2k\pi i: \ k \in \mathbb{Z}\} \end{array}$$

with $2k\pi i$ eigenvalue, with eigenfunction $\xi \mapsto ce^{2k\pi i \xi}$, and, for $\lambda \notin \{2k\pi i, \ k \in \mathbb{Z}\}$,

$$(R(\lambda, C)f)(\xi) = \frac{e^{\lambda\xi}}{e^\lambda - 1} \int_0^1 e^{\lambda(1-\eta)} f(\eta) d\eta - \int_0^\xi e^{\lambda(\xi-\eta)} f(\eta) d\eta.$$

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